

A Negative Answer to Harrach's Question on the Potential Map in the Unified Eddy-Current Formulation

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Abstract

In the unified variational formulation of the parabolic-elliptic eddy-current equations due to Arnold and Harrach, the electric field is written as $E = A + \nabla\varphi_A$, where A is divergence free and φ_A solves $\operatorname{div}(\sigma\nabla\varphi_A) = -\operatorname{div}(\sigma A)$. To solve this equation, Arnold and Harrach assume a conductivity σ that is bounded below on its support, and a support made of finitely many Lipschitz domains with disjoint closures. Harrach asked whether the map $A \mapsto \varphi_A$ can be defined for general nonnegative $\sigma \in L^\infty(\mathbb{R}^3)$. We show that it cannot. Let σ be the indicator function of a cyclic necklace of $n \geq 3$ closed balls in which consecutive balls touch at one point, or a smooth function that is positive exactly on the open balls. Then there is a smooth, compactly supported, divergence-free field A^* for which no $\varphi \in H_{\operatorname{loc}}^1(\mathbb{R}^3)$ solves $\operatorname{div}(\sigma(A^* + \nabla\varphi)) = 0$. A solid torus whose conductivity is positive almost everywhere but vanishes at least linearly across one cross-section ($\sigma \leq C|\sin(\theta/2)|$, with θ the azimuth) gives the same conclusion. So neither hypothesis can simply be dropped, although we do not claim that either is necessary. In both examples the eddy-current equation with zero initial data and a smooth divergence-free source that vanishes near the conductor has no solution in $L^2(0, T; W(\operatorname{curl}))$. On the positive side, the product $\sigma(A + \nabla\varphi_A)$, which is what the unified formulation uses, can be defined for every $\sigma \geq 0$ by a weighted orthogonal projection. With this definition the unified formulation stays uniquely solvable and uniformly coercive, and it controls every solution of the eddy-current equation. The eddy-current equation is solvable exactly when, for almost every t , the potential equation for the solution $A(t)$ of the unified formulation has a solution $\varphi(t) \in H_{\operatorname{loc}}^1(\mathbb{R}^3)$, with $\nabla\varphi(\cdot) \in L^2(0, T; L_\rho^2)$. The ingredients are classical: points have zero capacity, and the energy between touching balls diverges logarithmically. This is an unrefereed note.

1 Introduction

The transient eddy-current equation

$$\partial_t(\sigma E) + \operatorname{curl}(\mu^{-1} \operatorname{curl} E) = -J_t \quad \text{in } \mathbb{R}^3 \times (0, T) \quad (1)$$

describes low-frequency electromagnetic fields in the presence of conductors. Here $\sigma \geq 0$ is the conductivity, μ the permeability and $J_t = \partial_t J$ the time derivative of an excitation current. The equation is parabolic where $\sigma > 0$ and elliptic where $\sigma = 0$. Inverse problems require a solution theory that is uniform in σ . Arnold and Harrach [AH12] obtained one by writing $E = A + \nabla\varphi_A$ with A divergence free and φ_A a solution of

$$\operatorname{div}(\sigma\nabla\varphi_A) = -\operatorname{div}(\sigma A), \quad (2)$$

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and by deriving a coercive variational problem for A alone. To solve (2) they assume that σ lies in the class $L_R^\infty(\mathbb{R}^3)$ of [AH12, p. 566]:

- the support Ω of σ is a finite union of bounded Lipschitz domains Ω_i whose closures are pairwise disjoint;
- $\mathbb{R}^3 \setminus \overline{\Omega}$ is connected;
- $\sigma \in L_+^\infty(\Omega)$, that is, σ is bounded below on Ω by a positive constant.

In his Oberwolfach abstract on this work, Harrach calls it an open problem whether the mapping $A \mapsto \varphi_A$ “can be defined for general non-negative σ ” in $L^\infty(\mathbb{R}^3)$ [Har12, p. 631]. He restated the question in later talks as the lack of a solution theory for (2) with general $\sigma \geq 0$ [Har13, PDF pp. 18 and 26], [Har14, PDF p. 15]. The 2014 slides add that the theory does not cover continuous transitions between conducting and insulating regions. The question is record OWR-11578-001 of the UnsolvedMath dataset [ula26]. We show that the answer is negative, and we determine what survives.

Reading of the question. In [AH12, Lemma 3.1] the map $A \mapsto \nabla \varphi_A$ takes values in the space $H(\text{curl}0, \mathbb{R}^3) = \{F \in L^2(\mathbb{R}^3)^3 : \text{curl} F = 0\}$, so that $A + \nabla \varphi_A$ lies in the solution space $W(\text{curl})$ recalled in Section 2. Since σ has bounded support, it makes no difference for existence whether one asks for $\nabla \varphi \in L^2(\mathbb{R}^3)^3$ or only for $\varphi \in H_{\text{loc}}^1(\mathbb{R}^3)$: multiply φ by a cutoff function that equals 1 near $\text{supp } \sigma$. We therefore read the question as follows. Does the weak form of (2),

$$\int_{\mathbb{R}^3} \sigma(A + \nabla \varphi) \cdot \nabla \psi \, dx = 0 \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^3), \quad (3)$$

have a solution $\varphi \in H_{\text{loc}}^1(\mathbb{R}^3)$ for every divergence-free $A \in W(\text{curl})$? We also ask whether some other definition of φ_A , or of the product $\sigma(A + \nabla \varphi_A)$, keeps the conclusions of [AH12, Thm. 3.2].

Two examples. Fix $n \geq 3$ and put $r = \sin(\pi/n)$, $a_k = 2\pi k/n$ and $c_k = (\cos a_k, \sin a_k, 0)$ for $k \in \mathbb{Z}/n$. Let $B_k = B(c_k, r)$ be the open ball of radius r about c_k . The *necklace*

$$N := \bigcup_{k \in \mathbb{Z}/n} \overline{B_k}$$

is a closed ring of n balls. The balls $\overline{B_k}$ and $\overline{B_{k+1}}$ touch at exactly one point p_k , and all other pairs are disjoint (Lemma 4.1). Write $\varrho(x) = (x_1^2 + x_2^2)^{1/2}$ and let

$$\mathbb{T} := \{x \in \mathbb{R}^3 : (\varrho(x) - 1)^2 + x_3^2 \leq 1/4\}$$

be a closed solid torus. In Section 2.4 we fix a field $A^* \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$ with $\text{div } A^* = 0$ that equals $\varrho^{-1}e_\theta$ on an open set $U \supset N \cup \mathbb{T}$. Here $e_\theta = \varrho^{-1}(-x_2, x_1, 0)$, so $\varrho^{-1}e_\theta$ is the gradient of the multivalued azimuth θ , and A^* has circulation 2π around the x_3 -axis.

Theorem 1.1 (no potential for the necklace). *Let $\sigma \in L^\infty(\mathbb{R}^3)$ satisfy $0 \leq \sigma \leq C\chi_N$ for some constant C , and $\sigma > 0$ almost everywhere on N . Then (3) with $A = A^*$ has no solution $\varphi \in H_{\text{loc}}^1(\mathbb{R}^3)$. This applies to $\sigma = \chi_N$, and to every bounded smooth $\sigma \geq 0$ with $\{\sigma > 0\} = \text{int } N$.*

Theorem 1.2 (no potential for a degenerate torus). *Let $\sigma \in L^\infty(\mathbb{R}^3)$ satisfy $0 \leq \sigma \leq C|\sin(\theta/2)|\chi_{\mathbb{T}}$ for some constant C , and $\sigma > 0$ almost everywhere on $\mathbb{T}$. For example, $\sigma(x) = \sin^2(\theta/2) = \frac{1}{2}(1 - x_1/\varrho(x))$ on $\mathbb{T}$ and $\sigma = 0$ elsewhere. Then (3) with $A = A^*$ has no solution $\varphi \in H_{\text{loc}}^1(\mathbb{R}^3)$.*

So σ may vanish linearly across the half-disc $\{x_2 = 0, x_1 > 0\} \cap \mathbb{T}$ and be positive elsewhere on $\mathbb{T}$. Since $\sin^2(\theta/2) = \frac{1}{2}(1 - \cos \theta)$ is smooth away from the x_3 -axis, σ can also be chosen in $C^\infty(\mathbb{R}^3)$, with $\{\sigma > 0\}$ equal to the interior of $\mathbb{T}$ minus that half-disc.

Theorem 1.3 (no solution of the eddy-current equation). *Let σ be as in Theorem 1.1 or Theorem 1.2, let $\mu \in L_+^\infty(\mathbb{R}^3)$ and $T > 0$. Define $J_t \in L^2(0, T; W(\text{curl})')$ by*

$$\langle J_t(t), \Phi \rangle := -t \int_{\mathbb{R}^3} \mu^{-1} \text{curl } A^* \cdot \text{curl } \Phi \, dx.$$

Then J_t is divergence free and $J_t(t)$ vanishes on the open set U , which contains the conductor. For $\mu \equiv 1$, $J_t(x, t) = -t \text{curl curl } A^(x)$ is a smooth field with compact support in $\mathbb{R}^3 \setminus U$. The eddy-current equation (1) with zero initial data, in its natural variational form (6), has no solution $E \in L^2(0, T; W(\text{curl}))$.*

On the positive side, let G_σ be the closure in $L^2(\mathbb{R}^3)^3$ of $\{\sqrt{\sigma} \nabla \psi : \psi \in C_c^\infty(\mathbb{R}^3)\}$ and let P_σ be the orthogonal projection onto its orthogonal complement. We define

$$\sigma(A + \nabla \varphi_A) := \sqrt{\sigma} P_\sigma \sqrt{\sigma} A. \quad (4)$$

Theorem 1.4 (what survives for general σ). *Let $\sigma \in L^\infty(\mathbb{R}^3)$ be nonnegative with bounded support, let $\mu \in L_+^\infty(\mathbb{R}^3)$, and let $J_t \in L^2(0, T; W(\text{curl})')$ be divergence free. Read the unified formulation (7) with the definition (4).*

- (a) *If $\varphi \in H_{\text{loc}}^1(\mathbb{R}^3)$ solves (3), then $\sigma(A + \nabla \varphi) = \sqrt{\sigma} P_\sigma \sqrt{\sigma} A$. In particular (4) agrees with [AH12] for $\sigma \in L_R^\infty(\mathbb{R}^3)$.*
- (b) *The unified formulation (7) has a unique solution $A \in L^2(0, T; V)$. It is uniformly coercive, with a constant that does not depend on σ , and $\|A\|_{L^2(0, T; V)} \leq \sqrt{5} \max\{\|\mu\|_\infty, 2\} \|J_t\|_{L^2(0, T; W(\text{curl})')}$.*
- (c) *Every solution $E \in L^2(0, T; W(\text{curl}))$ of (6) satisfies $\text{curl } E = \text{curl } A$ and $\sqrt{\sigma} E = P_\sigma \sqrt{\sigma} A$. Moreover, for almost every t , $E(t) - A(t) = \nabla \varphi(t)$, where $\varphi(t) \in H_{\text{loc}}^1(\mathbb{R}^3)$ solves (3) with $A(t)$ in place of A .*
- (d) *Conversely, let $G \in L^2(0, T; L_\rho^2(\mathbb{R}^3)^3)$, and suppose that for almost every t we have $G(t) = \nabla \varphi(t)$ for some $\varphi(t) \in H_{\text{loc}}^1(\mathbb{R}^3)$ solving (3) with $A(t)$ in place of A . Then $E := A + G$ solves (6).*

Thus the parts of [AH12, Thm. 3.2] that concern the unified formulation (with zero initial data) hold for every $\sigma \geq 0$. The exception is the statement that $A + \nabla \varphi_A$ solves the eddy-current equation. By (c) and (d), (6) has a solution exactly when (3) can be solved along the solution A with $\nabla \varphi(\cdot) \in L^2(0, T; L_\rho^2(\mathbb{R}^3)^3)$. That is, there must be $G \in L^2(0, T; L_\rho^2(\mathbb{R}^3)^3)$ such that, for almost every t , $G(t) = \nabla \varphi(t)$ for a solution $\varphi(t) \in H_{\text{loc}}^1(\mathbb{R}^3)$ of (3) with $A(t)$ in place of A . In (c) this holds with $G := E - A$, which lies in $L^2(0, T; L_\rho^2(\mathbb{R}^3)^3)$ because E and A do. Theorem 1.3 shows that the condition can fail.

Which hypotheses fail. For $\sigma = \chi_N$ the support N consists of n closed balls, $\sigma = 1$ on N , the interiors B_k are Lipschitz domains, and $\mathbb{R}^3 \setminus N$ is connected. The only property of $L_R^\infty(\mathbb{R}^3)$ that fails is that the closures of the components touch. For σ as in Theorem 1.2, the support $\mathbb{T}$ is the closure of a smooth domain with connected complement, and $\sigma > 0$ almost everywhere on it. The only property that fails is the positive lower bound. So neither of these two hypotheses of [AH12] can simply be dropped, and smoothness of σ does not help. The Oberwolfach abstract states the hypothesis more loosely than [AH12]: it asks that $\text{supp } \sigma$ be a disjoint union of finitely many Lipschitz domains with connected complements [Har12, p. 631]. If the components are read as open sets, $\sigma = \chi_N$ satisfies

this, and Theorem 1.3 shows that the disjoint-closure condition of [AH12] is needed for the conclusion that $A + \nabla\varphi_A$ solves the eddy-current equation. There is no conflict with the theorems of [AH12]. The follow-up papers [AH13a, AH13b] keep the same hypotheses, and so does the thesis of L. Simon (née Arnold) [Sim14]. The thesis restates the unified formulation with the class $L^\infty_R(\mathbb{R}^3)$ in its (3.12) and with the potential map of its Lemma 3.5, which is that of [AH12, Lemma 3.1]. It does not treat general $\sigma \geq 0$.

Relation to other work. Pauly, Picard, Trostorff and Waurick [PPTW21] treat a degenerate eddy-current model by the theory of evolutionary equations. Numbers in [PPTW21] refer to its arXiv version, arXiv:1810.07974v2. Their Assumption 4.3 requires the conducting set to be open, with finitely many components with pairwise disjoint closures, to satisfy the H^1 -extension condition [PPTW21, (23)], and to carry a strictly positive definite σ ; they cite [AH12] as motivation for letting σ vanish. The necklace violates both the disjoint-closure condition and the extension condition (23). Our results show that an assumption of this kind cannot simply be removed, but we do not claim that it is necessary (see Remark 7.1). We also screened other work on eddy-current and magnetoquasistatic models found in our searches, including works that cite [AH12]: [FFV20, VAMP26, RS23, CRS23, HY23, Nic15, NT14, TV18, KTL16, Tyu19, AGLR20c], and the numerical work [AGLR20a, AGLR20b] on degenerate parabolic problems. None of them treats (2) for general $\sigma \geq 0$. Those that analyse the equations assume a uniformly positive conductivity [FFV20, VAMP26] or a fixed conducting region with some regularity [RS23, CRS23, HY23, KTL16, Tyu19, AGLR20c].

The ingredients are classical. A point has zero capacity in $\mathbb{R}^3$, and two tangent balls do not form an H^1 -extension domain; see, for example, [Maz11] for capacities and for extension domains of Sobolev functions. The second fact also follows from Lemma 4.3: the function equal to 1 on one ball and to 0 on the other has no H^1 extension. The Dirichlet energy between two nearly touching conductors at distance h grows like $\log(1/h)$ [Kel63]. A weight that vanishes linearly across a surface makes that surface invisible to the weighted Dirichlet energy. What seems new is the combination of these facts into an answer to Harrach's question, and in particular the nonexistence result Theorem 1.3 for the eddy-current equation with touching conductors. The projection (4) generalises the remark after [AH12, Lemma 3.1] that $\sqrt{\sigma}\nabla u_E$ is uniquely determined there.

Organisation. Section 2 collects the function spaces, the projection and the two variational formulations. Section 3 proves Theorem 1.4. Sections 4 and 5 prove Theorems 1.1 and 1.2, and Section 6 proves Theorem 1.3. Section 7 contains further remarks, two of them only sketched, and the verification record.

2 Preliminaries

2.1 Function spaces

We use the spaces of [AH12, Sect. 2.1]:

$$\begin{aligned} L^2_\rho(\mathbb{R}^3) &:= \{u \in \mathcal{D}'(\mathbb{R}^3) : (1 + |x|^2)^{-1/2}u \in L^2(\mathbb{R}^3)\}, & \|u\|_\rho &:= \|(1 + |x|^2)^{-1/2}u\|_{L^2}, \\ W(\text{curl}) &:= \{E \in L^2_\rho(\mathbb{R}^3)^3 : \text{curl } E \in L^2(\mathbb{R}^3)^3\}, & \|E\|_{W(\text{curl})}^2 &:= \|E\|_\rho^2 + \|\text{curl } E\|_{L^2}^2, \\ W^1(\mathbb{R}^3) &:= \{u \in L^2_\rho(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)^3\}, & W^1_\diamond &:= \{E \in W^1(\mathbb{R}^3)^3 : \text{div } E = 0\}. \end{aligned}$$

We write $(\cdot, \cdot)$ and $\|\cdot\|$ for the inner product and the norm of $L^2(\mathbb{R}^3)^3$. $\mathcal{D}(\mathbb{R}^3)^3$ is dense in $W(\text{curl})$, and $\mathcal{D}(\mathbb{R}^3)$ is dense in $W^1(\mathbb{R}^3)$ [AH12, p. 561]. The space $W^1_\diamond$ is a Hilbert space with the norm $\|\nabla E\| = \|\text{curl } E\|$, and $\|E\|_\rho \leq 2\|\nabla E\|$ for $E \in W^1_\diamond$ [AH12, p. 567 and (3.10)]. Let

$$V := \{A \in W(\text{curl}) : \text{div } A = 0\}, \quad \|A\|_V := \|\text{curl } A\|.$$

$L_+^\infty(\mathbb{R}^3)$ denotes the functions in $L^\infty(\mathbb{R}^3)$ with positive essential infimum, and B_R the open ball of radius R about 0.

Lemma 2.1. (a) $V = W_\diamond^1$. For $A \in V$ we have $\|A\|_\rho \leq 2\|A\|_V$, $\|A\|_{W(\text{curl})} \leq \sqrt{5}\|A\|_V$ and $\|A\|_{L^2(B_R)} \leq C_R\|A\|_V$ with $C_R := 2(1 + R^2)^{1/2}$. In particular $(V, \|\cdot\|_V)$ is a Hilbert space.

(b) For every $E \in W(\text{curl})$ there is a unique $A_E \in V$ with $\text{curl } A_E = \text{curl } E$. The map $E \mapsto A_E$ is linear, $\|A_E\|_V = \|\text{curl } E\|$, and $E - A_E = \nabla\varphi$ for some $\varphi \in H_{\text{loc}}^1(\mathbb{R}^3)$.

Proof. We first show that a field $H \in L_\rho^2(\mathbb{R}^3)^3$ with $\text{div } H = 0$ and $\text{curl } H = 0$ vanishes. Such an H is a tempered distribution with $\Delta H = \nabla \text{div } H - \text{curl } \text{curl } H = 0$, so its Fourier transform is supported at the origin and H is a polynomial. No nonzero polynomial lies in $L_\rho^2(\mathbb{R}^3)$, so $H = 0$.

Existence in (b). Let $E \in W(\text{curl})$. By the Lax–Milgram theorem in the Hilbert space $W_\diamond^1$ there is $A \in W_\diamond^1$ with $(\text{curl } A, \text{curl } \Psi) = (\text{curl } E, \text{curl } \Psi)$ for all $\Psi \in W_\diamond^1$. Put $D := \text{curl}(A - E) \in L^2(\mathbb{R}^3)^3$. Then $\text{div } D = 0$. Every $\Phi \in \mathcal{D}(\mathbb{R}^3)^3$ can be written as $\Phi = \Psi + \nabla\phi$ with $\Psi \in W_\diamond^1$ and $\phi \in W^1(\mathbb{R}^3)$ [AH12, Lemma 3.5(a)], so $(D, \text{curl } \Phi) = (D, \text{curl } \Psi) = 0$. Hence $\text{curl } D = 0$ in $\mathcal{D}'(\mathbb{R}^3)$, and $D = 0$ by the first paragraph. So $A_E := A$ has $\text{curl } A_E = \text{curl } E$ and $\|A_E\|_V = \|\text{curl } E\|$.

Uniqueness in (b) and (a). If $A_1 \in V$ and $\text{curl } A_1 = \text{curl } E$, then $A_1 - A_E$ is divergence free, curl free and in L_ρ^2 , so $A_1 = A_E$. Applying this with $E := A_1$ for an arbitrary $A_1 \in V$ shows $A_1 \in W_\diamond^1$. Hence $V \subset W_\diamond^1$, and the reverse inclusion is clear. The inequalities follow from $\|A\|_\rho \leq 2\|\nabla A\| = 2\|\text{curl } A\|$ and $(1 + |x|^2)^{1/2} \leq (1 + R^2)^{1/2}$ on B_R .

Finally, $F := E - A_E \in L_\rho^2 \subset L_{\text{loc}}^2$ is curl free. On each ball $B(0, j)$ ($j \in \mathbb{N}$) there is $\varphi_j \in H^1(B(0, j))$ with $F = \nabla\varphi_j$ [GR86, Ch. I, Thm. 2.9], unique up to a constant. Normalise $\int_{B(0,1)} \varphi_j = 0$. Then $\varphi_{j+1} = \varphi_j$ on $B(0, j)$, and the φ_j define $\varphi \in H_{\text{loc}}^1(\mathbb{R}^3)$ with $\nabla\varphi = F$. $\square$

2.2 The weighted projection

Let $\sigma \in L^\infty(\mathbb{R}^3)$ be nonnegative with $\text{supp } \sigma \subset B_R$. Define

$$G_\sigma := \overline{\{\sqrt{\sigma}\nabla\psi : \psi \in C_c^\infty(\mathbb{R}^3)\}}^{L^2(\mathbb{R}^3)^3},$$

let Q_σ be the orthogonal projection of $L^2(\mathbb{R}^3)^3$ onto G_σ , and let $P_\sigma := I - Q_\sigma$.

Lemma 2.2. (a) If $\varphi \in H_{\text{loc}}^1(\mathbb{R}^3)$, then $\sqrt{\sigma}\nabla\varphi \in G_\sigma$.

(b) Let $A \in L_{\text{loc}}^2(\mathbb{R}^3)^3$ and $\varphi \in H_{\text{loc}}^1(\mathbb{R}^3)$. Then φ solves (3) if and only if $\sqrt{\sigma}\nabla\varphi = -Q_\sigma\sqrt{\sigma}A$. So (3) is solvable if and only if $Q_\sigma\sqrt{\sigma}A = \sqrt{\sigma}\nabla\varphi$ for some $\varphi \in H_{\text{loc}}^1(\mathbb{R}^3)$.

Proof. (a) Choose $\chi \in C_c^\infty(\mathbb{R}^3)$ with $\chi = 1$ on a neighbourhood of $\text{supp } \sigma$. Then $\chi\varphi \in H^1(\mathbb{R}^3)$, and there are $\psi_j \in C_c^\infty(\mathbb{R}^3)$ with $\psi_j \rightarrow \chi\varphi$ in $H^1(\mathbb{R}^3)$. Hence $\sqrt{\sigma}\nabla\psi_j \rightarrow \sqrt{\sigma}\nabla(\chi\varphi) = \sqrt{\sigma}\nabla\varphi$.

(b) Condition (3) says that $\sqrt{\sigma}A + \sqrt{\sigma}\nabla\varphi$ is orthogonal to all $\sqrt{\sigma}\nabla\psi$, that is, $Q_\sigma(\sqrt{\sigma}A + \sqrt{\sigma}\nabla\varphi) = 0$. By (a), $Q_\sigma\sqrt{\sigma}\nabla\varphi = \sqrt{\sigma}\nabla\varphi$. $\square$

Definition 2.3. For $A \in L_{\text{loc}}^2(\mathbb{R}^3)^3$ we define $\sigma(A + \nabla\varphi_A) := \sqrt{\sigma}P_\sigma\sqrt{\sigma}A$, as in (4).

The element $w_A := -Q_\sigma\sqrt{\sigma}A$ is the unique $w \in G_\sigma$ with $\sqrt{\sigma}A + w \perp G_\sigma$; this is (3) with $\sqrt{\sigma}\nabla\varphi$ replaced by an element of the closure G_σ . Equivalently, $\sqrt{\sigma}A + w_A = P_\sigma\sqrt{\sigma}A$ minimises $\|\sqrt{\sigma}A + w\|$ over $w \in G_\sigma$. If φ solves (3), then $\sqrt{\sigma}\nabla\varphi = w_A$ by Lemma 2.2, and so $\sigma(A + \nabla\varphi) = \sqrt{\sigma}(\sqrt{\sigma}A + w_A) = \sqrt{\sigma}P_\sigma\sqrt{\sigma}A$. This proves Theorem 1.4(a) in general. For $\sigma \in L_R^\infty(\mathbb{R}^3)$ it applies to the potential $u_A \in W^1(\mathbb{R}^3) \subset H_{\text{loc}}^1(\mathbb{R}^3)$ of [AH12, Lemma 3.1], which solves (3) by [AH12, (3.1)]. It also explains two facts noted in [AH12, after Lemma 3.1]: $\sqrt{\sigma}\nabla u_E$ is uniquely determined, and $\|\sqrt{\sigma}\nabla u_E\| \leq \|\sqrt{\sigma}E\|$, because $\sqrt{\sigma}\nabla u_E = -Q_\sigma\sqrt{\sigma}E$. For $A \in V$ we have, by Lemma 2.1(a),

$$\|\sqrt{\sigma}P_\sigma\sqrt{\sigma}A\| \leq \|\sigma\|_\infty\|A\|_{L^2(B_R)} \leq \|\sigma\|_\infty C_R\|A\|_V. \quad (5)$$

2.3 The two variational formulations

Throughout, $T > 0$, $\mu \in L_+^\infty(\mathbb{R}^3)$, $\sigma \in L^\infty(\mathbb{R}^3)$ is nonnegative with $\text{supp } \sigma \subset B_R$, and $J_t \in L^2(0, T; W(\text{curl})')$ is divergence free: $\langle J_t(t), \nabla \psi \rangle = 0$ for all $\psi \in \mathcal{D}(\mathbb{R}^3)$ and almost every t . By density of $\mathcal{D}(\mathbb{R}^3)$ in $W^1(\mathbb{R}^3)$, the same then holds for all $\psi \in W^1(\mathbb{R}^3)$. As in [AH12], $\mathcal{D}(\mathbb{R}^3 \times [0, T])$ denotes the restrictions to $\mathbb{R}^3 \times [0, T]$ of functions in $\mathcal{D}(\mathbb{R}^3 \times]-\infty, T])$.

The natural formulation. Find $E \in L^2(0, T; W(\text{curl}))$ with

$$-\int_0^T \int_{\mathbb{R}^3} \sigma E \cdot \partial_t \Phi \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} \mu^{-1} \text{curl } E \cdot \text{curl } \Phi \, dx \, dt = -\int_0^T \langle J_t, \Phi \rangle \, dt \quad \text{for all } \Phi \in \mathcal{D}(\mathbb{R}^3 \times [0, T])^3. \quad (6)$$

This is [AH12, (2.4)] with zero initial data $E_0 = 0$. It is the natural variational formulation of [Har12], which is written with the opposite sign on both sides. By [AH12, Thm. 2.1 and Lemma 2.4], which are proved there for every nonnegative $\sigma \in L^\infty$ with bounded support, E solves (6) if and only if it solves (1) in the sense of distributions with $\sqrt{\sigma}E(0) = 0$. Moreover, by the implication (d) $\Rightarrow$ (c) of [AH12, Lemma 2.4], a solution of (6) satisfies (6) for all $\Phi \in H^1(0, T; W(\text{curl}))$ with $\Phi(T) = 0$.

The unified formulation. Let $\mathcal{H} := L^2(0, T; V)$ and $\mathcal{V} := \{\Phi \in H^1(0, T; V) : \Phi(T) = 0\}$. For $A \in \mathcal{H}$ and $\Phi \in \mathcal{V}$ put

$$a_\sigma(A, \Phi) := \int_0^T \left[-(P_\sigma \sqrt{\sigma} A, \sqrt{\sigma} \partial_t \Phi) + (\mu^{-1} \text{curl } A, \text{curl } \Phi) \right] dt, \quad l(\Phi) := -\int_0^T \langle J_t, \Phi \rangle \, dt, \\ \text{find } A \in \mathcal{H} \text{ with } a_\sigma(A, \Phi) = l(\Phi) \text{ for all } \Phi \in \mathcal{V}. \quad (7)$$

By Definition 2.3, the first term is $-\int_0^T \int \sigma(A + \nabla \varphi_A) \cdot \partial_t \Phi$. For $\sigma \in L_R^\infty(\mathbb{R}^3)$, (7) is exactly [AH12, (3.5)] with $E_0 = 0$, by Theorem 1.4(a). The abstract [Har12] states the unified formulation with smooth divergence-free test fields; we use the test space $\mathcal{V} = H_{T0}^1(0, T; W_\diamond^1)$ of [AH12].

2.4 The test field

Fix $\lambda \in C^\infty([0, \infty))$ with $0 \leq \lambda \leq 1$, $\lambda = 0$ on $[0, 1/16]$, $\lambda = 1$ on $[1/8, 2]$ and $\lambda = 0$ on $[3, \infty)$. Fix $\kappa \in C_c^\infty(\mathbb{R})$ with $\kappa = 1$ on $[-1, 1]$. Let $m(s) := -\int_s^\infty \lambda(\tau) \tau^{-1} d\tau$. Then m is constant on $[0, 1/16]$, vanishes on $[3, \infty)$, and $m'(s) = \lambda(s)/s$. So $F(x) := -\kappa(x_3) m(\varrho(x))$ lies in $C_c^\infty(\mathbb{R}^3)$, and

$$A^* := \text{curl}(F e_3) = \kappa(x_3) \frac{\lambda(\varrho)}{\varrho} e_\theta \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3), \quad \text{div } A^* = 0. \quad (8)$$

Let $U := \{x : 1/8 < \varrho(x) < 2, |x_3| < 1\}$. On U we have $A^* = \varrho^{-1} e_\theta$. This field is curl free on U , and on every open subset of U where a smooth branch of the azimuth θ exists, it is the gradient of that branch. A direct computation gives

$$\text{curl } A^* = -\kappa'(x_3) \frac{\lambda(\varrho)}{\varrho} e_\varrho + \kappa(x_3) \frac{\lambda'(\varrho)}{\varrho} e_3,$$

which vanishes on U . In particular $A^* \in V$, and $\text{curl } A^*$ and $\text{curl curl } A^*$ vanish on U . Since $1 - \sin(\pi/3) > 1/8$, the necklace satisfies $N \subset \{1 - r \leq \varrho \leq 1 + r, |x_3| \leq r\} \subset U$ (Lemma 4.1), and $\mathbb{T} \subset \{1/2 \leq \varrho \leq 3/2, |x_3| \leq 1/2\} \subset U$.

3 The weighted projection for general conductivities

In this section we prove Theorem 1.4. Part (a) was proved after Definition 2.3. Let $M\Phi := P_\sigma \sqrt{\sigma} \Phi$. Since $\|P_\sigma \sqrt{\sigma} \Phi\| \leq \|\sqrt{\sigma} \Phi\| \leq \|\sigma\|_\infty^{1/2} C_R \|\Phi\|_V$ by Lemma 2.1(a), M is a bounded linear map from V to L^2 . Hence M maps $H^1(0, T; V)$ into $H^1(0, T; L^2)$ with $\partial_t M\Phi = M\partial_t \Phi$. Since P_σ is an orthogonal projection, $(P_\sigma u, v) = (P_\sigma u, P_\sigma v)$, and so for $\Phi \in H^1(0, T; V)$

$$\int_0^s (P_\sigma \sqrt{\sigma} \Phi, \sqrt{\sigma} \partial_t \Phi) dt = \int_0^s (M\Phi, \partial_t M\Phi) dt = \frac{1}{2} \|M\Phi(s)\|^2 - \frac{1}{2} \|M\Phi(0)\|^2, \quad 0 \leq s \leq T. \quad (9)$$

Proof of Theorem 1.4(b), existence and coercivity. We apply the Lions–Lax–Milgram theorem in the form of [AH12, Lemma 3.6] (see [Sho97, Ch. III, Thm. 2.1 and Cor. 2.1]), with the Hilbert space $\mathcal{H}$ and the normed space $\mathcal{V}$ with the norm $|||\Phi|||^2 := \|\Phi\|_{\mathcal{H}}^2 + \|M\Phi(0)\|^2$.

- For fixed $\Phi \in \mathcal{V}$, the form $A \mapsto a_\sigma(A, \Phi)$ is continuous on $\mathcal{H}$. Indeed, $|(P_\sigma \sqrt{\sigma} A, \sqrt{\sigma} \partial_t \Phi)| \leq \|\sigma\|_\infty C_R^2 \|A\|_V \|\partial_t \Phi\|_V$ by Lemma 2.1(a), and $|(\mu^{-1} \operatorname{curl} A, \operatorname{curl} \Phi)| \leq (\operatorname{ess\,inf} \mu)^{-1} \|A\|_V \|\Phi\|_V$.
- By (9) with $s = T$ and $\Phi(T) = 0$,

$$a_\sigma(\Phi, \Phi) = \frac{1}{2} \|M\Phi(0)\|^2 + \int_0^T (\mu^{-1} \operatorname{curl} \Phi, \operatorname{curl} \Phi) dt \geq \frac{1}{2} \|M\Phi(0)\|^2 + \|\mu\|_\infty^{-1} \|\Phi\|_{\mathcal{H}}^2 \geq \alpha |||\Phi|||^2$$

with $\alpha := 1/\max\{\|\mu\|_\infty, 2\}$, which does not depend on σ . In particular $a_\sigma(\Phi, \Phi) \geq \|\mu\|_\infty^{-1} \|\Phi\|_{\mathcal{H}}^2$, which is the uniform coercivity of [AH12, Thm. 3.2(a)]. If $|||\Phi||| = 1$, then $\Phi \neq 0$ and $\sup_{\|E\|_{\mathcal{H}} \leq 1} |a_\sigma(E, \Phi)| \geq a_\sigma(\Phi, \Phi)/\|\Phi\|_{\mathcal{H}} \geq \alpha$, because $\|\Phi\|_{\mathcal{H}} \leq 1$.

- $|l(\Phi)| \leq \|J_t\|_{L^2(0, T; W(\operatorname{curl})')} \|\Phi\|_{L^2(0, T; W(\operatorname{curl}))} \leq \sqrt{5} \|J_t\|_{L^2(0, T; W(\operatorname{curl})')} |||\Phi|||$ by Lemma 2.1(a).

The theorem gives $A \in \mathcal{H}$ with $a_\sigma(A, \Phi) = l(\Phi)$ for all $\Phi \in \mathcal{V}$ and $\|A\|_{\mathcal{H}} \leq \alpha^{-1} \sqrt{5} \|J_t\|$. $\square$

Proof of Theorem 1.4(b), uniqueness. Let $D \in \mathcal{H}$ satisfy $a_\sigma(D, \Phi) = 0$ for all $\Phi \in \mathcal{V}$. Fix $s \in (0, T]$ and let $\Psi(t) := \int_t^s D(\tau) d\tau$ for $t \leq s$ and $\Psi(t) := 0$ for $t \geq s$. Then $\Psi \in \mathcal{V}$, with $\partial_t \Psi = -D$ on $(0, s)$ and $\partial_t \Psi = 0$ on (s, T) . The field $Z(t) := \operatorname{curl} \Psi(t) = \int_t^s \operatorname{curl} D d\tau$ lies in $H^1(0, s; L^2)$, with $Z' = -\operatorname{curl} D$ and $Z(s) = 0$. Hence

$$0 = a_\sigma(D, \Psi) = \int_0^s \|P_\sigma \sqrt{\sigma} D\|^2 dt - \int_0^s (\mu^{-1} Z', Z) dt = \int_0^s \|P_\sigma \sqrt{\sigma} D\|^2 dt + \frac{1}{2} (\mu^{-1} Z(0), Z(0)).$$

Both terms are nonnegative, so $Z(0) = \int_0^s \operatorname{curl} D d\tau = 0$ for every $s \in (0, T]$. Therefore $\operatorname{curl} D(t) = 0$ for almost every t , and $D = 0$ in $\mathcal{H}$. $\square$

Proof of Theorem 1.4(c). Let $E \in L^2(0, T; W(\operatorname{curl}))$ solve (6).

Step 1. Let $\psi \in \mathcal{D}(\mathbb{R}^3)$ and $\eta \in \mathcal{D}([0, T])$. The field $\Phi := \eta(t) \nabla \psi(x)$ lies in $\mathcal{D}(\mathbb{R}^3 \times [0, T])^3$ and is curl free, and $\langle J_t(t), \nabla \psi \rangle = 0$. So (6) gives $\int_0^T \eta'(t) g_\psi(t) dt = 0$ with $g_\psi(t) := \int \sigma E(t) \cdot \nabla \psi dx$. Taking $\eta \in \mathcal{D}(]0, T[)$ shows that g_ψ is almost everywhere equal to a constant c_ψ . Taking η with $\eta(0) = 1$ then gives $-c_\psi = 0$. So $g_\psi = 0$ almost everywhere.

Step 2. Choose a countable set $\{\psi_m\} \subset \mathcal{D}(\mathbb{R}^3)$ that is dense in $H^1(\mathbb{R}^3)$. Since $\|\sqrt{\sigma} \nabla \psi\| \leq \|\sigma\|_\infty^{1/2} \|\nabla \psi\|$, the fields $\sqrt{\sigma} \nabla \psi_m$ are dense in G_σ . By Step 1, for almost every t we have $(\sqrt{\sigma} E(t), \sqrt{\sigma} \nabla \psi_m) = 0$ for all m , that is, $Q_\sigma \sqrt{\sigma} E(t) = 0$.

Step 3. Let $A_E(t) := A_{E(t)}$ be as in Lemma 2.1(b). Since $E \mapsto A_E$ is bounded and linear, $A_E \in \mathcal{H}$, and $G := E - A_E \in L^2(0, T; L_\rho^2)$ with $G(t) = \nabla \varphi(t)$, $\varphi(t) \in H_{\operatorname{loc}}^1$. By Lemma 2.2(a),

$Q_\sigma \sqrt{\sigma} G(t) = \sqrt{\sigma} G(t)$. So Step 2 gives $\sqrt{\sigma} \nabla \varphi(t) = -Q_\sigma \sqrt{\sigma} A_E(t)$ and $\sqrt{\sigma} E(t) = P_\sigma \sqrt{\sigma} A_E(t)$. Also $\text{curl } E = \text{curl } A_E$.

Step 4. Let $\Phi \in \mathcal{V}$. Then $\Phi \in H^1(0, T; W(\text{curl}))$ by Lemma 2.1(a), and $\Phi(T) = 0$. By the last sentence of the paragraph on the natural formulation, (6) holds for Φ . Insert $\sigma E = \sqrt{\sigma} P_\sigma \sqrt{\sigma} A_E$ and $\text{curl } E = \text{curl } A_E$ to get $a_\sigma(A_E, \Phi) = l(\Phi)$. By uniqueness in (b), $A_E = A$. Steps 3 and 4 give the claims, and $\varphi(t)$ solves (3) with $A(t)$ by Lemma 2.2(b). $\square$

Proof of Theorem 1.4(d). By Lemma 2.2(b), $\sqrt{\sigma} G(t) = -Q_\sigma \sqrt{\sigma} A(t)$ for almost every t . So $E := A + G \in L^2(0, T; W(\text{curl}))$ satisfies $\text{curl } E = \text{curl } A$ and $\sqrt{\sigma} E = P_\sigma \sqrt{\sigma} A$. Let $\Phi \in \mathcal{D}(\mathbb{R}^3 \times [0, T])^3$. By [AH12, Lemma 3.5(b)] we can write $\Phi = \Psi + \nabla \phi$ with $\Psi \in \mathcal{V}$ and $\phi \in H^1(0, T; W^1(\mathbb{R}^3))$. We check the three terms of (6).

- $\int \sigma E \cdot \partial_t \Phi = (P_\sigma \sqrt{\sigma} A, \sqrt{\sigma} \partial_t \Psi) + (P_\sigma \sqrt{\sigma} A, \sqrt{\sigma} \nabla \partial_t \phi)$, and the last term vanishes because $\sqrt{\sigma} \nabla \partial_t \phi(t) \in G_\sigma$ by Lemma 2.2(a).
- $\text{curl } E = \text{curl } A$ and $\text{curl } \nabla \phi = 0$, so the curl term equals $\int_0^T (\mu^{-1} \text{curl } A, \text{curl } \Psi) dt$.
- $\langle J_t(t), \nabla \phi(t) \rangle = 0$, since $\phi(t) \in W^1(\mathbb{R}^3)$.

So (6) for Φ reduces to $a_\sigma(A, \Psi) = l(\Psi)$, which holds. $\square$

4 The necklace

4.1 Geometry

Recall $r = \sin(\pi/n)$, $a_k = 2\pi k/n$, $c_k = (\cos a_k, \sin a_k, 0)$ and $B_k = B(c_k, r)$. For $k \in \mathbb{Z}/n$ let S_k be $\{x : \varrho(x) > 0\}$ minus the closed half-plane of azimuth $a_k + \pi$, and let $\theta_k : S_k \rightarrow (a_k - \pi, a_k + \pi)$ be the smooth branch of the azimuth. Then $\nabla \theta_k = \varrho^{-1} e_\theta$ on S_k .

Lemma 4.1. (i) $|c_k - c_{k+j}| = 2 \sin(j\pi/n)$. Hence $\overline{B_k} \cap \overline{B_{k+1}} = \{p_k\}$ with $p_k = \frac{1}{2}(c_k + c_{k+1}) = \cos(\pi/n) (\cos(a_k + \pi/n), \sin(a_k + \pi/n), 0)$, and $\overline{B_k} \cap \overline{B_j} = \emptyset$ if $j \notin \{k-1, k, k+1\}$.

(ii) Every $x \in \overline{B_k}$ satisfies $1 - r \leq \varrho(x) \leq 1 + r$ and $|x_3| \leq r$. Moreover $\overline{B_k} \subset S_k$, and θ_k takes values in $[a_k - \pi/n, a_k + \pi/n]$ on $\overline{B_k}$.

(iii) $\theta_k(p_k) = a_k + \pi/n$ and $\theta_k(p_{k-1}) = a_k - \pi/n$. Consequently

$$\theta_k(p_k) - \theta_{k+1}(p_k) = 0 \quad (0 \leq k \leq n-2), \quad \theta_{n-1}(p_{n-1}) - \theta_0(p_{n-1}) = 2\pi. \quad (10)$$

Proof. (i) $|c_k - c_{k+j}|^2 = 2 - 2 \cos(2j\pi/n) = 4 \sin^2(j\pi/n)$. For $j = 1$ the distance is $2r$, so the two closed balls of radius r meet only at the midpoint of the centres, which is the stated point. For $2 \leq j \leq n-2$ we have $j\pi/n \in [2\pi/n, \pi - 2\pi/n]$, so $\sin(j\pi/n) > \sin(\pi/n)$ and the distance exceeds $2r$.

(ii) Write $x' = (x_1, x_2)$ and $c'_k = (\cos a_k, \sin a_k)$. Then $|x' - c'_k| \leq r$ and $|x_3| \leq r$, so $1 - r \leq |x'| \leq 1 + r$. The rays from 0 that meet the closed disc of radius $r < 1$ about c'_k , with $|c'_k| = 1$, make an angle at most $\arcsin r = \pi/n$ with c'_k . So the azimuth of x lies in $[a_k - \pi/n, a_k + \pi/n]$, which is contained in $(a_k - \pi, a_k + \pi)$.

(iii) The ray of azimuth $a_k + \pi/n$ has distance $\sin(\pi/n) = r$ from c'_k , so it is tangent to the disc, and the point of tangency is the foot of the perpendicular, at distance $\cos(\pi/n)$ from 0; this is p'_k . Since $a_k + \pi/n = a_{k+1} - \pi/n$, we get $\theta_k(p_k) = a_k + \pi/n$ and $\theta_{k+1}(p_k) = a_{k+1} - \pi/n$ for $0 \leq k \leq n-2$. For $k = n-1$ the neighbour is B_0 , and $\theta_0(p_{n-1}) = -\pi/n$ while $\theta_{n-1}(p_{n-1}) = 2\pi - \pi/n$. $\square$

Since $\overline{B_k} \subset S_k \cap U$, the field A^* equals $\nabla \theta_k$ on the open neighbourhood $S_k \cap U$ of $\overline{B_k}$.

4.2 Contact points are invisible to the weighted energy

Lemma 4.2. *Let $0 \leq \sigma \leq C\chi_N$. For each $k \in \mathbb{Z}/n$ let u_k be a C^∞ function on an open neighbourhood O_k of $\overline{B_k}$. Then $\sum_k \chi_{B_k} \sqrt{\sigma} \nabla u_k \in G_\sigma$. In particular $\sqrt{\sigma} A^* \in G_\sigma$, so $Q_\sigma \sqrt{\sigma} A^* = \sqrt{\sigma} A^*$ and $P_\sigma \sqrt{\sigma} A^* = 0$.*

Proof. Fix $\zeta \in C^\infty(\mathbb{R})$ with $0 \leq \zeta \leq 1$, $\zeta = 1$ on $(-\infty, 1]$ and $\zeta = 0$ on $[2, \infty)$. For small $\varepsilon > 0$ the balls $B(p_k, 2\varepsilon)$ are pairwise disjoint. Put $\eta_\varepsilon(x) := \sum_k \zeta(|x - p_k|/\varepsilon)$. Then $0 \leq \eta_\varepsilon \leq 1$ and, by scaling,

$$\|\nabla \eta_\varepsilon\|^2 = n\varepsilon \int_{\mathbb{R}^3} |\nabla(\zeta(|y|))|^2 dy \rightarrow 0 \quad (\varepsilon \rightarrow 0),$$

which expresses the fact that a point has zero capacity in $\mathbb{R}^3$.

The compact sets $K_k := \overline{B_k} \setminus \bigcup_j B(p_j, \varepsilon)$ are pairwise disjoint by Lemma 4.1(i), so they have positive mutual distances. Hence there are $\chi_k \in C_c^\infty(O_k)$ with disjoint supports and $\chi_k = 1$ near K_k . Put $v_\varepsilon := \sum_k \chi_k u_k \in C_c^\infty(\mathbb{R}^3)$ and $\psi_\varepsilon := (1 - \eta_\varepsilon)v_\varepsilon \in C_c^\infty(\mathbb{R}^3)$. On B_k we have $\psi_\varepsilon = (1 - \eta_\varepsilon)u_k$: where $\eta_\varepsilon < 1$ the point lies in K_k , so $v_\varepsilon = u_k$, and elsewhere both sides vanish. Let $L := \max_k \sup_{B_k} (|u_k| + |\nabla u_k|) < \infty$. Then

$$\|\nabla \psi_\varepsilon - \nabla u_k\|_{L^2(B_k)} \leq \|\eta_\varepsilon \nabla u_k\|_{L^2(B_k)} + \|u_k \nabla \eta_\varepsilon\|_{L^2(B_k)} \leq L |\text{supp } \eta_\varepsilon|^{1/2} + L \|\nabla \eta_\varepsilon\| \rightarrow 0.$$

Since $N \setminus \bigcup_k B_k$ is a null set and $\sigma \leq C\chi_N$,

$$\left\| \sqrt{\sigma} \nabla \psi_\varepsilon - \sum_k \chi_{B_k} \sqrt{\sigma} \nabla u_k \right\|^2 \leq C \sum_k \|\nabla \psi_\varepsilon - \nabla u_k\|_{L^2(B_k)}^2 \rightarrow 0,$$

and G_σ is closed. For the last claim take $u_k = \theta_k$ on $O_k = S_k \cap U$; then $\sqrt{\sigma} A^* = \sum_k \chi_{B_k} \sqrt{\sigma} \nabla \theta_k$ almost everywhere. $\square$

4.3 The gap lemma

Let $B_\pm := B(\pm re_3, r)$, two balls that touch at 0. Write $x = (x', x_3)$ and $g(s) := r - (r^2 - s^2)^{1/2}$ for $0 \leq s \leq r$. Near 0, the lower boundary of B_+ is the graph $x_3 = g(|x'|)$ and the upper boundary of B_- is $x_3 = -g(|x'|)$. Since $g(s) = s^2/(r + (r^2 - s^2)^{1/2})$, we have

$$s^2/(2r) \leq g(s) \leq s^2/r. \quad (11)$$

Lemma 4.3 (gap lemma). *Let $0 < b < r$ and $u \in H^1(B(0, b))$. Suppose that there are continuous functions $v_\pm$ on $\overline{B_\pm} \cap B(0, b)$ with $u = v_\pm$ almost everywhere on $B_\pm \cap B(0, b)$. Then $v_+(0) = v_-(0)$.*

Proof. Let $Z := \{|x'| < b/2, |x_3| < b/2\} \subset B(0, b)$. For $|x'| < b/2$ we have $g(|x'|) \leq b^2/(4r) < b/4$ by (11), and the segment $\{x'\} \times (g(|x'|), b/2)$ lies in B_+ , because $b/2 < r < 2r - g(|x'|)$. Likewise $\{x'\} \times (-b/2, -g(|x'|))$ lies in B_- .

By the characterisation of Sobolev functions as functions that are absolutely continuous on lines [Zie89, Thm. 2.1.4], u has a representative $\bar{u}$ that is absolutely continuous on almost every vertical segment in Z , with classical derivative $\partial_3 \bar{u} = \partial_3 u$ almost everywhere. By Fubini's theorem, for almost every x' with $|x'| < b/2$ we have $\bar{u}(x', \cdot) = v_+(x', \cdot)$ almost everywhere on $(g(|x'|), b/2)$. Both functions are continuous on $[g(|x'|), b/2)$, so they agree there, including at $x_3 = g(|x'|)$. In the same way $\bar{u}(x', -g(|x'|)) = v_-(x', -g(|x'|))$.

Suppose $\delta := |v_+(0) - v_-(0)| > 0$. The points $(x', \pm g(|x'|))$ tend to 0 as $x' \rightarrow 0$, so by continuity there is $b_0 \in (0, b/2)$ with $|v_+(x', g(|x'|)) - v_-(x', -g(|x'|))| \geq \delta/2$ for $|x'| < b_0$. For almost every such $x' \neq 0$, the Cauchy–Schwarz inequality on the segment gives

$$\frac{\delta^2}{4} \leq \left| \int_{-g}^g \partial_3 \bar{u}(x', s) ds \right|^2 \leq 2g(|x'|) \int_{-g}^g |\partial_3 u(x', s)|^2 ds,$$

so that, by (11), $\int_{-g}^g |\partial_3 u(x', s)|^2 ds \geq \delta^2 r / (8|x'|^2)$. Integrating over $0 < |x'| < b_0$ gives

$$\|\nabla u\|_{L^2(Z)}^2 \geq \frac{\delta^2 r}{8} \int_{|x'| < b_0} \frac{dx'}{|x'|^2} = \frac{\pi \delta^2 r}{4} \int_0^{b_0} \frac{ds}{s} = \infty,$$

a contradiction. $\square$

At each contact point p_k there is a rigid motion that maps p_k to 0, B_{k+1} to B_+ and B_k to B_- , because both balls have radius r and p_k is the midpoint of their centres. Rigid motions preserve H^1 and continuity, so Lemma 4.3 applies at every p_k .

4.4 Proof of Theorem 1.1

Suppose $\varphi \in H_{\text{loc}}^1(\mathbb{R}^3)$ solves (3) with $A = A^*$. By Lemma 2.2(b) and Lemma 4.2, $\sqrt{\sigma} \nabla \varphi = -Q_\sigma \sqrt{\sigma} A^* = -\sqrt{\sigma} A^*$. Since $\sigma > 0$ almost everywhere on N , we get $\nabla \varphi = -A^* = -\nabla \theta_k$ almost everywhere on each B_k . Each B_k is connected, so $\varphi = v_k$ almost everywhere on B_k with $v_k := -\theta_k + d_k$ for a constant $d_k \in \mathbb{R}$, and v_k is continuous on $\overline{B_k}$. By Lemma 4.3 at p_k , $v_k(p_k) = v_{k+1}(p_k)$, that is,

$$d_{k+1} - d_k = \theta_{k+1}(p_k) - \theta_k(p_k) \quad (k \in \mathbb{Z}/n).$$

Summing over $k \in \mathbb{Z}/n$, the left side gives 0, while by (10) the right side gives -2π . This contradiction proves Theorem 1.1. For the smooth example, let $\beta \in C^\infty(\mathbb{R})$ be positive on $(-\infty, r^2)$ and zero on $[r^2, \infty)$, for instance $\beta(s) = \exp(-1/(r^2 - s))$ for $s < r^2$, and put $\sigma(x) := \sum_k \beta(|x - c_k|^2)$. Then $\sigma \in C^\infty(\mathbb{R}^3)$, $\{\sigma > 0\} = \bigcup_k B_k = \text{int } N$, and $\sigma \leq \beta(0)\chi_N$. $\square$

Remark 4.4 (the obstruction is local). The same proof shows that there is no $\varphi \in H_{\text{loc}}^1(O)$ solving (3) for any open set $O \supset N$, with $\sigma \nabla \varphi$ extended by zero. Indeed, $\chi \varphi \in H^1(\mathbb{R}^3)$ for a cutoff $\chi \in C_c^\infty(O)$ equal to 1 near N , so Lemma 2.2(a) still applies. In contrast, the per-ball problems $\int_{B_k} (A + \nabla \varphi_k) \cdot \nabla v = 0$ for $v \in H^1(B_k)$ are solvable, with $\varphi_k = -\theta_k$ for $A = A^*$. They describe the element $w_A = -Q_\sigma \sqrt{\sigma} A$ of Definition 2.3, since $G_\sigma = \bigoplus_k \nabla H^1(B_k)$ for $\sigma = \chi_N$: the inclusion $\supset$ follows from Lemma 4.2 and the density of $C^\infty(\overline{B_k})$ in $H^1(B_k)$, and $\subset$ holds because the right side is closed. So the defect is that these per-ball potentials do not fit together into one H_{loc}^1 function.

5 The degenerate torus

Lemma 5.1. *For every integer $j \geq 2$ there is a nondecreasing $h_j \in C^\infty(\mathbb{R})$ with $h_j = 0$ on $(-\infty, -1/2]$, $h_j = 1$ on $[1/2, \infty)$ and $\int_{\mathbb{R}} |t| h_j'(t)^2 dt \leq 2/j$.*

Proof. Fix $\chi_0 \in C_c^\infty((-1/2, 1/2))$ with $0 \leq \chi_0 \leq 1$ and $\chi_0 = 1$ on $[-1/4, 1/4]$. Let $\beta_j(t) := \chi_0(t)(t^2 + e^{-2j})^{-1/2}$, $I_j := \int \beta_j$ and $h_j(t) := I_j^{-1} \int_{-\infty}^t \beta_j$. Then

$$I_j \geq \int_{-1/4}^{1/4} \frac{dt}{(t^2 + e^{-2j})^{1/2}} = 2 \operatorname{arsinh}(e^j/4) \geq 2(j - \log 2) \geq j,$$

$$\int |t| \beta_j^2 \leq \int_{-1/2}^{1/2} \frac{|t| dt}{t^2 + e^{-2j}} = \log(1 + e^{2j}/4) \leq 2j,$$

using $\operatorname{arsinh} y \geq \log(2y)$ and $j \geq 2$. Hence $\int |t| h_j'^2 = I_j^{-2} \int |t| \beta_j^2 \leq 2j/j^2$. $\square$

Let $\tilde{\theta} \in (-\pi, \pi]$ be the principal azimuth. It is smooth on $S := \{\varrho > 0\} \setminus \{x_2 = 0, x_1 < 0\}$, with $\nabla \tilde{\theta} = \varrho^{-1} e_\theta$.

Lemma 5.2. *Let $0 \leq \sigma \leq C|\sin(\theta/2)|\chi_{\mathbb{T}}$. Then $\sqrt{\sigma}A^* \in G_\sigma$, so $Q_\sigma\sqrt{\sigma}A^* = \sqrt{\sigma}A^*$ and $P_\sigma\sqrt{\sigma}A^* = 0$.*

Proof. Let h_j be as in Lemma 5.1, and put $\Theta_j := \tilde{\theta} - 2\pi h_j(\tilde{\theta})$ on S . On $S \cap \{\tilde{\theta} > 1/2\}$ we have $\Theta_j = \tilde{\theta} - 2\pi$, and on $S \cap \{\tilde{\theta} < -1/2\}$ we have $\Theta_j = \tilde{\theta}$. Near the half-plane $\{x_2 = 0, x_1 < 0\}$ the function equal to $\tilde{\theta}$ for $x_2 > 0$ and to $\tilde{\theta} + 2\pi$ for $x_2 < 0$ is a smooth branch of the azimuth, and Θ_j equals that branch minus 2π . So Θ_j extends to a C^∞ function on $\{\varrho > 0\}$, with

$$\nabla\Theta_j = (1 - 2\pi h'_j(\tilde{\theta}))\varrho^{-1}e_\theta,$$

where $h'_j(\tilde{\theta})$ is read as 0 near that half-plane. Choose $\chi_1 \in C_c^\infty(\{\varrho > 0\})$ with $\chi_1 = 1$ near $\mathbb{T}$ and put $\psi_j := \chi_1\Theta_j \in C_c^\infty(\mathbb{R}^3)$. Since $A^* = \varrho^{-1}e_\theta$ on $U \supset \mathbb{T}$, we have $\nabla\psi_j - A^* = -2\pi h'_j(\tilde{\theta})\varrho^{-1}e_\theta$ on $\mathbb{T}$. In cylindrical coordinates $(\varrho, \tilde{\theta}, x_3)$ the torus is $D_{\mathbb{T}} \times (-\pi, \pi]$ with the disc $D_{\mathbb{T}} := \{(\varrho-1)^2 + x_3^2 \leq 1/4\}$, and $\varrho \geq 1/2$ on $D_{\mathbb{T}}$, so that $\int_{D_{\mathbb{T}}} \varrho^{-1} d\varrho dx_3 \leq 2|D_{\mathbb{T}}| = \pi/2$. Using $|\sin(t/2)| \leq |t|/2$, $\text{supp } h'_j \subset [-1/2, 1/2]$ and Lemma 5.1,

$$\|\sqrt{\sigma}(\nabla\psi_j - A^*)\|^2 \leq 4\pi^2 C \int_{D_{\mathbb{T}}} \frac{d\varrho dx_3}{\varrho} \int_{-1/2}^{1/2} |\sin(t/2)| h'_j(t)^2 dt \leq 4\pi^2 C \cdot \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{2}{j} \rightarrow 0.$$

So $\sqrt{\sigma}A^* = \lim_j \sqrt{\sigma}\nabla\psi_j \in G_\sigma$. $\square$

Proof of Theorem 1.2. Suppose $\varphi \in H_{\text{loc}}^1(\mathbb{R}^3)$ solves (3) with $A = A^*$. By Lemma 2.2(b) and Lemma 5.2, $\sqrt{\sigma}(\nabla\varphi + A^*) = 0$. Since $\sigma > 0$ almost everywhere on $\mathbb{T}$, $\nabla\varphi = -\varrho^{-1}e_\theta$ almost everywhere on $\mathbb{T}$. The open set $\mathbb{T}^\circ \setminus \{x_2 = 0, x_1 \geq 0\}$ is connected, and on it the branch $\theta \in (0, 2\pi)$ of the azimuth is smooth with $\nabla\theta = \varrho^{-1}e_\theta$. Hence $\varphi = c - \theta$ almost everywhere there, for a constant c . Now consider the cube $Q := (3/4, 5/4) \times (-1/4, 1/4)^2$, which lies in $\mathbb{T}^\circ$. On $Q \cap \{x_2 > 0\}$ we have $\theta = \arctan(x_2/x_1)$, and on $Q \cap \{x_2 < 0\}$ we have $\theta = \arctan(x_2/x_1) + 2\pi$. So

$$\varphi(x) = c - \arctan(x_2/x_1) \quad \text{for } x_2 > 0, \quad \varphi(x) = c - 2\pi - \arctan(x_2/x_1) \quad \text{for } x_2 < 0,$$

almost everywhere in Q . By [Zie89, Thm. 2.1.4], for almost every (x_1, x_3) the function $x_2 \mapsto \varphi(x_1, x_2, x_3)$ has a representative that is absolutely continuous on $(-1/4, 1/4)$. It agrees almost everywhere with the two expressions above, so continuity at $x_2 = 0$ gives $c = c - 2\pi$, a contradiction. $\square$

The mechanism is the classical fact that a weight vanishing at least linearly across a surface gives that surface zero weighted capacity: the one-dimensional energy $\int |t|h'(t)|^2 dt$ of a transition from 0 to 1 can be made arbitrarily small, since $\int dt/|t| = \infty$. For weights $|t|^\gamma$ with $\gamma < 1$ this fails, and we do not know whether (3) is then always solvable.

6 Nonexistence for the eddy-current equation

Proof of Theorem 1.3. Let σ be as in Theorem 1.1 or Theorem 1.2. The map $\Phi \mapsto -t \int \mu^{-1} \text{curl } A^* \cdot \text{curl } \Phi$ is bounded on $W(\text{curl})$ with norm at most $t\|\mu^{-1}\|_\infty \|\text{curl } A^*\|$, so $J_t \in L^2(0, T; W(\text{curl})')$. It vanishes on gradients, so J_t is divergence free. It vanishes on fields supported in U , because $\text{curl } A^* = 0$ on U (Section 2.4). For $\mu \equiv 1$, integration by parts gives $J_t(t) = -t \text{curl } \text{curl } A^*$, a smooth field whose support is compact and disjoint from U .

The field $A(t) := tA^*$ lies in $\mathcal{H} = L^2(0, T; V)$ and solves the unified formulation (7). Indeed, $P_\sigma\sqrt{\sigma}A^* = 0$ by Lemma 4.2 or Lemma 5.2, so the first term of a_σ vanishes, and

$$a_\sigma(A, \Phi) = \int_0^T t(\mu^{-1} \text{curl } A^*, \text{curl } \Phi) dt = - \int_0^T \langle J_t(t), \Phi(t) \rangle dt = l(\Phi).$$

By uniqueness in Theorem 1.4(b), A is the solution of (7). Suppose that $E \in L^2(0, T; W(\text{curl}))$ solves (6). By Theorem 1.4(c) there is $t \in (0, T)$ and $\varphi(t) \in H_{\text{loc}}^1(\mathbb{R}^3)$ solving (3) with $A(t) = tA^*$ in place of A . Since (3) is linear, $u := \varphi(t)/t$ solves (3) with $A = A^*$. This contradicts Theorem 1.1 or Theorem 1.2. $\square$

Remark 6.1 (physical picture). With the source of Theorem 1.3, the unified solution $A = tA^*$ carries no eddy current: $\sigma(A + \nabla\varphi_A) = t\sqrt{\sigma}P_\sigma\sqrt{\sigma}A^* = 0$. Its curl $t \text{curl} A^*$ is the field of the non-conducting (magnetostatic) problem, and it vanishes near the conductor. Still, by Stokes' theorem the circulation of $E = A + \nabla\varphi$ around a loop through the ring would be $t \oint A^* \cdot dl = 2\pi t$; this is the voltage induced around the ring. In the necklace, no current can pass through a point contact, and each ball is simply connected. So by Theorem 1.4(c) any solution would have $E = 0$ in the balls, and the whole induced voltage would have to drop across the contact points. By Lemma 4.3 this costs infinite electric energy. In the torus, the voltage would have to drop across the cross-section where σ vanishes.

7 Further remarks

The first two remarks are sketches. They are not used in the proofs above, and we do not claim them as theorems.

Remark 7.1 (the exact obstruction for the necklace; sketch). Let $\sigma = \chi_N$ and $A \in V$. Let $\varphi_k \in H^1(B_k)$ be the Neumann potential, $\int_{B_k} (A + \nabla\varphi_k) \cdot \nabla v = 0$ for all $v \in H^1(B_k)$. Since $A|_{B_k} \in H^1$ and $\text{div} A = 0$, φ_k is harmonic with Neumann data $-A \cdot \nu \in H^{1/2}(\partial B_k)$, so by elliptic regularity $\varphi_k \in H^2(B_k) \subset C(\overline{B_k})$. Put $\ell(A) := \sum_k [\varphi_k(p_k) - \varphi_k(p_{k-1})]$. This is a continuous linear functional on V , and $\ell(A^*) = -2\pi$ by Lemma 4.1(iii). The proof of Theorem 1.1, with Remark 4.4 in place of Lemma 4.2, shows that (3) can be solvable only if $\ell(A) = 0$. Conversely, if $\ell(A) = 0$, choose constants $d_k \in \mathbb{R}$ with $\varphi_k(p_k) + d_k = \varphi_{k+1}(p_k) + d_{k+1}$ for all k . Near each contact point one can then glue H^2 extensions $\tilde{V}_\pm$ of the two potentials by $\tilde{V}_- + \chi(\tilde{V}_+ - \tilde{V}_-)$ with $\chi(x) = \tilde{\zeta}(x_3/g(|x'|))$, in the normal form of Lemma 4.3. Here $|\nabla\chi| \lesssim |x'|^{-2}$ on a layer of thickness $\asymp |x'|^2$, and $|\tilde{V}_+ - \tilde{V}_-| \lesssim |x'|^{1/2}$ there, so the extra energy is bounded by a multiple of $\int |x'| |x'|^{-4} |x'|^2 dx' < \infty$. We have not written out this gluing in full, including the removability of the axis $x' = 0$ and the measurability and boundedness of the resulting linear map. If it holds, then (3) is solvable exactly for $A \in \ker \ell$, a closed hyperplane of V , and by Theorem 1.4(c),(d) the eddy-current equation is solvable if and only if $\ell(A(t)) = 0$ for almost every t . Chains of touching balls without a cycle should cause no obstruction.

Remark 7.2 (potentials below the energy level; sketch). The half-plane of azimuth $-\pi/n$ meets N only at p_{n-1} ; this is the equality case in the proof of Lemma 4.1(ii),(iii). Let $\hat{\theta}$ be the branch of the azimuth with this cut. Then $-\hat{\theta}$ has gradient $-A^*$ on N but jumps by 2π across the cut. One can smooth the jump outside N , and inside the gap at p_{n-1} with the cutoff χ of Remark 7.1. The resulting φ has $\nabla\varphi = -A^*$ almost everywhere on N and $|\nabla\varphi| \lesssim |x'|^{-2}$ on a layer of thickness $\asymp |x'|^2$, so that $\int |\nabla\varphi|^p \lesssim \int_0 s^{3-2p} ds < \infty$ exactly when $p < 2$. So (3) for A^* should have solutions with $\nabla\varphi \in L_{\text{loc}}^p$ for every $p < 2$. If so, the obstruction in Theorem 1.1 sits exactly at the energy space.

Remark 7.3 (the energy of a potential jump). The gap lemma measures the energy needed to hold two touching balls at different potentials. For two unit balls at distance h , held at potentials $\pm \frac{1}{2}$, the minimal Dirichlet energy on $\mathbb{R}^3$ is given by the classical bispherical series $\mathcal{E}(h) = 2\pi \sinh \alpha \sum_{m \geq 1} \text{csch}(m\alpha)$ with $\cosh \alpha = 1 + h/2$. The expansion $\sum_{m \geq 1} \text{csch}(m\alpha) = \alpha^{-1}(\log(2/\alpha) + \gamma) + O(\alpha)$, with γ Euler's constant, which we checked numerically, gives $\mathcal{E}(h) = \pi \log(1/h) + 2\pi(\log 2 + \gamma) + o(1)$. The vertical-segment bound of Lemma 4.3 (with $r = 1$ and $\delta = 1$) gives $\mathcal{E}(h) \geq LB(h) := \int_{|x'| < 1} dx' / (h + 2g(|x'|)) = -\pi + \frac{\pi}{2}(2 + h) \log(1 + 2/h)$, which has the same leading term. Hence $\mathcal{E} - LB \rightarrow \pi(1 + \log 2 + 2\gamma) =$

8.945932..., in agreement with the computed values; for example $\mathcal{E}(10^{-10}) = 80.3198$. This logarithmic growth is the behaviour found by Keller [Kel63] for nearly touching perfect conductors. The numbers are a computation, recorded below, and are not used in the proofs.

Remark 7.4 (a sufficient condition and open questions). If the set $\{\sqrt{\sigma}\nabla\varphi : \varphi \in H_{\text{loc}}^1(\mathbb{R}^3)\}$ is closed in $L^2(\mathbb{R}^3)^3$, then it equals G_σ , and by Lemma 2.2(b) (3) is solvable for every $A \in L_{\text{loc}}^2(\mathbb{R}^3)^3$. This holds in the class $L_R^\infty(\mathbb{R}^3)$. More generally it holds when $\sigma = 0$ outside a bounded open set Ω_c that has finitely many components and the H^1 -extension property (for example, a set that satisfies [PPTW21, (23)]), and σ is bounded above and below by positive constants on Ω_c . Indeed, by the extension property the restrictions to Ω_c of the functions in $H_{\text{loc}}^1(\mathbb{R}^3)$ form all of $H^1(\Omega_c)$, and $H^1(\Omega_c)$ embeds compactly into $L^2(\Omega_c)$. Hence the Poincaré inequality holds on each component, $\nabla H^1(\Omega_c)$ is closed in $L^2(\Omega_c)^3$, and multiplication by $\sqrt{\sigma}$ is an isomorphism of $L^2(\Omega_c)^3$. For $\sigma = \chi_N$ the set is not closed, by Theorem 1.1 and Remark 4.4. We do not know a characterisation of the conductivities for which (3) is solvable for all $A \in V$. We have not examined whether the connectedness of $\mathbb{R}^3 \setminus \overline{\Omega}$ in [AH12] can be dropped, nor whether a weaker notion of solution of the eddy-current equation, with E only locally in L^p for some $p < 2$ near the contact points, restores existence.

Verification. The proofs above are complete and by hand. The finite and algebraic facts they use were also checked by computer, as follows.

1. A program by the author, `verify_owr11578_lead.py` (sympy, mpmath, numpy and scipy; about fifteen seconds), checks the following.
 - Lemma 4.1 for $3 \leq n \leq 12$, symbolically and in 60-digit arithmetic. The branch values are evaluated with `atan2` at the actual contact points. This gives the mismatch vector $(0, \dots, 0, 2\pi)$ of (10) and $\ell(A^*) = -2\pi$. It also checks $N \subset U$ and that p_k lies on no other closed ball.
 - The formulas of Section 2.4 for A^* and $\text{curl} A^*$, symbolically, and that $\text{curl} A^*$ and $\text{curl curl} A^*$ vanish on U .
 - The bounds (11), the segment estimate and the logarithmic divergence in Lemma 4.3, exactly, and the scaling $\|\nabla\eta_\varepsilon\|^2 = \varepsilon n C_\zeta$ of Lemma 4.2 for a C^1 profile ($C_\zeta = 384\pi/35$).
 - The two integrals in the proof of Lemma 5.1, symbolically. Its bounds for $2 \leq j \leq 200$ are checked in 250-digit arithmetic, which resolves the smallest margin, about $8e^{-400}$ in $\text{arsinh } y \geq \log(2y)$; the inequalities themselves are elementary. It also computes $\int \sin^2(t/2) h'_j(t)^2 dt$ by quadrature for $j = 2, 5, 10, 20$.
 - The series and the closed form of $LB(h)$ in Remark 7.3 for $h = 10^{-1}, \dots, 10^{-10}$, the two limits stated there, and the expansion of $\sum \text{csch}(m\alpha)$.
2. The finder's programs, `verify_necklace.py` and `gap_energy.py`, check the geometry for $3 \leq n \leq 8$, the identities for A^* , the gap bounds and the exponents in Remarks 7.1 and 7.2. They also compute $\mathcal{E}(h)$ in two ways, by the bispherical series and by the image-charge recursion, which agree to all printed digits for $h \geq 10^{-6}$.
3. Three independent referees checked all proofs by hand. The second and the third wrote separate code before reading the finder's; the third wrote it from the text of the paper alone, before opening any other script.
 - The second referee's code checks the geometry for $3 \leq n \leq 12$, with `atan2` at the contact points, and the curl and divergence identities for A^* symbolically, including $\text{curl} A^* = 0$ on U . It computes $\mathcal{E}(h)$ by a third method, a multipole re-expansion linear system, which

agrees with the series to eight digits for $h \geq 10^{-4}$, and it checks the one-dimensional weighted capacities behind Section 5.

- The third referee’s code, `referee2_check.py` (47 checks, about ten seconds), uses its own cutoff functions. It rechecks Lemma 4.1 for $3 \leq n \leq 24$ in 60-digit arithmetic, the field A^* and the gap lemma. A deterministic quadrature near the contact points shows that the error in Lemma 4.2 is $O(\varepsilon)$. It checks Lemma 5.1 for a concrete χ_0 , the torus energies of Lemma 5.2 and the values in Remark 7.3. Its own image-charge recursion agrees with the series for $\mathcal{E}(h)$ to 10^{-9} (relative) for $10^{-8} \leq h \leq 10^{-1}$.

The first referee checked numerically that $A^* = \nabla\theta_k$ on the balls, the divergence and the circulation, and that the error in Lemma 4.2 is $O(\varepsilon)$ in a Monte Carlo test.

Scope and priority. Theorems 1.1 to 1.3 answer the question of [Har12, p. 631], restated in [Har13, Har14], negatively in the reading explained in the introduction. The answer stays negative for smooth σ , and for every notion of φ_A that is required to make $A + \nabla\varphi_A$ a solution of the eddy-current equation in $L^2(0, T; W(\text{curl}))$. Theorem 1.4 is the positive counterpart: it extends the unified formulation of [AH12] with zero initial data to all $\sigma \geq 0$, but not its link to the eddy-current equation. Remarks 7.1 and 7.2 are sketches. We read the Oberwolfach abstract [Har12], the author-hosted version of [AH12] and the slides [Har13, Har14]. We checked the hypotheses of the follow-ups [AH13a, AH13b] and of the thesis [Sim14] of Harrach’s coauthor L. Simon (née Arnold), which was found in the catalogue of the German National Library. The thesis keeps the class $L_R^\infty(\mathbb{R}^3)$ (its (3.12)) and the potential map of [AH12] (its Lemma 3.5), and it does not treat general $\sigma \geq 0$. We searched arXiv, Crossref, OpenAlex, zbMATH and that catalogue, and ran one web search per agent. We screened the twenty works that OpenAlex lists as citing [AH12] and the thirteen documents that zbMATH lists, both queried on 30 September 2026. We found no work that treats (2) for general $\sigma \geq 0$, no counterexample with touching or degenerate conductors, and no nonexistence result for the eddy-current equation of this kind. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the counterexamples and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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