

A Negative Answer to a Question of Santos on the Dual Diameter of Surfaces with Boundary

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Abstract

Let $\bar{f}(2, n)$ be the maximum diameter of the dual graph of a triangulated surface with boundary on n vertices. At a 2012 Oberwolfach workshop, Santos asked whether $\bar{f}(2, n) \leq n - 3$. We show that the answer is no. A triangulation M_7 of the real projective plane minus two disjoint open discs has 7 vertices, 10 triangles and dual diameter 5. The same complex appears in work of Holmes on Stanley–Reisner rings with Serre’s property (S_2) , where it is not identified as a surface. Gluing copies of M_7 along boundary edges gives $\bar{f}(2, n) \geq n - 3 + \lfloor (n - 2)/5 \rfloor$ for all $n \geq 3$, so the excess over $n - 3$ is unbounded. An orientable surface of genus 2 with 10 vertices and dual diameter 8 gives the bound $n - 3 + \lfloor (n - 2)/8 \rfloor$ for orientable surfaces. A short layering argument shows $\bar{f}(2, n) \leq \max(n - 3, 2n - 8)$, and a theorem of Holmes gives $\max(2n - 10, n - 2)$. Computer searches with DRAT-certified unsatisfiability proofs, confirmed by an exhaustive enumeration of the surfaces with at most 10 vertices, show that $\bar{f}(2, n) = n - 3$ for $n \leq 6$ and $\bar{f}(2, n) = n - 2$ for $7 \leq n \leq 10$ (for $6 \leq n \leq 9$ these values also follow from results of Holmes), that M_7 is the only counterexample with at most 7 vertices, and that orientable surfaces with at most 9 vertices satisfy the bound. This is an unrefereed note.

1 Introduction

The *dual graph* of a pure simplicial complex has the facets as vertices, two facets being adjacent when they share a ridge; its diameter is the (*dual*) *diameter* of the complex. For the dual complex of a simple polytope this is the diameter of the polytope, and dual diameters of simplicial spheres, balls and manifolds are studied as relaxations of the Hirsch question [San13, AB14]. Without a manifold condition the diameter can be much larger: Criado and Santos [CS17] constructed $(d - 1)$ -dimensional pseudomanifolds without boundary on n vertices with diameter of order n^{d-1} .

In the problem session of the Oberwolfach workshop *Triangulations* (2012), Santos posed the following question [JLSS12, p. 1480, problem 9]. Let $f(d, n)$ be the maximal dual diameter of a closed simplicial d -manifold with n vertices, and $\bar{f}(d, n)$ the same for manifolds with boundary. Santos recalls that $f(2, n) \leq n - 3$ and $\bar{f}(2, n) \leq 2n$, and asks whether surfaces with boundary satisfy

$$\bar{f}(2, n) \leq n - 3. \tag{1}$$

This is the Hirsch-type bound $n - (d + 1)$ for $d = 2$. In dimension 3 the analogous bound fails for balls: the unbounded simple 4-polyhedron of Klee and Walkup with 8 facets and diameter 5 [KW67] is dual to a simplicial 3-ball with 8 vertices and dual diameter $5 > 8 - 4$ (for this duality see [San13, Sect. 1 and Prop. 2.4]). The question is record OWR-11786-023 of the UnsolvedMath dataset [ula26], where it is listed as open. We show that the answer is no, already for $n = 7$, and more.

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Theorem 1.1. (a) The complex M_7 on $\{0, \dots, 6\}$ with the ten triangles 012, 014, 035, 036, 045, 123, 235, 245, 246, 346 is a triangulation of the real projective plane with two disjoint open discs removed, and its dual diameter is $5 > 7 - 3$.

(b) $\bar{f}(2, n) \geq n - 3 + \lfloor (n - 2)/5 \rfloor$ for every $n \geq 3$.

(c) There is an orientable surface G_{10} of genus 2 with two boundary circles, 10 vertices and dual diameter 8. The maximal dual diameter $\bar{f}_{\text{or}}(2, n)$ of orientable surfaces with boundary on n vertices satisfies $\bar{f}_{\text{or}}(2, n) \geq n - 3 + \lfloor (n - 2)/8 \rfloor$ for every $n \geq 3$.

(d) Every surface, with or without boundary, with $n \geq 3$ vertices has dual diameter at most $\max(n - 3, 2n - 8)$. Hence $\frac{6}{5} \leq \liminf \bar{f}(2, n)/n \leq \limsup \bar{f}(2, n)/n \leq 2$.

Theorem 1.2 (computer-assisted). (a) $\bar{f}(2, n) = n - 3$ for $3 \leq n \leq 6$, and $\bar{f}(2, n) = n - 2$ for $7 \leq n \leq 10$.

(b) Every surface with at most 7 vertices and dual diameter greater than $n - 3$ is isomorphic to M_7 . There are exactly two isomorphism classes of surfaces with 8 vertices and dual diameter greater than 5.

(c) $\bar{f}_{\text{or}}(2, n) = n - 3$ for $3 \leq n \leq 9$, and $\bar{f}_{\text{or}}(2, 10) = 8$.

The upper bounds hold for all surfaces with or without boundary and rest on DRAT proofs checked by three separately written checkers; all three parts were also confirmed by an exhaustive enumeration, without SAT solvers, of the surfaces with at most 10 vertices (Section 6).

Prior work. Holmes [Hol18a] studied the largest diameter $\mu(3, n)$ of dual graphs of 2-dimensional complexes on n vertices whose Stanley–Reisner rings satisfy Serre’s condition (S_2) . By [Hol18a, Thm. 16, Def. 17] these are the complexes in which any two facets are joined by a path of facets containing their intersection (the normal complexes of [San13]). Surfaces are such complexes: two triangles sharing an edge are adjacent, and the triangles containing a vertex induce a connected subgraph (Lemma 2.2(b)). So $\bar{f}(2, n) \leq \mu(3, n)$. The complex of Figure 4 in [Hol18a], which shows $\mu(3, 7) \geq 5$ there (Prop. 39), is the image of M_7 under $0, 1, \dots, 6 \mapsto A, B, C, E, D, G, F$, and the examples for $\mu(3, 8) \geq 6$ and $\mu(3, 9) \geq 7$ (Props. 40, 41) are M_7 with one and two pendant triangles. Holmes proves $\mu(3, 6) = 3$ [Hol18a, Prop. 37] and $\mu(3, n) \leq \max(2n - 10, n - 2)$ [Hol18a, Thm. 19], which is stronger than Theorem 1.1(d) for $n \geq 7$, and states $\mu(3, 9) = 7$ [Hol18a, Prop. 41]; the proof of this value is omitted in [Hol18a] and given in his dissertation [Hol18b, Prop. 3.6.7]. So his results already give a negative answer and, for $6 \leq n \leq 9$, the values of $\bar{f}(2, n)$ in Theorem 1.2(a); but [Hol18a, Hol18b] do not observe that these complexes are surfaces, and do not mention Santos’ question. Holmes’ glued families of diameter about $\frac{5}{4}n$ [Hol18a, Thm. 31] have edges in three triangles, so they are not surfaces. What is new here is the observation, Theorem 1.1(b),(c), the short proof of (d) (a variant of Holmes’ block argument), the value $\bar{f}(2, 10) = 8$, and parts (b) and (c) of Theorem 1.2. Our computer-assisted proof of the values for $n \leq 9$ does not use [Hol18a, Thm. 19] or [Hol18b]. Finally, M_7 is not flag (026 is a missing triangle with all its edges present), as it must be by [AB14].

2 Preliminaries

A *pure 2-complex* K is a finite set of 3-element sets, its *triangles*; its vertices and edges are the 1- and 2-element subsets of triangles, and n is the number of vertices. The dual graph $G(K)$ has the triangles as vertices, adjacent when they share an edge; d_K is its distance and $\text{diam } K$ its diameter. The *link* $\text{lk}_K(v)$ is the graph on the vertices w with vw an edge, with an edge ab for each triangle $vab \in K$, and the *star* $\text{st}_K(v)$ is the set of triangles containing v .

Definition 2.1. A *surface* is a connected pure 2-complex in which every vertex link is a path (with at least one edge) or a cycle. Edges in exactly one triangle are *boundary edges*; the surface is *closed* if there are none, and a *surface with boundary* otherwise. It is *orientable* if its triangles can be oriented so that any two triangles sharing an edge induce opposite orientations on it. Let $\bar{f}(2, n)$ and $\bar{f}_{\text{or}}(2, n)$ be the maximum of $\text{diam } K$ over all surfaces, respectively orientable surfaces, with boundary and n vertices.

These are exactly the pure 2-complexes whose realization is a compact connected 2-manifold, possibly with boundary, and boundary, orientability and $\chi = f_0 - f_1 + f_2$ agree with the topological notions.

Lemma 2.2. *Let K be a surface.*

- (a) *Every edge lies in one or two triangles.*
- (b) *For every vertex v , $\text{st}_K(v)$ induces a path or a cycle in $G(K)$, and $G(K)$ is connected.*
- (c) *Fix a triangle X and put $\ell(T) = d_K(X, T)$. If some triangle has $\ell \geq D$, then some triangle has $\ell = D$. For every vertex v the values $\ell(T)$, $T \ni v$, form an interval of integers.*

Proof. (a) The edge vw lies in $\deg_{\text{lk}(v)}(w) \in \{1, 2\}$ triangles. (b) The triangles vab containing v correspond to the edges ab of $\text{lk}(v)$, and vab, vcd share an edge iff $\{a, b\} \cap \{c, d\} \neq \emptyset$. So $\text{st}(v)$ induces the line graph of a path or a cycle. If uv is an edge, $\text{st}(u) \cap \text{st}(v) \neq \emptyset$; as K is connected, so is $G(K)$. (c) Take a shortest path for the first claim. For the second, ℓ changes by at most 1 along dual edges and $\text{st}(v)$ is connected by (b). $\square$

3 The surface M_7

Proof of Theorem 1.1(a). The vertex links are the paths $\text{lk}(0) = 214536$, $\text{lk}(1) = 3204$, $\text{lk}(2) = 013546$, $\text{lk}(3) = 125064$, $\text{lk}(4) = 105263$, $\text{lk}(6) = 0342$ and the cycle $\text{lk}(5) = (0324)$. So M_7 is a surface. It has 12 interior edges 01, 03, 04, 05, 12, 23, 24, 25, 35, 36, 45, 46 and 6 boundary edges 02, 06, 26, 13, 14, 34 (so $2 \cdot 12 + 6 = 3 \cdot 10$), forming the disjoint circles 026 and 134. Thus $\chi(M_7) = 7 - 18 + 10 = -1$. Adding the triangles 026 and 134 closes the two boundary paths in each link and gives a closed surface with $\chi = 1$, a projective plane. So M_7 is the projective plane minus the interiors of two disjoint triangles; in particular it is not orientable.

The 12 interior edges give the 12 dual edges 012–014, 012–123, 014–045, 123–235, 045–035, 045–245, 235–035, 235–245, 035–036, 245–246, 036–346, 246–346. The breadth-first layers from 012 are

L_0	L_1	L_2	L_3	L_4	L_5
012	014, 123	045, 235	035, 245	036, 246	346

and every dual edge joins consecutive layers. So every path from 012 to 346 has at least 5 steps, and 012, 123, 235, 245, 246, 346 has 5. Hence $d(012, 346) = 5$. A breadth-first search from the other triangles shows that no other pair is at distance 5, so $\text{diam } M_7 = 5$. $\square$

4 Edge sums and two infinite families

Lemma 4.1 (edge sum). *Let K_1, K_2 be surfaces with boundary on disjoint vertex sets, with n_1 and n_2 vertices, and let $T_i \in K_i$ be a triangle with a boundary edge $u_i v_i$. Let K be obtained by identifying u_2 with u_1 and v_2 with v_1 (call them u, v). Then K is a surface with boundary with $n_1 + n_2 - 2$ vertices*

and $\chi(K) = \chi(K_1) + \chi(K_2) - 1$, and it is orientable if K_1, K_2 are. Moreover, $G(K)$ is the disjoint union of $G(K_1)$ and $G(K_2)$ plus the edge T_1T_2 . Hence distances inside each K_i are unchanged, and $d_K(A, B) = d_{K_1}(A, T_1) + 1 + d_{K_2}(T_2, B)$ for $A \in K_1, B \in K_2$.

Proof. Only u, v are identified, so no two triangles coincide, and the only common face of K_1, K_2 is the edge uv , which now lies in exactly T_1 and T_2 . Other edges keep their triangles, and links of vertices other than u, v are unchanged. As u_1v_1 lies in one triangle, $\text{lk}_{K_1}(u_1)$ is a path ending at v_1 ; likewise $\text{lk}_{K_2}(u_2)$ is a path ending at v_2 , and the two paths share no other vertex. So $\text{lk}_K(u)$ is a path, and similarly $\text{lk}_K(v)$; and K is connected. The counts f_0, f_1, f_2 change by $-2, -1, 0$. A triangle of K_1 and one of K_2 can only share uv , so T_1T_2 is the only new dual edge, and it is a bridge. For orientability, reverse the orientation of K_2 if necessary so that T_1, T_2 induce opposite orientations on uv . $\square$

With K_2 a single triangle, Lemma 4.1 shows that a *pendant triangle* glued to a boundary edge of $T \in K$ has distance $d_K(A, T) + 1$ from every $A \in K$, and that its two new edges are boundary edges. The *strip* with triangles $\{i, i+1, i+2\}$, $0 \leq i \leq n-3$, is a disc whose dual graph is a path, so $\tilde{f}_{\text{or}}(2, n) \geq n-3$ for all $n \geq 3$.

Proof of Theorem 1.1(b). In M_7 , $X = 012$ has the boundary edge 02, $Y = 346$ has the boundary edge 34, and $d(X, Y) = 5$. Glue k copies of M_7 , the edge 34 of $Y^{(i)}$ to the edge 02 of $X^{(i+1)}$. By Lemma 4.1 this is a surface with $5k+2$ vertices and $d(X^{(1)}, Y^{(k)}) = 5k + (k-1)$. Glue to the still free edge 34 of $Y^{(k)}$ a chain of $r \in \{0, \dots, 4\}$ pendant triangles, each on a new boundary edge of the previous one. The result has $n = 5k+2+r$ vertices and two triangles at distance $6k-1+r = n-3+k$. Take $k = \lfloor (n-2)/5 \rfloor$ for $n \geq 7$, and the strip for $n \leq 6$. (For $k=2, r=0$ this is a surface with 12 vertices and dual diameter $11 = n-1$.) $\square$

Proof of Theorem 1.1(c). Let G_{10} have the following 20 triangles, each written in a cyclic order that gives a coherent orientation:

$$\begin{array}{cccccccccccc} 012 & 031 & 043 & 054 & 065 & 096 & 078 & 089 & 192 & 236 \\ 283 & 245 & 294 & 258 & 267 & 349 & 356 & 395 & 598 & 687 \end{array}$$

The vertex links are the paths $\text{lk}(0) = 213456987$, $\text{lk}(1) = 3029$, $\text{lk}(2) = 019458367$, $\text{lk}(3) = 10495628$, $\text{lk}(6) = 8723509$, $\text{lk}(7) = 0862$, $\text{lk}(8) = 3259076$, $\text{lk}(9) = 12435806$ and the cycles $\text{lk}(4) = (03925)$, $\text{lk}(5) = (0428936)$. So G_{10} is an orientable surface with 34 edges. Its 8 boundary edges form the circles 027 and 13869, so $\chi = -4 = 2 - 2g - 2$ and $g = 2$. The breadth-first layers from 012 are

$L_0 = \{012\}$	$L_3 = \{045, 245, 349\}$	$L_6 = \{089, 236\}$
$L_1 = \{013, 129\}$	$L_4 = \{056, 258, 359\}$	$L_7 = \{078, 267\}$
$L_2 = \{034, 249\}$	$L_5 = \{069, 238, 356, 589\}$	$L_8 = \{678\}$

Each of the 26 dual edges joins consecutive layers, except 045–245 inside L_3 . So $d(012, 678) \geq 8$, with equality by the star of 0 followed by 678: 012, 013, 034, 045, 056, 069, 089, 078, 678. A breadth-first search from every triangle shows that this is the only pair at distance 8, so $\text{diam } G_{10} = 8$. The triangles 012 and 678 have the boundary edges 02 and 68. Gluing k copies, the edge 68 of one to the edge 02 of the next, and adding $r \leq 7$ pendant triangles gives by Lemma 4.1 an orientable surface with $n = 8k+2+r$ vertices and two triangles at distance $9k-1+r = n-3+k$. For $n \leq 9$ use the strip. $\square$

5 An upper bound

Proof of Theorem 1.1(d). Let K be a surface with n vertices and X, Y triangles with $d(X, Y) = D = \text{diam } K$. Put $\ell(T) = d(X, T)$ and $L_i = \{T : \ell(T) = i\}$ for $0 \leq i \leq D$ (all non-empty), and let $V(L_i)$ be the set of vertices of the triangles in L_i . Define blocks: $s_1 = 0$; l_j is the largest $i \leq D$ with $V(L_i) \cap V(L_{s_j}) \neq \emptyset$; $s_{j+1} = l_j + 1$; stop at the first k with $l_k = D$. Block j consists of $L_{s_j}, \dots, L_{l_j}$, and n_j is the number of vertices of its triangles.

(i) Some vertex v_j lies in $V(L_{s_j}) \cap V(L_{l_j})$. By Lemma 2.2(c) it lies in every layer of block j , and by the choice of l_j in no L_i with $l_j < i \leq D$.

(ii) $n_j \geq l_j - s_j + 3$. This is clear if $l_j - s_j \leq 1$ (one triangle has 3 vertices, two have at least 4). If $l_j - s_j \geq 2$, choose in the path or cycle $\text{st}(v_j)$ (Lemma 2.2(b)) a simple arc from a triangle of L_{s_j} to one of L_{l_j} . Let A be its last triangle with $\ell \leq s_j$, and B the first triangle after A with $\ell = l_j$. As ℓ changes by at most 1 per step, the sub-arc from A to B has $t \geq l_j - s_j + 1$ triangles, all in block j . It is not a whole cycle, since $|\ell(A) - \ell(B)| \geq 2$ means that A, B are not adjacent. So its triangles have $t + 1$ distinct link vertices of v_j , and $n_j \geq t + 2$.

(iii) Every vertex lies in at most two blocks, and these are consecutive: if w lies in blocks j and $j' \geq j + 2$, then by Lemma 2.2(c) it lies in $L_{s_{j+1}}$ and in $L_{l_{j+1}+1}$, contradicting the choice of l_{j+1} . The three vertices of X lie only in block 1, by the choice of l_1 .

Now $D = \sum_j (l_j - s_j) + k - 1 \leq \sum_j n_j - 2k - 1$ by (ii), and $\sum_j n_j \leq 2(n - 3) + 3$ by (iii). If $k = 1$, $D \leq n_1 - 3 \leq n - 3$; if $k \geq 2$, $D \leq 2n - 3 - 2k - 1 \leq 2n - 8$. Finally, part (b) gives $\bar{f}(2, n) \geq (6n - 21)/5$, while $\bar{f}(2, n) \leq 2n - 8$ for $n \geq 5$. $\square$

6 Small values by computer

6.1 The encoding

Fix $n \geq 4$, $D \geq 1$ and $[n] = \{0, \dots, n - 1\}$. Triangles $S, T \subset [n]$ are *adjacent* if $|S \cap T| = 2$. The formula $E(n, D)$ has variables x_T (“ T is present”) for all triangles, u_e for all edges, $p_{k,T}$ (“ T is the k -th triangle of a walk”) and $a_{k,v}$ for $0 \leq k \leq D$, and $g_{T,j}$ (“ $\ell(T) \geq j$ ”) for $1 \leq j \leq D$. Its clauses are:

- (E) $\neg x_R \vee \neg x_S \vee \neg x_T$ for any three triangles with a common edge; $x_T \rightarrow u_e$ for $T \supset e$, and $u_e \rightarrow \bigvee_{T \supset e} x_T$;
- (P) the units $p_{0,012}, p_{1,013}, \neg p_{0,T} (T \neq 012), \neg p_{1,T} (T \neq 013)$; for $1 \leq k \leq D$ the clause $\bigvee_T p_{k,T}$ and all clauses $\neg p_{k,S} \vee \neg p_{k,T}; p_{k,T} \rightarrow x_T$ for $0 \leq k \leq D$; and $p_{k,T} \rightarrow \bigvee_{S \text{ adjacent to } T} p_{k+1,S}$ for $0 \leq k < D$;
- (S) $a_{0,v} \leftrightarrow [v \leq 2]$ (units); for $k \geq 1$, $a_{k,v} \leftrightarrow a_{k-1,v} \vee \bigvee_{T \ni v} p_{k,T}$ and $a_{k,v} \rightarrow a_{k,v-1} (v \geq 1)$;
- (L) $g_{T,j+1} \rightarrow g_{T,j}; \neg g_{012,1}; p_{k,T} \rightarrow g_{T,k} (k \geq 1)$; and $x_S \wedge x_T \wedge g_{S,j+1} \rightarrow g_{T,j}$ for all ordered adjacent pairs and $1 \leq j < D$.

$E_{\text{or}}(n, D)$ adds variables o_T and, for each pair of adjacent triangles, two clauses forcing opposite induced orientations on the common edge when both are present. A *link cut* is a clause $\neg u_{va} \vee \neg u_{vb} \vee \bigvee_{s \in C, s' \in C'} x_{\{v, s, s'\}}$, where $[n] \setminus \{v\} = C \sqcup C'$, $a \in C$ and $b \in C'$.

Lemma 6.1. *If $E(n, D)$ together with some set of link cuts is unsatisfiable, then every surface with at most n vertices has dual diameter at most $D - 1$. The same holds for $E_{\text{or}}(n, D)$ and orientable surfaces.*

Proof. Suppose a surface K with at most n vertices has dual diameter at least D . By Lemma 2.2(c) it has two triangles at distance exactly D ; let $Q_0, \dots, Q_D$ be a shortest path between them. Label $Q_0 \cap Q_1$ by 0, 1, the third vertices of Q_0 and Q_1 by 2 and 3, then the new vertices of $Q_2, \dots, Q_D$ in order of first appearance (each Q_k has at most one vertex outside Q_{k-1}), and the rest arbitrarily. Put $x_T = [T \in K]$, $u_e = [e \text{ is an edge of } K]$, $p_{k,T} = [T = Q_k]$, $a_{k,v} = [v \in Q_0 \cup \dots \cup Q_k]$ and $g_{T,j} = [T \in K, \min(d(Q_0, T), D) \geq j]$, and take o_T from a coherent orientation if K is orientable. (E) holds by Lemma 2.2(a), (P) as $Q_0, \dots, Q_D$ is a walk with $Q_0 = 012$ and $Q_1 = 013$, (S) by the labelling, and (L) since $d(Q_0, \cdot)$ changes by at most 1 between adjacent present triangles and $d(Q_0, Q_k) = k$. For a link cut with $va, vb \in K$, the vertices a, b lie in the connected graph $\text{lk}_K(v)$, so a path from a to b in it has an edge ss' with $s \in C$, $s' \in C'$, i.e. $vss' \in K$. So all clauses hold, a contradiction. $\square$

Conversely, if a model of $E(n, D)$ has all vertex links of $K = \{T : x_T\}$ connected, then the component of K containing 012 is a surface (by (E) links have maximum degree 2), and by (L) and the walk, $d(012, P_D) = D$. The search therefore adds cuts lazily: whenever a model has a vertex whose link is disconnected, it adds for each ordered pair of components the link cut with C the first component and a, b the least vertices of the two components, and solves again.

6.2 Certificates and results

We used Glucose 4.1 [AS09] through PySAT [IMMS18] with proof logging. When the final formula ($E(n, D)$ or $E_{\text{or}}(n, D)$ plus the cuts added so far) is unsatisfiable, it is written out with its DRAT proof [WHH14]. Table 1 lists the certificates. No lemma needed the RAT rule.

n	D	or.	vars	clauses	cuts	lemmas	n	D	or.	vars	clauses	cuts	lemmas
4	2	–	42	135	0	1	10	9	–	2545	97524	2098	1609977
5	3	–	110	588	2	14	7	5	yes	518	6883	64	691
6	4	–	245	2128	14	108	8	6	yes	924	17907	254	7192
7	6	–	560	7722	42	823	9	7	yes	1536	41803	786	82267
8	7	–	988	19811	202	6965	10	9	yes	2665	99960	2014	1448064
9	8	–	1629	45838	720	70764							

Table 1: The DRAT certificates of unsatisfiability. “Lemmas” counts the proof lemmas checked up to the refutation.

Proof of Theorem 1.2. (a), (c): The upper bounds follow from Table 1 and Lemma 6.1; also $\bar{f}(2, 3) = \bar{f}_{\text{or}}(2, 3) = 0$. Strips give $n - 3$, M_7 with a chain of $n - 7$ pendant triangles on the edge 34 gives $n - 2$ for $7 \leq n \leq 10$, and G_{10} gives $\bar{f}_{\text{or}}(2, 10) \geq 8$.

(b): For $(n, D) = (7, 5)$ and $(8, 6)$ the search enumerates: each time the model is a surface, a clause excluding exactly its set of present triangles is added, until the formula is unsatisfiable. The final formulas (6461 clauses with 58 cuts and 4 excluding clauses, and 17101 clauses with 272 cuts and 16 excluding clauses) have checked DRAT proofs. By the proof of Lemma 6.1, every surface with at most n vertices and two triangles at distance D is isomorphic to an excluded complex. For $(7, 5)$ all four are isomorphic to M_7 ; with the values in (a) this proves the first claim. For $(8, 6)$ the 16 excluded complexes form two classes: M_7 with a pendant triangle, and the surface with triangles 012, 013, 023, 124, 135, 246, 256, 257, 346, 347, 356, 457 (non-orientable, $\chi = -1$, two boundary circles). $\square$

Uncertified computations. For $n = 11$, CaDiCaL 1.9.5 [BFF⁺24] reports $E(11, 10)$ with its lazy cuts unsatisfiable after about 27 minutes, and $E_{\text{or}}(11, 10)$ after about 41–44 minutes (two runs each, same outcome). With Theorem 1.1, this indicates $\bar{f}(2, 11) = \bar{f}_{\text{or}}(2, 11) = 9$, without a certificate. The case $n = 12$, $D = 12$ was not decided.

6.3 A check without SAT solvers

In the independent AI-assisted verification run (item 4 of the Verification paragraph at the end of the note), a separately written program enumerated the surfaces with at most 10 vertices directly, without SAT solvers. It decides the triangles on $[n]$ in lexicographic order, keeps every edge in at most two triangles, and tests each vertex link for connectivity as soon as the link is complete. For $n \leq 7$ it lists all labelled complexes on $[n]$. For $5 \leq n \leq 10$ it fixes a vertex 0 of maximal degree d , with link the path $1, 2, \dots, d$ or the cycle $(1\ 2 \dots d)$, and bounds all other degrees by d ; this reaches every surface with exactly n vertices up to isomorphism. All dual diameters are computed by breadth-first search. As a check of completeness, the closed surfaces found for $n = 4, \dots, 10$ form 1, 1, 3, 9, 43, 655, 42426 isomorphism classes, the numbers of triangulated closed surfaces with these numbers of vertices found by Lutz [Lut08]. The maximal dual diameters of surfaces with boundary and exactly $n = 3, \dots, 10$ vertices are 0, 1, 2, 3, 5, 6, 7, 8, and those of closed surfaces with $n = 4, \dots, 10$ vertices are 1, 2, 3, 3, 4, 5, 5. The surfaces with n vertices and dual diameter at least $n - 2$ form 1, 2, 35 and 621 isomorphism classes for $n = 7, 8, 9, 10$. For $n = 7, 8$ these are the classes of Theorem 1.2(b), for $n \leq 9$ none of them is orientable, and for $n = 10$ the only orientable one is G_{10} . So G_{10} is the only orientable surface with at most 10 vertices that violates (1). This enumeration carries no certificate, but it does not depend on the encoding of Section 6.1.

7 Concluding remarks

Problem 7.1. (a) Does $\bar{f}(2, n)/n$ converge, and to what? Is $\bar{f}(2, n) = n - 3 + \lfloor (n - 2)/5 \rfloor$ for all n ? This holds for $n \leq 10$ (and $n = 11$ by the uncertified computation). (b) What is the growth of $\bar{f}_{\text{or}}(2, n) \geq n - 3 + \lfloor (n - 2)/8 \rfloor$? (c) Which topological types violate (1)? The surface M_7 is non-orientable with $\chi = -1$, and G_{10} is orientable of genus 2; by Section 6.3, G_{10} is the only orientable violator with at most 10 vertices. We do not know whether an orientable surface of genus 0 or 1 can violate (1).

Verification. The claims were checked as follows. Items 1–3 were done while the note was prepared, item 4 in an independent AI-assisted verification run.

1. The explicit complexes (M_7 , G_{10} , the chains for $n \leq 60$, the two 8-vertex classes, and Figures 4 and 6 of [Hol18a]) were checked by two separately written programs: all vertex links, χ , the boundary, orientability, and all dual distances (by breadth-first search, and by the Floyd–Warshall algorithm). The chains have dual diameter exactly $n - 3 + \lfloor (n - 2)/5 \rfloor$, respectively $n - 3 + \lfloor (n - 2)/8 \rfloor$. The second program also tests Lemma 4.1 on 150 random pairs of surfaces, and the steps (i)–(iii) of Section 5 from every triangle of 173 surfaces.
2. All proofs of Table 1 and both enumeration proofs were checked by a forward DRAT checker that applies deletions, and by a separately written RUP checker, with deletions ignored and with deletions applied; both reject corrupted formulas and proofs in self-tests. A further program regenerates $E(n, D)$ and $E_{\text{or}}(n, D)$ from Section 6.1 and confirms that each certified formula consists of exactly these clauses followed by link cuts (and, for the enumerations, excluding clauses). The certificates for $n \leq 9$ are in the package. The two proofs for $n = 10$ (about 420 and 390 MB) are not, but the search is deterministic: a rerun from a clean copy reproduced them bit for bit, and their SHA-256 hashes are recorded.
3. Uncertified cross-checks: a second, separately written encoding (eager link connectivity by reachability inside each link, breadth-first layer variables, no symmetry breaking beyond 012, 013),

solved with CaDiCaL, gives the same answers for all entries of Table 1 and the same isomorphism classes for (7, 5) and (8, 6). A third encoding, based on reachability layers, gives the same maxima for $6 \leq n \leq 9$. The scripts of the search were rerun from a clean copy, with identical outputs.

4. An independent AI-assisted verification run read the source and all proofs and wrote new programs, without using the code of items 1–3. These check all explicit complexes and Lemma 4.1 once more; a third RUP checker (deletions ignored) verifies all certificates of Table 1 and both enumeration proofs, including the two proofs for $n = 10$ regenerated from the package; a regeneration of $E(n, D)$ and $E_{\text{or}}(n, D)$ from the text of Section 6.1 confirms that every certified formula consists of these clauses followed by valid link cuts (and, for the enumerations, excluding clauses for complexes isomorphic to those of Theorem 1.2(b)); and the enumeration of Section 6.3 is independent of the SAT encoding. The run also reran the scripts of the package from a fresh copy, except the long uncertified solver runs (the second encoding, the maxima for $n \geq 10$ and $n = 11$); all outputs and all certificate files agreed with the recorded ones, apart from running times.

Scope and priority. Theorem 1.1(a) answers the question as posed in [JLSS12, p. 1480] (read in the published report) and in [ula26]. As explained above, M_7 and M_7 with pendant triangles already appear in [Hol18a, Hol18b] as (S_2) complexes of maximal diameter, and with Holmes’ upper bounds they give $f(2, n)$ for $6 \leq n \leq 9$. We found no source that identifies them as surfaces or relates them to Santos’ question. We searched arXiv, Crossref, zbMATH, Semantic Scholar, the Kansas repository and the web; OpenAlex citation queries were rate-limited. Semantic Scholar lists five works citing [Hol18a]: Holmes’ dissertation [Hol18b], whose text we searched, and four papers on commutative algebra and matroids, of which we read the abstracts. We did not see the work of Adler and Dantzig on abstract polytopes [AD74], the survey of Klee and Kleinschmidt [KK87], or the discussion papers published with [San13], and we did not check the proof of [Hol18b, Prop. 3.6.7]. Older unrecognized examples cannot be excluded; this negative search is not a proof of priority. The literature note attached to the record in [ula26] concerns a different problem (Lagrangian structures and the Novikov conjecture) and links to the neighbouring report 10.4171/OWR/2012/25; the statement of the record matches the source.

AI-assisted tools were used to search the problem corpus and the literature, to find the counterexamples and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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