

Blocking Shortest Paths between Three Terminals Is NP-Hard: A Reduction from Three-Terminal Cut

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Abstract

In the problem *Blocking Shortest Paths* one is given an undirected graph with edge costs and some pairs of vertices, and asks for an edge set of minimum cost whose deletion increases the distance of each pair by at least 1; that is, the set must meet every shortest path between the two vertices of each pair. Krause showed that the problem can be solved in polynomial time when all pairs have a vertex in common and that it is NP-hard for two disjoint pairs, and left open the remaining case of three pairs: the three pairs of three given vertices (Oberwolfach Report 50/2005; thesis, Braunschweig 2006). We show that this case is NP-hard, with edge costs and with unit costs. The proof is a short reduction from the minimum cut problem with three terminals: from a graph H with edge set D we build a bipartite graph with three terminals in which the minimum cost of a blocking set is $2w(D)$ plus the minimum weight of a three-terminal cut of H , for every positive integral weight w on D . So the question of record OWR-1111-001 of the UnsolvedMath collection, to determine the complexity of the problem for three vertices, with or without edge costs, has the answer: NP-hard in both cases, and the decision problem is NP-complete. The construction was tested on all connected graphs H with at most six vertices. The hardness of the three-terminal cut problem is the theorem of Dahlhaus, Johnson, Papadimitriou, Seymour and Yannakakis. No proof or statement of the result was found in the literature accessible to us; a search that finds nothing is not a proof of novelty. This is an unrefereed note.

1 Introduction

1.1 Blocking shortest paths

All graphs are finite, simple and undirected. For a graph $G = (V, E)$ and $a, b \in V$ we write $\text{dist}_G(a, b)$ for the number of edges of a shortest a - b path of G , and $\text{dist}_G(a, b) = \infty$ if there is no such path. For $F \subseteq E$ the graph $G - F$ has the vertex set V and the edge set $E \setminus F$.

Definition 1.1. An instance of BLOCKING SHORTEST PATHS (BSP) consists of a graph $G = (V, E)$, a cost $c_e \in \mathbb{Z}_{>0}$ for every edge e , and pairs $(s_1, t_1), \dots, (s_k, t_k)$ of distinct vertices with $\text{dist}_G(s_i, t_i) < \infty$. A set $F \subseteq E$ is *blocking* if every shortest s_i - t_i path of G contains an edge of F , for $i = 1, \dots, k$; equivalently, if $\text{dist}_{G-F}(s_i, t_i) \geq \text{dist}_G(s_i, t_i) + 1$ for all i . The task is to find a blocking set of minimum cost $c(F) = \sum_{e \in F} c_e$; this minimum is denoted by $\text{opt}(G)$, the costs and the pairs being understood. The instance has *unit costs* if $c_e = 1$ for all e . In the decision version an integer B is given and one asks whether $\text{opt}(G) \leq B$.

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The two conditions are equivalent because a path of $G - F$ is a path of G , so that the distance grows exactly when no shortest $s_i - t_i$ path of G survives. The distance is the number of edges; the costs play no role for it.

According to [Kra05, Kra06] the problem was proposed by Bienstock and Diaz [BD93]. It was studied by Krause in his thesis [Kra06], of which [Kra05] is a summary. For one pair, and more generally when all pairs have a vertex in common (*star instances*), a minimum blocking set is a minimum cut in the digraph of the shortest paths from the common vertex, and the problem is solved in polynomial time [Kra06, Theorem 4.1]. For two disjoint pairs the problem is NP-hard, even for grid graphs of maximum degree 3 and for planar bipartite graphs with unit costs [Kra06, Theorem 4.2, Corollaries 4.1 and 4.2]. A set of pairs in which no two pairs are disjoint is a star or consists of the three pairs of three vertices. So one configuration is left [Kra06, Section 4.3]:

Definition 1.2. A *triangle instance* of BSP is an instance with three pairs (r_1, r_2) , (r_1, r_3) , (r_2, r_3) , where r_1, r_2, r_3 are three distinct vertices, the *terminals*.

The first of the two open problems at the end of [Kra05] asks for the complexity of these instances: given an undirected graph with or without edge costs, to find a minimum (cost) edge set whose deletion increases the distance of any two of three given vertices by at least 1. In the thesis the section on triangle instances gives some remarks and says that the complexity is not determined there [Kra06, Section 4.3]. Record OWR-1111-001 of the UnsolvedMath collection [ula26] is this problem; it is listed there as open.

1.2 Result

Theorem 1.3. *The decision version of BLOCKING SHORTEST PATHS is NP-complete for triangle instances. More precisely, it is NP-complete for each of the following two classes of triangle instances.*

- (a) *Bipartite graphs with edge costs in which the three terminals are pairwise at distance 4.*
- (b) *Bipartite graphs with unit costs in which the three terminals are pairwise at distance 6.*

Class (b) is contained in the class of all triangle instances with edge costs, so (b) alone gives the hardness of both versions of the problem. We state (a) separately because its proof is the simplest form of the construction and because it uses less of the theorem on multiway cuts (Section 1.3).

The theorem follows from a reduction from the minimum cut problem with three terminals. For a graph $H = (U, D)$ with positive integer weights w_e on the edges and three distinct vertices a_1, a_2, a_3 , a *three-terminal cut* is a set $C \subseteq D$ such that no component of $H - C$ contains two of the vertices a_1, a_2, a_3 ; we write $\text{mwc}(H)$ for the minimum weight $w(C) = \sum_{e \in C} w_e$ of a three-terminal cut, the weights and the terminals being understood.

Theorem 1.4. *Let $H = (U, D)$ be a graph with at least one edge, with weights $w_e \in \mathbb{Z}_{>0}$ and with three distinct vertices a_1, a_2, a_3 .*

- (a) *The triangle instance $\Gamma(H)$ of Section 3 satisfies $\text{opt}(\Gamma(H)) = 2w(D) + \text{mwc}(H)$.*
- (b) *If $w_e = 1$ for all e , the triangle instance $\Gamma'(H)$ of Section 4, which has unit costs, satisfies $\text{opt}(\Gamma'(H)) = 2|D| + \text{mwc}(H)$.*

The graph $\Gamma(H)$ has $3|D| + |U| + 6$ vertices and $9|D| + 6$ edges, and $\Gamma'(H)$ has $O(|D|^2 + |U|)$ vertices and edges; both are constructed in polynomial time. From a blocking set of cost $2w(D) + k$ one obtains in polynomial time a three-terminal cut of H of weight at most k .

The idea is as follows. Each terminal r_i of $\Gamma(H)$ is joined to every vertex of H by paths of length 2, one path through a private middle vertex for every edge of H at that vertex. A shortest path between two terminals r_i and r_j then consists of two such paths which meet in a vertex u of H . A blocking set must therefore leave every vertex of H reachable from at most one terminal, which defines a labelling of the vertices of H by the terminals; and a middle vertex which belongs to r_i and to an edge uv of H can keep its edge to r_i only if both u and v have the label i . So an edge of H costs twice its weight if its ends have the same label and three times its weight otherwise. The reduction does not imitate the proof of [DJP⁺94]; it uses the cut problem itself, in the form of labellings (Lemma 2.1).

1.3 Conventions that could change the answer

Lengths and costs. As in [Kra05, Kra06] all edges have length 1: the distance is the number of edges, and the costs are used only to measure the deleted set. Costs are positive integers; an instance with positive rational costs is brought to this form by scaling.

“At least 1”. The distance of a pair may grow by more than 1, and the pair may become disconnected. The problem in which each distance must grow by exactly 1 is a different one [Kra06, Section 9.2] and is not treated here.

“With or without edge costs”. Both versions are covered: (b) of Theorem 1.3 is the version without costs.

The complexity. “NP-hard” refers to the optimisation problem and “NP-complete” to the decision version of Definition 1.1. The theorem is about arbitrary graphs. For planar graphs the cut problem with three terminals is solvable in polynomial time [DJP⁺92, Theorem 1], [DJP⁺94], so our reduction gives nothing for planar triangle instances; their complexity is not determined here (Remark 5.3).

What is used from the literature. Part (a) uses that the three-terminal cut problem with weights is NP-hard, and part (b) that it is NP-hard for unit weights. Both are due to Dahlhaus, Johnson, Papadimitriou, Seymour and Yannakakis: Theorem 3 of the extended abstract [DJP⁺92] states that the multiway cut problem for arbitrary graphs is NP-hard for every fixed $k \geq 3$ with all edge weights equal to 1, and its Section 3 notes that the case $k = 3$ suffices. We read the extended abstract; the full paper [DJP⁺94] was not accessible to us (Section 6).

1.4 What was known

The results of Krause on BSP were recalled in Section 1.1. For triangle instances [Kra06, Section 4.3] observes that the union of an optimal solution for one pair and an optimal solution for the star formed by the other two pairs is a feasible solution whose cost is at most twice the optimum and can be that bad (see also [Kra06, Theorem 7.1]); that two natural set functions which can be evaluated in polynomial time, and whose minimisation would solve triangle instances, are not submodular; and that the proof of Dahlhaus et al. for the multiway cut problem with three terminals could not be adapted, because that proof needs that all paths between two terminals are cut. The summary of the thesis in zbMATH (Zbl 1238.05002) says that the triangle case is the only case of the chapter on complexity whose status stays open. For the directed version of the problem, triangle instances are NP-hard [Kra06, Theorem 9.3]; the proof derives this from the case of two disjoint pairs by adding a new vertex and two directed paths, and in an undirected graph two pairs with a common vertex form a star instance. The concluding section of [DJP⁺92] observes that the multicut problem for the three pairs of three terminals, in which all paths are to be cut, is the three-terminal cut problem and hence NP-hard; for BSP, where only the shortest paths are to be cut, this gives nothing directly, and a reduction of its own is needed.

We found no later work on triangle instances and no work that cites the thesis; see “Scope and priority” at the end of Section 6 for the searches. New, to our knowledge and relative to the literature

we found, are Theorems 1.3 and 1.4.

2 Three-terminal cuts as labellings

Let $H = (U, D)$ be a graph with weights $w_e \in \mathbb{Z}_{>0}$ and three distinct vertices a_1, a_2, a_3 . A *labelling* is a map $L: U \rightarrow \{1, 2, 3\}$ with $L(a_i) = i$ for $i = 1, 2, 3$. For a labelling L let $\delta(L)$ be the set of the edges of H whose two ends have different labels.

Lemma 2.1. $\text{mwc}(H) = \min_L w(\delta(L))$, the minimum being taken over all labellings. More precisely, $\delta(L)$ is a three-terminal cut for every labelling L , and every three-terminal cut contains $\delta(L)$ for some labelling L .

Proof. Let L be a labelling. Along a path of $H - \delta(L)$ the label does not change, so a component of $H - \delta(L)$ contains at most one of a_1, a_2, a_3 . Conversely let C be a three-terminal cut. Put $L(u) = i$ if u lies in the component of $H - C$ which contains a_i , and $L(u) = 1$ if u lies in a component without terminal. This is a labelling, and it is constant on every component of $H - C$; hence an edge of H whose ends have different labels belongs to C . $\square$

Theorem 2.2 (Dahlhaus, Johnson, Papadimitriou, Seymour, Yannakakis [DJP⁺92, Theorem 3], [DJP⁺94]). *It is NP-complete to decide, for a graph H with positive integer weights on the edges, three distinct vertices of H and an integer $k \geq 0$, whether $\text{mwc}(H) \leq k$. This remains true when all weights are equal to 1.*

The theorem is stated in [DJP⁺92] for every fixed number of terminals $k \geq 3$ and unit weights (Theorem 3 there); the weighted version follows. According to the subsection “Near-Optimal Multiway Cuts” of [DJP⁺92] (pp. 249–250) the proof also shows that the problem with three terminals is MAX SNP-hard (Theorem 5 there). The decision problem is in NP. The extended abstract gives no details of the unit-weight instances of its Theorem 3; if they have parallel edges, subdividing every parallel edge once leaves mwc unchanged, so that H may be taken simple (alternatively, Remark 5.1 applies with w_e the multiplicity of e).

3 The construction with edge costs

Let $H = (U, D)$, w and a_1, a_2, a_3 be as in Theorem 1.4. Put $W = w(D)$ and $M = 3W + 1$. The graph $\Gamma = \Gamma(H)$ is defined as follows (Figure 1).

- Its vertices are three new vertices r_1, r_2, r_3 , the *terminals*; the vertices of H ; a vertex $x_{i,e}$ for every $i \in \{1, 2, 3\}$ and every $e \in D$, the *hubs*; and three vertices p_1, p_2, p_3 .
- For $i \in \{1, 2, 3\}$ and $e \in D$ the edge $r_i x_{i,e}$ is an edge of Γ , with cost w_e . These $3|D|$ edges are the *light* edges.
- For $i \in \{1, 2, 3\}$, $e \in D$ and each of the two ends u of e , the edge $x_{i,e}u$ is an edge of Γ ; and for $i \in \{1, 2, 3\}$ the edges $r_i p_i$ and $p_i a_i$ are edges of Γ . These $6|D| + 6$ edges are the *heavy* edges; each of them has the cost M .

No edge of H is an edge of Γ . The pairs are (r_1, r_2) , (r_1, r_3) , (r_2, r_3) . We write $A_i = \{x_{i,e} : e \in D\} \cup \{p_i\}$, the *arm* of r_i .

Lemma 3.1. (a) Γ is bipartite, with the classes $\{r_1, r_2, r_3\} \cup U$ and $A_1 \cup A_2 \cup A_3$.

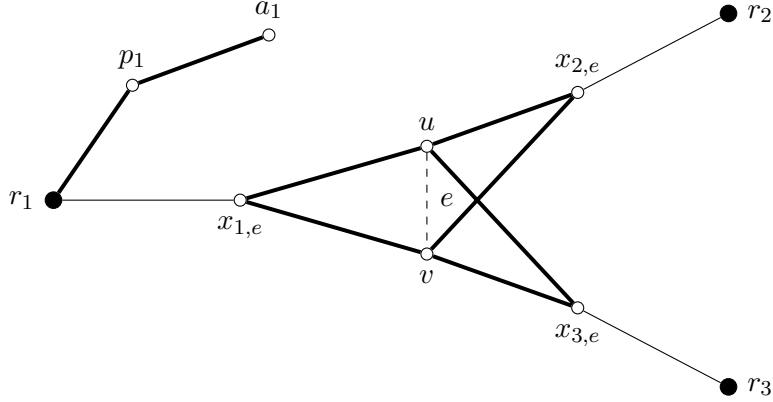


Figure 1: The part of $\Gamma(H)$ that belongs to one edge $e = uv$ of H (dashed; it is not an edge of Γ), and the two edges that join r_1 to the terminal a_1 of H . Thin edges are light (cost w_e), thick edges are heavy (cost M).

- (b) *The set of neighbours of r_i is A_i , and every vertex of A_i has all its neighbours other than r_i in U .*
- (c) *Let $i \neq j$. Then $\text{dist}_\Gamma(r_i, r_j) = 4$, and the shortest r_i - r_j paths are exactly the paths (r_i, y, u, y', r_j) with $u \in U$, $y \in A_i$, $y' \in A_j$, where y and y' are adjacent to u .*

Proof. (a) and (b) are read off from the definition. For (c) let P be an r_i - r_j path. By (b) its second vertex y lies in A_i and its last vertex before r_j , say y' , lies in A_j . The sets A_i and A_j are disjoint, and by (b) the vertex y is not adjacent to y' . So P has at least 4 edges, and if it has exactly 4, its middle vertex u is a common neighbour of y and y' other than r_i , hence $u \in U$ by (b). Every sequence of this form is a path. There is one: H has an edge e , and for an end u of e the sequence $(r_i, x_{i,e}, u, x_{j,e}, r_j)$ is a path of Γ . $\square$

For a labelling L of H put

$$F_L = \{ r_i x_{i,e} : i \in \{1, 2, 3\}, e = uv \in D, \text{ not } L(u) = L(v) = i \}.$$

Proposition 3.2. (a) *For every labelling L the set F_L is a blocking set of Γ , and $c(F_L) = 2W + w(\delta(L))$.*

- (b) *Let F be a blocking set of Γ which contains no heavy edge. Then there is a labelling L with $F_L \subseteq F$.*
- (c) $\text{opt}(\Gamma) = 2W + \text{mwc}(H)$.

Proof. (a) Let $e = uv \in D$. If $L(u) = L(v)$, the set F_L contains the edge $r_i x_{i,e}$ for the two indices $i \neq L(u)$; otherwise it contains it for all three indices. So $c(F_L) = \sum_{e \notin \delta(L)} 2w_e + \sum_{e \in \delta(L)} 3w_e = 2W + w(\delta(L))$. Suppose that a shortest r_i - r_j path contains no edge of F_L . By Lemma 3.1(c) it has the form (r_i, y, u, y', r_j) . If $y = x_{i,e}$, then u is an end of e and, since $r_i x_{i,e} \notin F_L$, both ends of e have the label i ; if $y = p_i$, then $u = a_i$. In both cases $L(u) = i$. In the same way $L(u) = j$, a contradiction.

(b) For $i \in \{1, 2, 3\}$ let U_i be the set which consists of a_i and of the ends of all edges $e \in D$ with $r_i x_{i,e} \notin F$. For every $u \in U_i$ there is a path (r_i, y, u) with $y \in A_i$ which contains no edge of F : for $u = a_i$ take $y = p_i$, and if u is an end of e with $r_i x_{i,e} \notin F$ take $y = x_{i,e}$; the other edges of these paths are heavy. If some u belonged to $U_i \cap U_j$ with $i \neq j$, the two paths (r_i, y, u) and (r_j, y', u) would give a

path (r_i, y, u, y', r_j) without edge of F , which is a shortest r_i-r_j path by Lemma 3.1(c). So U_1, U_2, U_3 are pairwise disjoint. Put $L(u) = i$ for $u \in U_i$ and $L(u) = 1$ for the vertices of U in none of the three sets. Then L is a labelling, because $a_i \in U_i$. If $e = uv \in D$ and $r_i x_{i,e} \notin F$, then $u, v \in U_i$ and so $L(u) = L(v) = i$. Hence $F_L \subseteq F$.

(c) By (a) and Lemma 2.1, $\text{opt}(\Gamma) \leq 2W + \text{mwc}(H) \leq 3W < M$. So a blocking set F of minimum cost contains no heavy edge, and by (b), (a) and Lemma 2.1 we get $c(F) \geq c(F_L) = 2W + w(\delta(L)) \geq 2W + \text{mwc}(H)$. $\square$

The proof of (b) is constructive: from a blocking set F without heavy edges, of cost $2W + k$, it gives a labelling L with $w(\delta(L)) \leq k$, and $\delta(L)$ is a three-terminal cut by Lemma 2.1. A blocking set of cost $2W + k$ contains no heavy edge when $2W + k < M$, that is, when $k \leq W$; if $k > W$, the edge set D itself is a three-terminal cut of weight $W \leq k$. This proves Theorem 1.4(a) and the last sentence of that theorem.

4 Unit costs

Now let $w_e = 1$ for all $e \in D$, and put $K = 3|D| + 1$. A *bundle of length ℓ* between two vertices y and z consists of K paths with ℓ edges from y to z whose inner vertices are new vertices, all distinct. The graph $\Gamma' = \Gamma'(H)$ has the terminals r_1, r_2, r_3 , the vertices of H , the hubs $x_{i,e}$ and the inner vertices of the following bundles, and its edges are:

- the *light* edges $r_i x_{i,e}$ for $i \in \{1, 2, 3\}$ and $e \in D$;
- a bundle of length 2 between $x_{i,e}$ and u , for $i \in \{1, 2, 3\}$, $e \in D$ and each end u of e ;
- a bundle of length 3 between r_i and a_i , for $i \in \{1, 2, 3\}$.

All edges have the cost 1. The pairs are again $(r_1, r_2), (r_1, r_3), (r_2, r_3)$. So Γ' arises from Γ when every heavy edge at a hub is replaced by a bundle of length 2 and the two heavy edges at p_i by a bundle of length 3. Let B_i be the set of the hubs $x_{i,e}$, of the inner vertices of the bundles at these hubs and of the inner vertices of the bundle between r_i and a_i .

Lemma 4.1. (a) Γ' is bipartite.

- (b) *Let $u \in U$. Every r_i-u path of Γ' has at least 3 edges. The r_i-u paths with exactly 3 edges are the paths which consist of a light edge $r_i x_{i,e}$, where u is an end of e , and of a path of the bundle between $x_{i,e}$ and u ; and, if $u = a_i$, the paths of the bundle between r_i and a_i .*
- (c) *Let $i \neq j$. Then $\text{dist}_{\Gamma'}(r_i, r_j) = 6$, and the shortest r_i-r_j paths are exactly the paths which consist of an r_i-u path with 3 edges and of a $u-r_j$ path with 3 edges, for some $u \in U$.*

Proof. The vertex set of Γ' is the disjoint union of U and the three sets $B_i \cup \{r_i\}$. No edge joins $B_i \cup \{r_i\}$ to $B_j \cup \{r_j\}$ for $i \neq j$, and no edge has both ends in U . Give the vertices of $B_i \cup \{r_i\}$ a level: r_i has level 0; the hubs $x_{i,e}$ and the neighbours of r_i on the bundle between r_i and a_i have level 1; the inner vertices of the bundles at the hubs and the neighbours of a_i on the bundle between r_i and a_i have level 2. Every edge inside $B_i \cup \{r_i\}$ joins vertices whose levels differ by 1, and a vertex of $B_i \cup \{r_i\}$ which has a neighbour in U has level 2.

(a) The vertices of level 1 in the three sets and the vertices of U form one class, the remaining vertices the other.

(b) Let P be an r_i-u path and u_0 its first vertex in U . The part of P from r_i to u_0 runs inside $B_i \cup \{r_i\}$ up to its last edge, so it passes through vertices of levels 0, 1, $\dots$ and reaches level 2 before

u_0 ; it has at least 3 edges, and exactly 3 only if its vertices have the levels 0, 1, 2 in this order. The paths with this property are those of the statement, with u_0 in place of u . If $u_0 \neq u$, then P has more than 3 edges.

(c) An r_i – r_j path P meets U . Let u be its first and u' its last vertex in U . By (b), applied to r_i and to r_j , the part of P from r_i to u and the part from u' to r_j have at least 3 edges each. So P has at least 6 edges, with equality only if $u = u'$ and both parts have 3 edges. Conversely an r_i – u path with 3 edges and a u – r_j path with 3 edges have only u in common, since their other vertices lie in $B_i \cup \{r_i\}$ and in $B_j \cup \{r_j\}$; so together they form a path with 6 edges. Such paths exist, as in the proof of Lemma 3.1. $\square$

The set F_L of Section 3 consists of light edges, so it is also a set of edges of Γ' .

Proposition 4.2. (a) *For every labelling L the set F_L is a blocking set of Γ' , and $|F_L| = 2|D| + |\delta(L)|$.*

(b) *Let F be a blocking set of Γ' with $|F| < K$. Then there is a labelling L with $F_L \subseteq F$.*

(c) $\text{opt}(\Gamma') = 2|D| + \text{mwc}(H)$.

Proof. (a) The count is that of Proposition 3.2(a). Suppose that a shortest r_i – r_j path contains no edge of F_L , and let u be its vertex in U (Lemma 4.1(c)). By Lemma 4.1(b) its part from r_i to u begins with a light edge $r_i x_{i,e} \notin F_L$, where u is an end of e , or $u = a_i$; in both cases $L(u) = i$. In the same way $L(u) = j$.

(b) The K paths of a bundle are pairwise edge-disjoint. As $|F| < K$, every bundle contains a path without edge of F . Define U_i as in the proof of Proposition 3.2(b). For every $u \in U_i$ there is an r_i – u path with 3 edges and without edge of F : a path of the bundle between r_i and a_i if $u = a_i$, and otherwise a light edge $r_i x_{i,e} \notin F$ followed by a path of the bundle between $x_{i,e}$ and u . If some u belonged to $U_i \cap U_j$ with $i \neq j$, two such paths would form a shortest r_i – r_j path without edge of F , by Lemma 4.1(c). The rest is as in the proof of Proposition 3.2(b).

(c) By (a), $\text{opt}(\Gamma') \leq 2|D| + \text{mwc}(H) \leq 3|D| < K$, and by (b) and (a) a blocking set F of minimum size satisfies $|F| \geq |F_L| = 2|D| + |\delta(L)| \geq 2|D| + \text{mwc}(H)$. $\square$

This proves Theorem 1.4(b); the last sentence of that theorem holds for Γ' as well, since a blocking set of size $2|D| + k$ has fewer than K edges when $k \leq |D|$, and D is a cut of size $|D| \leq k$ otherwise. The graph Γ' has $3|D|$ light edges, $6|D|$ bundles of length 2 and three bundles of length 3, hence $3|D| + 12|D|K + 9K$ edges.

Proof of Theorem 1.3. The decision version of BSP is in NP: for a given set F the distances in G and in $G - F$ are found by breadth-first search. Let (H, w, a_1, a_2, a_3, k) be an instance of the problem of Theorem 2.2. If H has no edge, then $\text{mwc}(H) = 0 \leq k$, and the instance is mapped to a fixed instance of the class with the answer yes, for instance $\Gamma(P_3)$ with the bound $3w(D)$, where P_3 is the path with three vertices, all three terminals, and unit weights. Otherwise, by Theorem 1.4(a), $\text{mwc}(H) \leq k$ if and only if $\text{opt}(\Gamma(H)) \leq 2w(D) + k$; the graph $\Gamma(H)$ and its costs are computed in polynomial time, and $\Gamma(H)$ is bipartite with the terminals at distance 4 (Lemma 3.1). This proves (a). For unit weights, $\text{mwc}(H) \leq k$ if and only if $\text{opt}(\Gamma'(H)) \leq 2|D| + k$ by Theorem 1.4(b), and $\Gamma'(H)$ is bipartite with the terminals at distance 6 (Lemma 4.1); an edgeless H is mapped to $\Gamma'(P_3)$ with the bound $3|D|$. This proves (b). $\square$

5 Remarks

Remark 5.1 (weights and unit costs). For integer weights w_e one obtains a unit cost instance with optimum $2w(D) + \text{mwc}(H)$ by taking w_e hubs for every pair (i, e) , each with its light edge and its two bundles, and $K = 3w(D) + 1$. The proofs are the same. The size is polynomial in $w(D)$. A second way to unit costs is the replacement of every edge of cost c by c paths of length 2 between its ends, which Krause uses for two disjoint pairs [Kra06, proof of Corollary 4.1]; applied to $\Gamma(H)$ it gives, by the argument given there, bipartite unit cost instances of size polynomial in $|D|$ and $w(D)$, with the same optimum and with the terminals at distance 8.

Remark 5.2 (an edge serves at most two pairs). In a triangle instance no edge lies on shortest paths of all three pairs; this holds for arbitrary positive edge lengths ℓ . Indeed, let the edge ab lie on a shortest r_1 – r_2 path which passes a before b , so that $\text{dist}(r_1, b) = \text{dist}(r_1, a) + \ell_{ab}$ and $\text{dist}(a, r_2) = \ell_{ab} + \text{dist}(b, r_2)$. A shortest r_1 – r_3 path through ab passes a before b as well, by the first equation, and so $\text{dist}(a, r_3) = \ell_{ab} + \text{dist}(b, r_3)$. A shortest r_2 – r_3 path through ab passes b before a , by the second equation, and so $\text{dist}(b, r_3) = \ell_{ab} + \text{dist}(a, r_3)$. The last two equations contradict each other. Every edge of $\Gamma(H)$ lies on shortest paths of exactly two pairs: an edge at a vertex of A_i lies on shortest paths of the two pairs which contain r_i (an edge $x_{i,e}u$ lies on the path $(r_i, x_{i,e}, u, x_{j,e}, r_j)$, and $p_i a_i$ on $(r_i, p_i, a_i, p_j, r_j)$).

Remark 5.3 (what the reduction does not give). (i) *Planar graphs*. The graphs $\Gamma(H)$ are not planar in general, and the three-terminal cut problem is polynomial for planar graphs [DJP⁺92, Theorem 1], [DJP⁺94]. We do not know the complexity of triangle instances on planar graphs. For two disjoint pairs BSP is NP-hard on planar graphs [Kra06, Corollary 4.1].

(ii) *Approximation*. The reduction keeps the additive error: a blocking set of cost $\text{opt} + \varepsilon$ gives a three-terminal cut of weight at most $\text{mwc}(H) + \varepsilon$ (for $\text{mwc}(H) + \varepsilon \leq w(D)$ by Proposition 3.2(b), and trivially otherwise). Because of the term $2w(D)$ this alone does not transfer the MAX SNP-hardness of the three-terminal cut problem [DJP⁺92, Theorem 5] to a bound for the approximation ratio. We make no statement on the approximability of triangle instances; the union of the optimal solutions of two star instances is within a factor 2 of the optimum [Kra06, Section 4.3].

(iii) *Small distances*. In the instances of Theorem 1.3 the terminals are at distance 4 and 6. We did not examine triangle instances with smaller distances between the terminals.

6 Computations and reproducibility

The proofs of this note use no computation. The identities of Theorem 1.4 were in addition tested by programs, which are in the reproducibility package with their command lines and recorded outputs:

<code>bsp_core.py</code>	distances; the test whether an edge set is blocking; the optimum of a BSP instance by enumeration of edge sets; $\text{mwc}(H)$ by enumeration of labellings and, independently, from the definition by components
<code>bsp_milp.py</code>	the optimum of a BSP instance by an integer program with potentials, the formulation of [Kra06, Section 6.2] (needs SciPy); the edge set that is returned is tested with <code>bsp_core.py</code>
<code>reduction.py</code>	the graphs $\Gamma(H)$ and $\Gamma'(H)$, the sets F_L , the labelling of the proof of Proposition 3.2(b), and the replacement of Remark 5.1
<code>test_reduction.py</code>	the tests described below

The tests cover all connected graphs H with 3 to 6 vertices and all choices of three terminals: 2476 instances with unit weights, and the same instances with random weights in $\{1, 2, 3\}$. For each of them the distances of Lemma 3.1 were confirmed, the set F_L of an optimal labelling was confirmed to be blocking and of cost $2w(D) + \text{mwc}(H)$, the optimum of $\Gamma(H)$ found by the integer program was equal to $2w(D) + \text{mwc}(H)$, and the labelling obtained from the optimal solution as in the proof of Proposition 3.2(b) had weight $\text{mwc}(H)$. For the 222 unit weight instances in which H has at most five edges, the optimum of $\Gamma(H)$ was also found by enumeration of all sets of light edges, and the optima of $\Gamma'(H)$ and of the instance of Remark 5.1 with distance 8 were found by the integer program; all were equal to $2|D| + \text{mwc}(H)$. A further run covered the disconnected graphs H with 3 to 5 vertices and at least one edge, in which isolated vertices and isolated terminals occur: 822 instances for $\Gamma(H)$ and 127 for $\Gamma'(H)$, with the same result. The two methods for the optimum of a BSP instance were compared on 298 random triangle instances with at most seven vertices. With $K = 1$ in place of $K = 3|D| + 1$ the identity for Γ' fails already for H a triangle on the three terminals (8 instead of 9); this shows what the bundles are for. These are tests on small graphs. They are not part of the proofs.

Verification. The reduction was found and written with the programs of the table above. Two independent verification runs, both AI-assisted, followed; they did not see each other, and each wrote its own programs from the statements of the note. The first run compared Definitions 1.1 and 1.2 and the sentences of Sections 1.1 and 1.4 with the Oberwolfach abstract and the thesis, re-derived Lemmas 2.1, 3.1 and 4.1 and Propositions 3.2 and 4.2 and Remark 5.2 from scratch, examined the boundary cases (isolated vertices and isolated terminals of H , a single edge, adjacent terminals, heavy edges in a blocking set) and the two hand examples of a star and a triangle, and checked the identities of Theorem 1.4 with an integer program and by enumeration on further small cases. It found no gap in the proofs; it required the case $k > w(D)$ in the last sentence of Theorem 1.4, the treatment of an edgeless H in the proof of Theorem 1.3, and corrections of citations. The second run recomputed the identities with solvers of its own which do not use the formulation of `bsp_milp.py`: an exact hitting-set computation over all enumerated shortest paths, a branch and bound, and an integer program over paths. It covered all connected H with 3 to 7 vertices and all choices of three terminals (2476 + 29855 instances with unit weights and as many with random weights), disconnected and random instances, the unit cost graphs $\Gamma'(H)$ for $|D| \leq 7$ and the instances of Remark 5.1, and it confirmed that with $K = 1$ the identity for Γ' fails (54 failures) while with $K \geq 2$ it held in every case tested. It checked Remark 5.2 on 60 327 random weighted edges and proved it by hand, read the extended abstract [DJP⁺92] and the thesis against the note, verified the bibliography, and searched the literature; it required the citations by subsection and the remark on parallel edges after Theorem 2.2. Neither run found a mathematical error, and all corrections were made.

A third independent run, also AI-assisted, examined the final text. It re-derived every statement and proof from the definitions, including the boundary cases (isolated vertices or terminals, parallel edges, the threshold $k \leq w(D)$, the constant $K = 3|D| + 1$, the level argument of Lemma 4.1), and found one boundary case to repair, an edgeless H with a negative bound k , which led to the condition $k \geq 0$ in Theorem 2.2. It compared the sentences about the sources with the Oberwolfach abstract, the thesis, the extended abstract [DJP⁺92] and the record of the collection, and the bibliographic data with Crossref and DataCite. With a program written from the statements of the note alone (enumeration of all shortest paths and an exact hitting-set computation, cross-checked by a branch and bound and by an integer program over paths) it recomputed the identities of Theorem 1.4 for $\Gamma(H)$ on all 1416 instances with connected H on three to five vertices and all ordered choices of the terminals, each with unit weights and two random weightings, and on 822 instances with disconnected H , and for $\Gamma'(H)$ on 529 instances; all identities held, every optimal blocking set consisted of light

edges, and the labelling of Proposition 3.2(b) gave a three-terminal cut of the predicted weight. It reran `test_reduction.py` from a copy and reproduced the recorded output. Its programs and outputs are in the package (`independent_run_2/`).

Scope and priority. The result of this note is Theorem 1.3 with the identities of Theorem 1.4: triangle instances of BLOCKING SHORTEST PATHS are NP-hard, with edge costs and with unit costs. The problem, the notion of a triangle instance and the question are due to Krause [Kra05, Kra06]; the problem BSP goes back to Bienstock and Diaz [BD93] according to [Kra05, Kra06]. The hardness of the three-terminal cut problem is due to Dahlhaus, Johnson, Papadimitriou, Seymour and Yannakakis [DJP⁺94]; our proof is a reduction from that problem and contains no new hardness argument of its own. The replacement of an edge of cost c by c paths of length 2 in Remark 5.1 is taken from [Kra06]. New, to our knowledge and relative to the literature we found, is the answer for triangle instances. The complexity for planar graphs and the approximability are not settled (Remark 5.3). We claim no priority.

What was read: the abstract [Kra05] (pp. 2856–2858) in the publisher’s file; in [Kra06] Chapter 1, Sections 3.2 and 3.5, of Chapter 4 the Sections 4.1, 4.3, 4.4 and 4.5 and, in Section 4.2, the statements and the proofs of the two corollaries, Section 7.1, Section 9.1 and the beginning of Section 9.2, in the file of the German National Library; the summary of [Kra06] in zbMATH; in version 1 of the arXiv version of [vBBC⁺17] the proof of Theorem 8 up to the description of the construction; of [DJP⁺92] the introduction with the statements of Theorems 1 to 5, the beginning of Section 3, the subsection “Near-Optimal Multiway Cuts” (pp. 249–250) and the concluding section “Related Results and Open Problems” (pp. 250–251), in a copy of the publisher’s file. Not read: [DJP⁺94], the full version of [DJP⁺92], which was not accessible to us (its abstract was read in the record of Crossref); [BD93], which is cited as in [Kra05, Kra06]; the remaining parts of [Kra06], [DJP⁺92] and [vBBC⁺17].

The searches were made on 13 October 2026: queries to the arXiv, zbMATH Open, Crossref and OpenAlex, among them the list of the works which cite [Kra06] in OpenAlex (no entry) and the publications of its author in zbMATH, and five web searches. MathSciNet, Google Scholar and Semantic Scholar were not searched, and the author of the problem was not contacted. No proof and no statement of the complexity of triangle instances was found in the literature accessible to us; a search that finds nothing is not a proof of novelty.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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