

Favourite Distances in Three-Space: Exact Suspension Counts and a Three-Arc Gap

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Abstract

Assign a positive radius to each point of a finite set in Euclidean three-space and count directed pairs at the assigned distance. Put $B(n) = \lceil n^2/4 + 5n/2 \rceil + 1$. We prove that the exact maximum among n -point suspensions is $B(n)$ for every $n \geq 13$, using Swanepoel's construction for sharpness. A maximum principle on the finitely generated group of circle angles supplies a rational increment, and a rational sine-product obstruction excludes equality in the earlier upper bound. We then prove that a suspension core with at least eight circle points and one genuine exceptional point has at most $B(n)$ arcs; with two exceptions it has at most $B(n)+3$. Combining this with Swanepoel's structural theorem gives $B(n) \leq f_3(n) \leq B(n)+3$ for all sufficiently large n . Two exceptions in a large extremizer must be reflections across the circle plane. The exact unrestricted maximum and the conjecture that every sufficiently large extremizer is a suspension remain unresolved.

1 The exact restricted maximum

For a finite $S \subset \mathbb{R}^3$ and a function $r : S \rightarrow (0, \infty)$, let

$$E_r(S) = \{(x, y) \in S^2 : |x - y| = r(x)\}, \quad e_r(S) = |E_r(S)|.$$

For subsets $A, B \subseteq S$, write $e_r(A, B) = |E_r(S) \cap (A \times B)|$. Let $f_3(n)$ be the maximum of $e_r(S)$ over all n -point sets and all radius assignments. Throughout, put

$$B(n) = \left\lfloor \frac{(n+5)^2}{4} \right\rfloor - 5 = \left\lceil \frac{n^2}{4} + \frac{5n}{2} \right\rceil + 1. \quad (1)$$

A *suspension* consists of a circle $\mathcal{C}$ with symmetry axis $\mathcal{L}$, a finite set $S \subseteq \mathcal{C} \cup \mathcal{L}$, and radii satisfying $r(z) = d(z, \mathcal{C})$ for $z \in S \cap \mathcal{L}$. Put $C = S \cap \mathcal{C}$, $L = S \cap \mathcal{L}$, $c = |C|$ and $\ell = |L|$.

Theorem 1. *Every suspension with $c > 0$ satisfies*

$$e_r(S) \leq \ell c + \ell + 4c - 1.$$

Consequently, for every integer $n \geq 13$, the maximum of $e_r(S)$ over all n -point suspensions is

$$\left\lceil \frac{n^2}{4} + \frac{5n}{2} \right\rceil + 1.$$

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Swanepoel's Proposition 1 gives an upper bound one larger, while his Theorem 2 gives the matching lower bound used here [1]. The remark following that theorem suggests the unattainability of the larger bound and points to a result of Newman [2]. The missing step supplied below is a rational angular increment in an otherwise unrestricted finite circle configuration. Swanepoel's Theorem B reduces a sufficiently large unrestricted extremizer to a suspension with at most two exceptions. Our estimates for these exceptions also give the following improvement of the constant in his Theorem A, whose upper bound is $B(n) + 11$.

Theorem 2. *For all sufficiently large n ,*

$$B(n) \leq f_3(n) \leq B(n) + 3.$$

If an extremizer has two genuine exceptional points relative to a suspension core with at least eight circle points, then, for $n \geq 23$, the two exceptions are reflections of one another across the circle plane.

Here a *genuine exception* is off both the supporting circle and its axis. The theorem does not assert that the exceptions can be removed.

2 A maximum principle for circle angles

We regard circle angles as elements of $\mathbb{R}/\mathbb{Z}$, measured in full turns.

Lemma 3. *Let $A \subset \mathbb{R}/\mathbb{Z}$ be finite and nonempty. Suppose that for every $a \in A$ there are distinct $b, c \in A \setminus \{a\}$ with $b + c = 2a$. Then for at least one such triple the difference $b - a$ is a nonzero torsion element of order different from two.*

Proof. Let Γ be the subgroup generated by A and let H be its torsion subgroup. The group Γ/H is free abelian of finite rank. By sending a basis to real numbers linearly independent over $\mathbb{Q}$, we obtain a homomorphism $h : \Gamma \rightarrow \mathbb{R}$ with kernel H . When the rank is zero, take $h = 0$.

Choose $a \in A$ maximizing $h(a)$. The relation $h(b) + h(c) = 2h(a)$, together with $h(b), h(c) \leq h(a)$, forces $h(b) = h(c) = h(a)$. Thus $b - a \in H$. It is nonzero because $b \neq a$. If it had order two, then $c - a = -(b - a) = b - a$, contradicting $b \neq c$. $\square$

Two distinct points on a circle at the same positive distance from a third point have opposite angular increments from it. Thus the lemma applies whenever every circle point has two chosen-distance neighbours. It produces one rational increment, not rationality of all angles.

3 An elementary rational angle obstruction

We include the special sine-product fact needed in the proof. It belongs to the classical root-of-unity setting of Newman [2].

Lemma 4. *If ζ is a root of unity and $\alpha = \zeta + \zeta^{-1}$ has all its algebraic conjugates nonnegative, then $\alpha \in \{0, 1, 2\}$.*

Proof. Assume $\alpha \neq 0$, and let N be the order of ζ . Every conjugate of α is then strictly positive, since an embedding cannot take a nonzero element to zero. The norm of the algebraic integer α is a positive integer. Hence the arithmetic mean of its distinct conjugates is at least their geometric mean, which is at least one.

Averaging over the primitive N th roots gives the same mean, since each conjugate occurs with the same multiplicity. The sum of the primitive N th roots is $\mu(N)$, by Möbius inversion of the sums of all roots. It follows that

$$1 \leq \frac{2\mu(N)}{\varphi(N)}.$$

Thus $\varphi(N) \leq 2$, so $N \in \{1, 2, 3, 4, 6\}$. The respective values of $\zeta + \zeta^{-1}$ are $2, -2, -1, 0, 1$. The nonnegative possibilities, including the previously excluded zero case, are exactly those stated. $\square$

Lemma 5. *Suppose $\theta/\pi, \psi/\pi \in \mathbb{Q}$, $0 < \theta < \pi$ and $0 < \psi < \pi/2$. If $\sin \theta \sin \psi = 1/2$, then*

$$(\theta, \psi) \in \{(\pi/4, \pi/4), (\pi/2, \pi/6), (3\pi/4, \pi/4)\}.$$

Proof. Put $u = \theta - \psi$, $v = \theta + \psi$, $x = 2 \cos u$ and $y = 2 \cos v$. Then $x - y = 2$. Both x and y are sums of a root of unity and its inverse. In a common cyclotomic field every conjugate of either lies in $[-2, 2]$, and every embedding preserves $x - y = 2$. Every conjugate of x is therefore nonnegative; embeddings of its subfield extend to the common field. Lemma 4 gives $x \in \{0, 1, 2\}$ and $y = x - 2$.

The inequalities $-\pi/2 < u < \pi$ and $0 < v < 3\pi/2$ now leave three possibilities consistent with $0 < \psi < \pi/2$. If $(x, y) = (0, -2)$, then $(u, v) = (\pi/2, \pi)$. If $(x, y) = (1, -1)$, the allowed pair is $(u, v) = (\pi/3, 2\pi/3)$. If $(x, y) = (2, 0)$, then $(u, v) = (0, \pi/2)$. Recovering $\theta = (u + v)/2$ and $\psi = (v - u)/2$ gives the assertion. $\square$

4 Axis dynamics and the strict bound

Normalize $\mathcal{C}$ to radius one in the plane $z = 0$ and identify $\mathcal{L}$ with the real height coordinate. For $z \in L$ we have $r(z) = \sqrt{1 + z^2}$. An axis arc $z \rightarrow w$ satisfies

$$(w - z)^2 = 1 + z^2.$$

There is no such arc ending at zero. For $w \neq 0$ its only possible predecessor is

$$p(w) = \frac{1}{2} \left(w - \frac{1}{w} \right). \quad (2)$$

In particular $e_r(L, L) \leq \ell$.

Lemma 6. *If $e_r(L, L) = \ell$, then $0, 1, -1 \notin L$ and every $w \in L$ can be written $w = \cot \theta$ with $0 < \theta < \pi$ and $\theta/\pi \in \mathbb{Q}$.*

Proof. Equality forces every vertex to have its predecessor in L . Thus $0 \notin L$ and $p(L) \subseteq L$. Since $p(1) = p(-1) = 0$, the heights $1, -1$ are also excluded.

The identity $p(\cot \theta) = \cot(2\theta)$ conjugates the predecessor map to angle doubling modulo π . Every forward orbit in the finite set L repeats. Consequently, for some integers $b > a \geq 0$, $(2^b - 2^a)\theta \in \pi\mathbb{Z}$, proving the rationality assertion. $\square$

Proof of Theorem 1. The four arc classes satisfy

$$e_r(L, C) = \ell c, \quad e_r(L, L) \leq \ell, \quad e_r(C, L) \leq 2c, \quad e_r(C, C) \leq 2c.$$

For the last two estimates, a sphere meets the axis in at most two points, and a sphere centred on $\mathcal{C}$ meets $\mathcal{C}$ in at most two points. The centre cannot be the centre of the circle.

Suppose their sum attains $\ell c + \ell + 4c$. Every circle point must then have two circle neighbours and two axis neighbours, and the hypothesis of Lemma 6 holds. Lemma 3 supplies a circle point whose two neighbours have angular increments $\pm 2\psi$, where $\psi/\pi \in \mathbb{Q}$ and $0 < \psi < \pi/2$. The upper endpoint is excluded because it would give only one antipodal neighbour. Its chosen radius is $2 \sin \psi$.

Its two axis neighbours have opposite heights $h, -h$ with $h > 0$. Write $h = \cot \theta$, $0 < \theta < \pi/2$. Lemma 6 makes θ/π rational, and equality of the circle and axis distances gives

$$2 \sin \psi = \sqrt{1 + h^2} = \frac{1}{\sin \theta}.$$

Lemma 5 forces $\theta = \pi/4$, hence $h = 1$. This contradicts Lemma 6. Equality is impossible, and integrality yields $e_r(S) \leq \ell c + \ell + 4c - 1$.

For $c > 0$ and $n = c + \ell$,

$$e_r(S) \leq (\ell + 4)(c + 1) - 5 \leq \left\lfloor \frac{(n + 5)^2}{4} \right\rfloor - 5 = \left\lceil \frac{n^2}{4} + \frac{5n}{2} \right\rceil + 1.$$

If $c = 0$, then $e_r(S) \leq n$, which also satisfies the final bound for $n \geq 1$. Swanepoel's Theorem 2 [1] supplies an n -point suspension attaining the bound for every $n \geq 13$. This completes the proof. $\square$

5 Exceptional points and deficits

Let $S = C \cup L \cup T$, where $C \cup L$ is a suspension core and every point of T is a genuine exception. Write $t = |T|$ and $n = c + \ell + t$. In the rest of the argument assume $c \geq 8$ and $t \leq 2$. Suppress the subscript r and define the nonnegative integer deficits

$$a = \ell - e(L, L), \quad b = 2c - e(C, L), \quad d = 2c - e(C, C), \quad D = a + b + d.$$

The core has $\ell c + \ell + 4c - D$ arcs, and Theorem 1 gives $D \geq 1$.

For $z \in T$, let $q(z) = e(C, \{z\})$. Rotate about the axis to write $z = (\rho, 0, h)$ with $\rho > 0$. If a circle point $x = (\cos \theta, \sin \theta, 0)$ reaches both z and an axis height w , then equality of the two squared distances gives

$$\cos \theta = \frac{\rho^2 + h^2 - w^2}{2\rho}. \quad (3)$$

For each height w this allows at most two circle points. Summing the axis incidences of the $q(z)$ circle points therefore gives

$$2q(z) - b \leq 2\ell, \quad q(z) \leq \min\{c, \ell + \lfloor b/2 \rfloor\}. \quad (4)$$

This uses the total deficit b , including points not reaching z .

An arc from an axis height w to z requires $2hw = \rho^2 + h^2 - 1$. There is at most one such height: when $h = 0$, there is none because z is not on the unit circle. A sphere centered at z meets each of the circle and axis in at most two points, so there are at most four outgoing arcs from z to the core. Hence

$$e(S) \leq \ell c + \ell + 4c - D + \sum_{z \in T} q(z) + 5t + t(t - 1). \quad (5)$$

If $b = 0$ and $0 \in L$, no circle point uses height zero, because two axis neighbors have distinct opposite nonzero heights. In this case the incidence count improves to

$$q(z) \leq \ell - 1. \quad (6)$$

Lemma 7. *If $a \leq 1$ and $h, -h \in L$ with $h > 0$, then $h = \cot \theta$ for some $\theta/\pi \in \mathbb{Q}$.*

Proof. The functional graph sending w to $p(w)$ when the latter lies in L has exactly a vertices with no image, called roots. Write $h = \cot \theta$, $0 < \theta < \pi/2$. If either of the orbits starting at $h, -h$ reaches a cycle, angle doubling makes θ/π rational. Otherwise both orbits terminate. This is impossible when $a = 0$; when $a = 1$ they reach the same root after, say, i and j steps. Thus $2^i \theta \equiv -2^j \theta \pmod{\pi}$, which again proves rationality. An orbit reaching zero stops; p is never applied to zero. $\square$

The lemma concerns paired heights. It does not make every axis angle rational when $a = 1$: a separate rooted component can have irrational angles.

Use a homomorphism $\eta : \Gamma \rightarrow \mathbb{R}$ with torsion kernel as in Lemma 3. A maximum or minimum fiber of η on C is called an *extreme fiber*. At a point in such a fiber with two circle neighbors, the neighbors lie in that same fiber, and their angular increments are rational. Call a circle point *good* if it has both two circle neighbors and two axis neighbors. If $a \leq 1$, a good point in an extreme fiber has radius $\sqrt{2}$ and axis neighbors ± 1 , by Lemmas 7 and 5. Its circle neighbors are its quarter-turn translates.

Lemma 8. *If $a = 0$, then $d = 0$ implies $b \geq 12$, and $d = 1$ implies $b \geq 6$.*

Proof. By Lemma 6, every axis angle is rational and $0, \pm 1$ are absent. An extreme-fiber point with two circle neighbors has no axis neighbor at all. Indeed, any nonzero height neighbor would satisfy the rational sine-product relation with its rational chord angle, forcing absolute height one by Lemma 5; zero is excluded separately. Such a point contributes two to b .

If $d = 0$, an extreme fiber has at least three points, all with two circle neighbors. When η is nonconstant, its two disjoint extreme fibers give $b \geq 12$; when it is constant, $b = 2c \geq 16$. If $d = 1$, precisely one point lacks one circle neighbor. In the nonconstant case an extreme fiber avoiding that point has at least three points, so $b \geq 6$. In the constant case all other $c - 1$ points contribute two each, also giving the stated bound. $\square$

Lemma 9. *If $D = 1$ and $1 \leq t \leq 2$, then $e(S) \leq B(n) - 2$.*

Proof. Lemma 8 forces $a = 1$ and $b = d = 0$. Every extreme-fiber point is good, so its quarter-turn neighbors are in that fiber and good as well. Each extreme fiber therefore contains a full square of radius- $\sqrt{2}$ vertices. If η is nonconstant there are at least eight such vertices in two disjoint extreme fibers. The single sphere of radius $\sqrt{2}$ centered at a genuine exception meets the entire circle at most twice, hence $q(z) \leq c - 6$. If η is constant, every circle radius is $\sqrt{2}$, so $q(z) \leq 2 \leq c - 6$.

By (5),

$$\begin{aligned} e(S) &\leq \ell c + \ell + (4 + t)c + t^2 - 2t - 1 \\ &= (\ell + 4 + t)(c + 1) + t^2 - 3t - 5 \\ &\leq B(n) + t(t - 3) = B(n) - 2. \end{aligned}$$

$\square$

6 One exceptional point

Theorem 10. *If $c \geq 8$ and $t = 1$, then $e(S) \leq B(n)$. Equality requires*

$$(a, b, d) = (2, 0, 0) \quad \text{or} \quad (0, 0, 2). \quad (7)$$

It also requires $|c - \ell - 2| \leq 1$, a symmetric axis set L excluding zero, and, for the exception z ,

$$q(z) = \ell, \quad e(L, \{z\}) = 1, \quad e(\{z\}, L) = e(\{z\}, C) = 2.$$

Proof. Put $K = a + d + \lceil b/2 \rceil$. Equations (4) and (5) give

$$e(S) \leq (\ell + 4)(c + 2) - 3 - K \leq B(n) + 2 - K. \quad (8)$$

Thus $K \geq 3$ gives strict inequality. Lemma 8 rules out $a = 0$, $K \leq 2$ except for $(a, b, d) = (0, 0, 2)$. For $a \geq 2$ only $(2, 0, 0)$ remains. With $a = 1$, the possibilities are $(b, d) = (0, 0), (1, 0), (2, 0), (0, 1)$. The first is excluded by Lemma 9.

Suppose $a = 1, d = 0, b \in \{1, 2\}$. At most b circle points are not good. Every extreme fiber has at least four points. A closed three-point fiber would be an equilateral triangle, with all radii $\sqrt{3}$, and so would have no good point; this contradicts $b \leq 2$. Smaller fibers cannot give every point two distinct other neighbors. If η is nonconstant, the two extreme fibers contain at least $8 - b \geq 6$ good vertices, all of radius $\sqrt{2}$; hence $q(z) \leq c - 4$. If η is constant, at least $c - b$ vertices have that radius, so $q(z) \leq b + 2 \leq 4 \leq c - 4$. Therefore

$$e(S) \leq \ell c + \ell + 5c - b = (\ell + 5)(c + 1) - 5 - b \leq B(n) - b. \quad (9)$$

Now suppose $a = 1, d = 1, b = 0$. There is a good vertex in an extreme fiber: use the fiber avoiding the unique circle-deficient vertex if η is nonconstant, and any other vertex if it is constant. This supplies axis heights ± 1 . If zero were absent, both would be roots, contradicting $a = 1$. Equation (6) applies, giving

$$e(S) \leq (\ell + 4)(c + 2) - 6 \leq B(n) - 1.$$

Only the profiles in (7) remain; they have $K = 2$, and (8) proves the bound. Equality requires every incident-arc bound to be tight and the two product factors $\ell + 4, c + 2$ to differ by at most one. This gives the listed counts and balance condition. Since $b = 0$, the $q(z) = \ell$ circle points use exactly 2ℓ axis incidences. Each height permits at most two, so every height is used exactly twice. Zero cannot be used, and a used nonzero height comes with its negative in L . This proves the remaining necessary conditions, not their attainability. $\square$

7 Two exceptional points

Theorem 11. *If $c \geq 8$ and $t = 2$, then $e(S) \leq B(n) + 3$. If the exceptions are not reflected in the circle plane and $n \geq 23$, then $e(S) < B(n)$.*

Proof. Applying (4) twice in (5) gives

$$e(S) \leq (\ell + 4)(c + 3) - a - d - (b \bmod 2) \leq B(n) + 5 - a - d - (b \bmod 2). \quad (10)$$

If $a + d \geq 2$, this is enough. To handle $a + d \leq 1$, observe that, for integers $s \geq 0$ and $b \geq 2s$,

$$-b + 2 \min\{c, \ell + \lfloor b/2 \rfloor\} \leq 2 \min\{c - s, \ell\}. \quad (11)$$

For even $b = 2j$ the left side is $2 \min\{c-j, \ell\}$; an odd b gives one less than the preceding even value. Put $k = c - \ell$.

If $a = d = 0$, Lemma 8 gives $b \geq 12$. Use $s = 6$ in (11). For $k \geq 6$ the resulting upper bound is $(\ell + 4)(c + 3)$, whose factors differ by $k - 1 \geq 5$, so it is at least six below their integer product maximum and at most $B(n) - 1$. For $k \leq 5$ the bound is $(\ell + 6)(c + 1) - 6 \leq B(n) - 1$.

If $a = 0, d = 1$, then $b \geq 6$. Use $s = 3$. For $k \geq 3$ the bound is $(\ell + 4)(c + 3) - 1$; the factors differ by at least two, giving at most $B(n) + 3$. For $k \leq 2$ it is $(\ell + 6)(c + 1) - 1$; the factors differ by at least three, giving at most $B(n) + 2$.

It remains to treat $a = 1, d = 0$. If $b = 0$, use Lemma 9. If $b = 1$ or 2 , the good-vertex count in the proof of Theorem 10 gives $q(z) \leq c - 4$ for each exception, so

$$e(S) \leq (\ell + 6)(c + 1) - 3 - b \leq B(n) + 2 - b.$$

Every odd $b \geq 3$ gives $B(n) + 3$ directly in (10); every even $b \geq 6$ is handled by $s = 3$ in (11), just as in the preceding paragraph.

For $b = 4$, there are at least three radius- $\sqrt{2}$ vertices in the extreme fibers. To see this, suppose first that η is nonconstant. Each fiber containing a good vertex has at least four points, since a three-point closed fiber is equilateral. If both extreme fibers contain good vertices, at least $8 - 4 = 4$ of their vertices are good. If one has only bad vertices, it uses at least three of the at most four bad vertices, leaving at least three good vertices in the other fiber. Both fibers cannot consist only of bad vertices. If η is constant, at least $c - 4 \geq 4$ vertices are good. Thus $q(z) \leq c - 1$, and (4) also gives $q(z) \leq \ell + 2$. Consequently

$$e(S) \leq \ell c + \ell + 4c + 7 + 2 \min\{c - 1, \ell + 2\}.$$

For $k \geq 3$ this is at most $(\ell + 4)(c + 3) - 1 \leq B(n) + 3$; for $k \leq 2$ it is at most $(\ell + 6)(c + 1) - 1 \leq B(n) + 2$. This proves the uniform bound.

For the geometric assertion write the exceptions as $z_i = (u_i, h_i)$ with $u_i \in \mathbb{R}^2$. A circle point x reaching both satisfies

$$2x \cdot (u_1 - u_2) = |z_1|^2 - |z_2|^2.$$

If $u_1 \neq u_2$, this is a line meeting the circle at most twice. If $u_1 = u_2$ and the squared norms differ, there are no such points. The remaining possibility, equal projections and equal norms for two distinct points, is exactly $h_1 = -h_2 \neq 0$. Thus a non-reflected pair has at most $c + 2$ incoming arcs from C . Using (5),

$$e(S) \leq (\ell + 5)(c + 1) + 9 - D \leq \left\lfloor \frac{(n + 4)^2}{4} \right\rfloor + 9 - D.$$

The difference from $B(n)$ is at least

$$\left\lfloor \frac{n + 5}{2} \right\rfloor - 14 + D \geq \left\lfloor \frac{n + 5}{2} \right\rfloor - 13 > 0 \quad (n \geq 23).$$

□

Proof of Theorem 2. For a sufficiently large extremizer, Swanepoel's Theorem B [1] supplies a suspension core after deleting at most two points. Its circle part has $c \geq 8$ for sufficiently large n : if $c \leq 7$, the core has at most $8\ell + 28$ arcs, and the omitted points add at most $4(n - 1)$, contradicting the quadratic lower bound $B(n)$.

Absorb any omitted circle point into the core, and any omitted axis point whose radius is its distance to the circle. An omitted axis point with a different radius reaches no core circle point, at most two core axis points and at most one other omitted point. Changing only its radius gives at least eight outgoing arcs instead of at most three and leaves its incoming arcs unchanged, contrary to extremality. All remaining exceptions are therefore genuine.

Theorem 1 handles no exceptions, Theorem 10 handles one, and Theorem 11 handles two. Swanepoel’s construction gives the lower bound. The reflection assertion follows from Theorem 11 and the lower bound. $\square$

8 Scope and reproducibility

The exact unrestricted value of $f_3(n)$ is not determined here. The one-exception equality profiles in (7) and the two-reflected-exception case with $D \geq 2$ remain unresolved. Even an exact formula $f_3(n) = B(n)$ would not by itself rule out nonsuspension extremizers tying the bound. The conjecture that all sufficiently large extremizers are suspensions remains open.

The accompanying standard-library Python checker uses exact cyclotomic polynomial remainders, rational angles and algebraic coordinates. It tests the sine-product classification on 144008 rational-angle pairs, the finite-group maximum principle on small configurations, the axis dynamics, the sharpness construction and the integer rounding. Normal and optimized runs agree on all 169675 diagnostic checks. A second checker agrees in both modes on 3955936 exact diagnostics for the exceptional-point bounds, integer deficit profiles, finite free-plus-torsion models and rational circle incidences. These finite tests supplement the written proof; they are not a proof of its universal statements.

AI tools assisted the mathematical development, literature search, writing and exact calculations. This preprint is self-audited and unrefereed; it has not undergone independent review or formal verification. The credited construction and classical inputs are not claimed as new, and no absolute priority claim is made.

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