

Maximal Gaps between PL and Linear Embedding Dimensions

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Abstract

For a finite simplicial complex K , let $e_{\text{PL}}(K)$ and $e_{\text{L}}(K)$ denote its minimum piecewise-linear and simplex-wise linear embedding dimensions. We prove that, for every $d \geq 3$, a finite d -complex satisfies $e_{\text{PL}}(K) = d$ and $e_{\text{L}}(K) = 2d + 1$; this is the largest possible gap in that dimension. In dimension two we obtain the pair $(3, 5)$. These examples answer the minimum-dimension question recorded by Brehm and Sarkaria in the 2006 Oberwolfach report. The proof adapts Newman's Radon-pair, order-type and Janson method to make a random nonembedding obstruction survive every small set of facet deletions. Removing overlaps then produces a linear hypergraph, to which the PL realization lemma of Lee and Nevo applies. The construction is probabilistic and existential; no manifold hypothesis or explicit small facet list is asserted.

1 The dimension gap

A linear embedding of a finite abstract simplicial complex means an injective map of its realization that is affine on every simplex of the given complex. A piecewise-linear embedding is linear on a subdivision. Define $e_{\text{L}}(K)$ and $e_{\text{PL}}(K)$ to be the respective least Euclidean dimensions. If $\dim K = d$, then

$$d \leq e_{\text{PL}}(K) \leq e_{\text{L}}(K) \leq 2d + 1. \quad (1)$$

For the last bound, place the vertices in affine general position in $\mathbb{R}^{2d+1}$. For the first, an embedded subdivision of a d -simplex must still contain a nondegenerate d -simplex.

Brehm's report of joint work with Sarkaria asks whether the difference $e_{\text{L}}(K) - e_{\text{PL}}(K)$ can exceed one, with particular interest in dimensions two and three [1, p. 702]. Earlier fixed-ambient separations appear in [2]. We give the following exact dimension pairs.

Theorem 1. *For each integer $d \geq 3$, there is a finite d -dimensional simplicial complex K such that*

$$e_{\text{PL}}(K) = d, \quad e_{\text{L}}(K) = 2d + 1.$$

There is also a finite two-dimensional complex K such that

$$e_{\text{PL}}(K) = 3, \quad e_{\text{L}}(K) = 5.$$

Consequently the gap is unbounded, and for every $d \geq 3$ it attains the upper bound $d + 1$ in (1).

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Our complexes are generated by uniform hypergraphs: for a hypergraph H on $[n]$, write $K(H)$ for the complex containing all subsets of its hyperedges and all its vertices. A hypergraph is called *linear* when distinct hyperedges meet in at most one vertex. This combinatorial adjective is separate from linear embeddability.

The two main inputs have distinct roles. Newman's proof of random linear nonembeddability uses balanced Radon pairs, an order-type count and Janson's inequality [3, Section 4]. We make the obstruction uniform over a family of potential deletions. Lee and Nevo's PL realization theorem for linear hypergraphs [4, Lemma 3.1] then gives a much smaller PL dimension. All these inputs are credited below.

2 An obstruction uniform over small deletions

We first recall two geometric facts in the precise form needed. The affine van Kampen–Flores theorem implies that any $2r + 1$ points in $\mathbb{R}^{2r-2}$ contain two disjoint r -subsets whose convex hulls intersect. Double counting gives at least

$$\frac{1}{2r+1} \binom{n}{2r} \quad (2)$$

such pairs among n points in general position: each pair is contained in at most $n - 2r$ subsets of size $2r + 1$. This is [3, Corollary 10], which cites the van Kampen–Flores input of Blagojević, Frick and Ziegler. The Goodman–Pollack order-type bound, in the form recorded in [3, Lemma 12], says that the number of simple labelled order types in any fixed dimension is $\exp(O(n \log n))$. An order type determines all Radon partitions and hence all the pairs counted in (2).

We use Janson's inequality with the following convention. For events A_i that specified sets of independent Bernoulli coordinates are all present, put

$$X = \sum_i \mathbf{1}_{A_i}, \quad \mu = \mathbb{E}X, \quad \Delta = \sum_{\substack{i \neq j \\ \text{coordinate sets overlap}}} \mathbb{P}(A_i \cap A_j),$$

where the sum is over ordered pairs. Then

$$\mathbb{P}(X = 0) \leq \exp\left(-\frac{\mu^2}{2(\mu + \Delta)}\right). \quad (3)$$

See [5, Theorem 8.2.2 and equation (8.1)]. The random coordinates in our application will be whole hyperedges, not their constituent vertices.

Proposition 2. *Fix $r \geq 3$. Independently select each r -subset of $[n]$ with probability*

$$p = n^{-r+5/4},$$

and call the resulting hypergraph H . Put $t = \lfloor n^{3/4} \rfloor$. With probability tending to one as $n \rightarrow \infty$, for every $F \subseteq \binom{[n]}{r}$ with $|F| \leq t$, the complex $K(H \setminus F)$ has no linear embedding in $\mathbb{R}^{2r-2}$.

Proof. Write $\mathcal{U} = \binom{[n]}{r}$ and $N = |\mathcal{U}|$. Fix a simple order type q of n points in $\mathbb{R}^{2r-2}$. Its crossing graph G_q has vertex set $\mathcal{U}$, and an edge $\{A, B\}$ whenever A, B are disjoint r -sets with intersecting convex hulls. By (2), for sufficiently large n ,

$$|E(G_q)| \geq c_r n^{2r}$$

for a constant $c_r > 0$ independent of q .

Fix also $F \subseteq \mathcal{U}$ with $|F| \leq t$. Removing F from the crossing graph loses at most tN edges. Since $r + 3/4 < 2r$,

$$tN = O_r(n^{r+3/4}) = o(n^{2r}),$$

so $G = G_q[\mathcal{U} \setminus F]$ still has at least $(c_r/2)n^{2r}$ edges, uniformly in q, F .

Let X count edges of G whose two endpoints were selected into H . A counted edge is a forbidden intersection of surviving facets. Its expectation satisfies

$$\mu = |E(G)|p^2 = \Theta_r(n^{5/2}). \quad (4)$$

Two distinct events in this sum are independent unless they share a vertex of G , that is, the same *entire* r -subset. Sharing an original vertex of $[n]$ is not enough to create dependence. The number of ordered pairs of distinct edges of G sharing an endpoint is

$$\sum_{u \in V(G)} \deg_G(u)(\deg_G(u) - 1) \leq N^3.$$

Each joint event has probability p^3 , and therefore

$$\Delta \leq N^3 p^3 = O_r(n^{15/4}). \quad (5)$$

Equations (3)–(5) give a constant $a_r > 0$ such that

$$\mathbb{P}(X = 0) \leq \exp(-a_r n^{5/4}), \quad (6)$$

uniformly over the fixed choices q, F .

There are $\exp(O_r(n \log n))$ choices of q . Also

$$\#\{F \subseteq \mathcal{U} : |F| \leq t\} \leq (t+1)N^t = \exp(O_r(n^{3/4} \log n)).$$

Taking a union bound over both choices bounds the probability of any exception by

$$\exp(O_r(n \log n) + O_r(n^{3/4} \log n) - a_r n^{5/4}) = o(1).$$

Thus every general-position placement retains a crossing after every permitted deletion.

Finally, general position entails no loss. In any linear embedding of a finite complex, the convex hulls of each pair of vertex-disjoint faces have positive distance. The minimum of these finitely many positive distances remains positive under sufficiently small perturbations of the vertices. This disjoint-face condition also guarantees injectivity: cancel common coefficients in an equality of two distinct convex combinations to obtain an intersection between convex hulls of disjoint faces. The same argument excludes affine degeneracies within a simplex. Hence any linear embedding could be perturbed to a generic one. The preceding obstruction rules it out. $\square$

3 Alteration and the PL dimension

Lemma 3. *For every fixed $r \geq 3$ and all sufficiently large n , there is a linear r -uniform hypergraph H' on $[n]$ with $\Omega_r(n^{5/4})$ hyperedges such that*

$$e_L(K(H')) = 2r - 1.$$

Proof. Sample H as in Proposition 2. Let B count unordered pairs of distinct selected hyperedges meeting in at least two vertices. Exactly,

$$\mathbb{E}B = \frac{p^2}{2} \binom{n}{r} \sum_{j=2}^{r-1} \binom{r}{j} \binom{n-r}{r-j} = O_r(n^{2r-2}p^2) = O_r(n^{1/2}).$$

Thus Markov's inequality gives $\mathbb{P}(B > \lfloor n^{3/4} \rfloor) = O_r(n^{-1/4}) = o(1)$. The number of selected hyperedges is binomial, with mean $\binom{n}{r}p = \Theta_r(n^{5/4})$ and variance at most its mean. Chebyshev's inequality implies that it is at least half its mean with probability $1 - o(1)$.

Choose an outcome for which these two conclusions and Proposition 2 hold simultaneously. For each bad pair, mark one of its hyperedges for deletion. The union F of marked hyperedges has $|F| \leq B \leq t$. Then $H' = H \setminus F$ is linear and has

$$|E(H')| \geq \frac{1}{2} \binom{n}{r} p - t = \Omega_r(n^{5/4}).$$

Although F depends on H , Proposition 2 applies: it was proved simultaneously for every potential F . Hence $K(H')$ does not linearly embed in $\mathbb{R}^{2r-2}$. The general-position upper bound for an $(r-1)$ -complex gives a linear embedding in $\mathbb{R}^{2r-1}$, proving the equality. $\square$

Lemma 4 (Lee–Nevo). *If $d \geq 3$, the complex generated by a linear $(d+1)$ -uniform hypergraph has a PL embedding in $\mathbb{R}^d$.*

This is precisely [4, Lemma 3.1]. Their construction embeds the incidence graph and enlarges its facet vertices into polyhedral cells; linearity controls the intersections. We use the theorem as a credited input.

Proof of Theorem 1. For $d \geq 3$, take $r = d + 1$ in Lemma 3. Lemma 4 gives $e_{\text{PL}}(K(H')) \leq d$, and (1) gives the reverse inequality. The linear dimension is $2r - 1 = 2d + 1$.

For dimension two, take $r = 3$. Extend each triangle e of H' to a four-element hyperedge $e \cup \{w_e\}$ by adding a distinct private vertex w_e . Distinct extended hyperedges still meet in at most one vertex. By Lemma 4 with $d = 3$, their generated complex embeds PL in $\mathbb{R}^3$. Its subcomplex $K(H')$ therefore does too.

Let $m = |E(H')|$. The one-skeleton of $K(H')$ has exactly $3m$ edges, since different triangles share no edge. For sufficiently large n , the lower bound $m = \Omega(n^{5/4})$ gives $m > n$. The planar graph bound of $3n - 6$ edges excludes a PL embedding of this one-skeleton in $\mathbb{R}^2$. Consequently $e_{\text{PL}}(K(H')) = 3$. Lemma 3 gives $e_{\text{L}}(K(H')) = 5$. $\square$

4 Scope and reproducibility

The result concerns arbitrary finite simplicial complexes. The examples need not be manifolds, and the proof does not give a small explicit facet list or optimal vertex counts. For each fixed dimension all constants are allowed to depend on that dimension. The all-deletion estimate is important: merely finding a crossing and then deleting overlapping facets would not show that any obstruction survives.

Newman's concluding comparison [3, Theorem 16] uses the Linial–Meshulam–Wallach complex with its complete lower skeleton and obtains PL/linear separation in the same ambient dimension $2d$. Here we keep only the complex generated by the repaired hyperedges and apply Lee–Nevo to lower its PL dimension. No inference from collapsibility or from an undecidability comparison is needed.

The accompanying standard-library Python script checks exact Radon dependencies on five rational configurations, the bad-pair counting formula, the ordered-dependency and deletion bounds, private-vertex extensions, and the exponent arithmetic. Its 2000 checks agree in ordinary and optimized Python. These finite diagnostics are not a substitute for the asymptotic existence proof, and none of the sampled hypergraphs is claimed to be a universal nonembedding witness. AI tools assisted the derivation, source search, verification code and exposition. This is an unrefereed preprint; no independent review, proof-assistant formalization or absolute priority certification is asserted.

References

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