

A Smooth Counterexample to Second-Difference Stability on Graded Meshes

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Abstract

We construct a counterexample to a proposed mesh-independent bound on scaled normal-derivative jumps of finite-element Ritz projections. The exact solution is the fixed polynomial $u(x, y) = xy(1 - x^2)(1 - y^2)$ on the unit square, with homogeneous Dirichlet data. The conforming piecewise affine meshes are uniformly shape-regular, their largest diameter tends to zero, and their Ritz projections converge in the energy norm. Nevertheless, on an interior edge of a corner patch,

$$h_E^{-3} \|[\partial_n u_h]\|_{L^2(E)}^2 \longrightarrow \infty.$$

The proof reduces the corner degrees of freedom to a tridiagonal recurrence. Positivity gives an explicit geometric lower bound. The construction disproves the unrestricted second-difference stability conjecture in Becker and Rannacher's 2005 Oberwolfach report. It also gives a nested counterexample family when the outer mesh is held fixed.

1 The proposed estimate

Let $\Omega = (0, 1)^2$, let $\mathcal{T}$ be a conforming triangulation, and let $V_{\mathcal{T}}$ be the continuous piecewise affine subspace of $H_0^1(\Omega)$. For $u \in H_0^1(\Omega)$, its Ritz projection $u_{\mathcal{T}} \in V_{\mathcal{T}}$ satisfies

$$\int_{\Omega} \nabla u_{\mathcal{T}} \cdot \nabla v = \int_{\Omega} \nabla u \cdot \nabla v \quad (v \in V_{\mathcal{T}}). \quad (1)$$

For an interior edge E , write $h_E = |E|$ and let $[\partial_n u_{\mathcal{T}}]$ be the jump of the normal derivative across E . The choice of normal changes only its sign.

Becker and Rannacher proposed, for $u \in C^2(\overline{\Omega})$, a bound of the form

$$\sum_{E \subset \partial K} h_E^{-3} \|[\partial_n u_{\mathcal{T}}]\|_{L^2(E)}^2 \leq C(u) \quad (K \in \mathcal{T}), \quad (2)$$

uniformly over the meshes under consideration [1, p. 2098, equation (5)]. They established the quasi-uniform case and asked for an extension to locally refined meshes. It is enough to consider the interior edges in (2); our obstruction occurs on one such edge and is independent of a convention for boundary jumps.

Here shape regularity alone does not suffice. Our argument uses ordinary conforming affine triangles, the exact Galerkin equations, and a single smooth solution. No numerical integration error or singularity of the exact solution is involved.

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Theorem 1. *Set*

$$u(x, y) = xy(1 - x^2)(1 - y^2).$$

There is a family $(\mathcal{T}_m)_{m \geq 2}$ of conforming triangulations of Ω with the following properties:

1. *Every triangle has angles between $\arctan(1/3)$ and $3\pi/4$, and $\max_{K \in \mathcal{T}_m} \text{diam } K = \sqrt{2} 2^{-m}$.*
2. *The Ritz projections u_m of u converge to u in $H_0^1(\Omega)$.*
3. *There is an interior edge E_m for which*

$$|E_m|^{-3} \|[\partial_n u_m]\|_{L^2(E_m)}^2 \geq \frac{4}{9} \left(\frac{(16/13)^8}{4} \right)^{2m} \rightarrow \infty. \quad (3)$$

Consequently (2) is false for general shape-regular locally refined meshes, even with vanishing maximum mesh size.

2 A graded corner patch

First fix $a = 1/M$, where $M \geq 4$ is an integer, and an integer $N \geq 1$. Partition the unit square into squares of side a . Except for $[0, a]^2$, split every square along its southwest to northeast diagonal. We replace the two triangles of $[0, a]^2$ by the following corner patch.

For $r_k = a2^{-k}$, $0 \leq k \leq N$, put

$$A_k = (r_k, 0), \quad B_k = (r_k, r_k), \quad C_k = (0, r_k).$$

For each $0 \leq k < N$, use the four triangles

$$(A_k, B_k, A_{k+1}), \quad (B_k, B_{k+1}, A_{k+1}), \quad (B_k, C_k, C_{k+1}), \quad (B_k, C_{k+1}, B_{k+1}). \quad (4)$$

They triangulate $[0, r_k]^2 \setminus [0, r_{k+1}]^2$. Complete the mesh with the two triangles

$$(O, A_N, B_N), \quad (O, B_N, C_N), \quad O = (0, 0). \quad (5)$$

Denote the resulting triangulation by $\mathcal{T}(a, N)$.

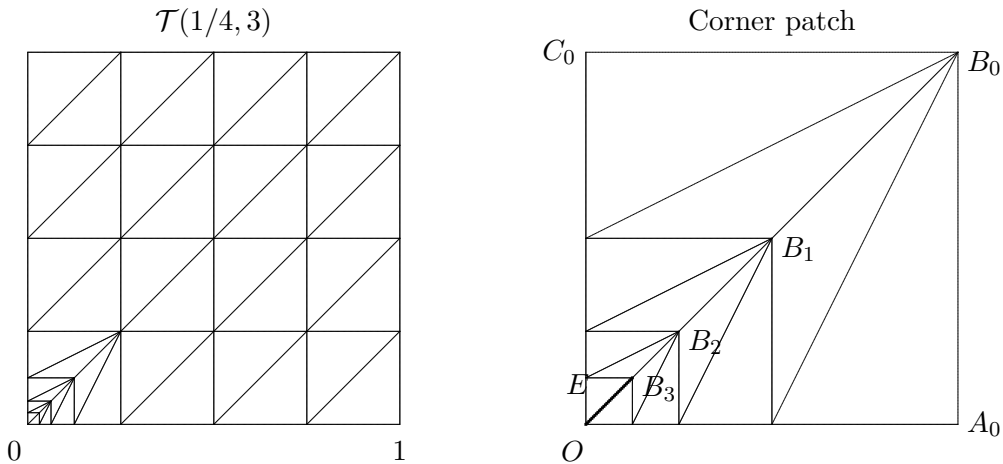


Figure 1: The finite mesh with three corner rings. The bold interior edge E is shared by the two cap triangles.

This is a conforming triangulation: the only new vertices on the boundary of $[0, a]^2$ lie on the coordinate axes, hence on $\partial\Omega$. All triangles are similar to a member of a fixed finite list. The

outer and cap triangles are right isosceles. The ring triangles have smallest angle $\arctan(1/3)$ and largest angle at most $3\pi/4$. Their diameters are at most $\sqrt{2}a$. Thus shape regularity is uniform in both a and N .

The new interior vertices are exactly $B_1, \dots, B_N$. The vertices A_k, C_k, O are Dirichlet vertices. The mesh has

$$2M^2 + 4N \quad \text{triangles,} \quad (M-1)^2 + N \quad \text{interior vertices.}$$

For fixed a , increasing N only subdivides the two cap triangles. In particular, the family $\mathcal{T}(a, N)$ is nested.

3 The corner stiffness equations

Let ϕ_i denote the nodal basis functions at interior vertices, and let S be the stiffness matrix. We record the local calculation responsible for the example.

Lemma 2. *One ring in (4), after omitting its Dirichlet vertices, contributes*

$$\begin{pmatrix} 3/2 & -2 \\ -2 & 5 \end{pmatrix} \tag{6}$$

to the pair (B_k, B_{k+1}) . The cap contributes 1 to the diagonal entry at B_N . The global Dirichlet stiffness matrix is symmetric positive definite and has nonpositive off-diagonal entries.

Proof. Scale the outer radius to 1. For the first two triangles, in the vertex order in (4), the local matrices are

$$L_1 = \begin{pmatrix} 5/4 & -1/4 & -1 \\ -1/4 & 1/4 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1/2 & -1 & 1/2 \\ -1 & 5/2 & -3/2 \\ 1/2 & -3/2 & 1 \end{pmatrix}.$$

These follow from $L_{ij} = |K| \nabla \phi_i \cdot \nabla \phi_j$. The other two triangles are their reflections across $x = y$. Omitting axis vertices leaves twice the corner contributions of these two matrices, giving (6). Scaling in two dimensions does not change a stiffness matrix. Each of the two cap triangles contributes $1/2$ at B_N .

In L_2 the only positive off-diagonal entries connect an interior corner to an axis vertex. They disappear when Dirichlet values are eliminated. All remaining ring off-diagonal entries are nonpositive. The outer grid consists of right triangles and also has nonpositive off-diagonal entries. Positive definiteness is the usual identity $z^T S z = \|\nabla \sum_i z_i \phi_i\|_{L^2(\Omega)}^2$. $\square$

The fixed exact solution in Theorem 1 satisfies

$$-\Delta u = f, \quad f(x, y) = 6xy(2 - x^2 - y^2) > 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0. \tag{7}$$

Consequently the load vector $b_i = \int_{\Omega} f \phi_i$ is strictly positive.

Lemma 3. *Every interior nodal value of the Ritz projection of u on $\mathcal{T}(a, N)$ is positive.*

Proof. For completeness, if S is symmetric positive definite with nonpositive off-diagonal entries and $Sv = b \geq 0$, write $v = v^+ - v^-$ with $v^{\pm} \geq 0$ and disjoint supports. If $v^- \neq 0$, then

$$0 \leq (v^-)^T b = (v^-)^T S v^+ - (v^-)^T S v^- < 0,$$

a contradiction. Thus $v \geq 0$. Since here $b_i > 0$, the equation $S_{ii}v_i = b_i - \sum_{j \neq i} S_{ij}v_j$ gives $v_i > 0$. $\square$

Write $v_k = u_{\mathcal{T}(a,N)}(B_k)$, and use b_k for the corresponding load component. Lemma 2 gives

$$-2v_{k-1} + \frac{13}{2}v_k - 2v_{k+1} = b_k > 0 \quad (1 \leq k < N), \quad (8)$$

$$-2v_{N-1} + 6v_N = b_N > 0. \quad (9)$$

We also need a lower bound at $B_0 = (a, a)$ that is independent of the depth N .

Lemma 4. For $a \leq 1/4$,

$$v_0 \geq \frac{2}{3}a^4, \quad v_N \geq \frac{2}{3}a^4(4/13)^N. \quad (10)$$

Proof. On the unmodified uniform grid, the diagonal stiffness entry at B_0 is 4. The two removed corner triangles contribute 1 to it, and the first new ring contributes $3/2$. Hence $S_{B_0B_0} = 9/2$.

Consider the unchanged grid square $Q = [a, 2a]^2$. For $(x, y) \in Q$ and $a \leq 1/4$,

$$f(x, y) \geq 6a^2(2 - 8a^2) \geq 9a^2.$$

The nodal function ϕ_{B_0} has integral $a^2/3$ over Q : both triangles contain B_0 , and the integral of a barycentric coordinate is one third of the triangle's area. Thus

$$b_0 \geq \int_Q f \phi_{B_0} \geq 3a^4.$$

By Lemma 3 and the off-diagonal signs, $(9/2)v_0 \geq b_0$, proving the first inequality.

Equation (8) and positivity imply $v_k \geq (4/13)v_{k-1}$ for $1 \leq k < N$. At the cap, (9) gives $v_N \geq v_{N-1}/3 \geq (4/13)v_{N-1}$. Iteration proves the second inequality. $\square$

4 An unbounded interior jump

Let $r = r_N$ and let $E = [O, B_N]$ be the common edge of the two cap triangles. Its relative interior lies in Ω . On the first cap triangle the Ritz projection is $v_N y/r$; on the second it is $v_N x/r$. Therefore

$$|E| = \sqrt{2}r, \quad |[\partial_n u_{\mathcal{T}(a,N)}]| = \sqrt{2}v_N/r.$$

Since the jump is constant along E ,

$$|E|^{-3} \|[\partial_n u_{\mathcal{T}(a,N)}]\|_{L^2(E)}^2 = \frac{v_N^2}{r^4} \geq \frac{4}{9}a^4(16/13)^{2N}. \quad (11)$$

This already diverges for the nested fixed- a family as $N \rightarrow \infty$.

Proof of Theorem 1. Take

$$a_m = 2^{-m}, \quad N_m = 8m, \quad \mathcal{T}_m = \mathcal{T}(a_m, N_m), \quad m \geq 2.$$

The geometry proved above gives uniform shape regularity and maximum diameter $\sqrt{2}a_m$. Substitution in (11) gives (3). The ratio is strictly larger than one because

$$16^4 = 65536 > 57122 = 2 \cdot 13^4, \quad \text{so} \quad (16/13)^8 > 4.$$

Finally, energy orthogonality gives $\|\nabla(u - u_m)\|_2 \leq \|\nabla(u - I_m u)\|_2$. Taylor's theorem on the finite set of reference triangle shapes bounds the latter by $C a_m \|D^2 u\|_{L^\infty(\Omega)}$. Poincaré's inequality gives convergence in $H_0^1(\Omega)$. $\square$

Remark 5. The globally shrinking family in the theorem is not asserted to be nested. The fixed- a family is nested and has the same divergence mechanism, but its maximum mesh size stays fixed. The source conjecture (2) specifies no marking rule or particular refinement algorithm. We make no claim about an additional, prescribed refinement rule or about families satisfying stronger grading restrictions.

Remark 6. There is no contradiction with energy convergence or first-gradient stability estimates. A scaled derivative jump tests a second-order quantity. Even though the corner nodal values tend to zero, they need not decay as fast as the square of the local edge length. The homogeneous part of (8) has decaying root $(13 - \sqrt{105})/8 > 1/4$, which also displays this mechanism; the proof uses only the simpler positive lower bound. The separate averaging hypothesis in Section 2.2 of [1] is not addressed.

5 Reproducibility

The accompanying script assembles the complete finite-element system with rational coordinates and exact polynomial integration. For five finite meshes it checks conformity, total area, stiffness signs, the corner equations, positivity, and the jump identity using exact fractions. A separate assembly uses Gauss–Duffy quadrature and a sparse numerical solver. Some values of v_N/r_N^2 , whose squares equal the edge term, are

m	N	v_N/r_N^2
2	16	148.8265
3	24	2133.0208
4	32	28099.0065
5	40	363730.6911

These computations supplement the all-parameter proof; the numerical values are not used to establish divergence. The exploratory four-spoke interior construction and both ring-diagonal choices are retained in the research archive.

The manuscript was prepared with AI assistance in the mathematical exploration, exact computations, and drafting. The proof was checked within that workflow. This is an unrefereed preprint; no independent review or absolute-priority certification is claimed.

References

- [1] R. Becker and R. Rannacher, Some thoughts about convergence of the DWR method, in *Mini-Workshop: Convergence of Adaptive Algorithms*, Oberwolfach Reports **2** (2005), report 37, 2097–2100; the full workshop report is 2091–2138. [doi:10.4171/owr/2005/37](https://doi.org/10.4171/owr/2005/37).