

Finite Adjoint-Transform Orbits for Funcoids and Reloids

Alper Ferudun*

25 October 2026

Abstract

For a monotone map between complete lattices, let its two adjoint transforms be the supremum of sublevel sets and the infimum of superlevel sets. Porton’s diagram problem asks for the missing transform of the outward reloid map and whether its iterated transforms form a finite set. Working in ZFC, we identify the missing map as the join of the induced funcoid with the product of the two Fréchet filters. We give a closed transition table containing at most twenty typed maps, thereby proving finiteness for arbitrary source and destination sets. If either set is finite, at most two maps occur. The proof uses Trimble’s representation of funcoids by closed subsets of a product of Stone spaces, together with two density statements about nonsingleton fibers of the canonical Stone map. We also give a counterexample to an extra arrow in the published diagram. The bound twenty is an upper bound, not a claim that all its labels are distinct.

1 The problem and the result

For a monotone map $f : L \rightarrow M$ between complete lattices, define

$$(\Phi_* f)(y) = \bigvee \{x \in L : f(x) \leq y\}, \quad (\Phi^* f)(y) = \bigwedge \{x \in L : y \leq f(x)\}. \quad (1)$$

Empty joins and meets are included. Both transforms are monotone maps $M \rightarrow L$. If the appropriate adjoint exists, these are its usual formulas; no adjoint existence is assumed here.

Fix sets A, B . Porton considers three maps between $\text{RLD}(A, B)$ and $\text{FCD}(A, B)$: the induced funcoid map (FCD) and the inward and outward reloid maps $(\text{RLD})_{\text{in}}$ and $(\text{RLD})_{\text{out}}$. His Open Problem Garden question [1] asks for

$$\text{other} = \Phi_*(\text{RLD})_{\text{out}}$$

and for the finiteness of its orbit under Φ^* and Φ_* . The later draft [2, Chapter 7] asks the same questions. We fix the source and destination throughout; each operation reverses the type of the map.

Write Ω_A for the Fréchet filter of cofinite subsets of A , allowing the improper filter when A is finite. We use Porton’s order convention: filters are ordered by reverse inclusion of their families of members.

Theorem 1. *In ZFC, for every reloid r from A to B ,*

$$(\Phi_*(\text{RLD})_{\text{out}})(r) = (\Omega_A \times^{\text{FCD}} \Omega_B) \vee (\text{FCD})(r). \quad (2)$$

If A, B are both infinite, all maps obtained from this map by arbitrary finite words in Φ^*, Φ_* belong to the twenty-label transition system in Tables 1 and 2. Thus its orbit is finite, of cardinality at most twenty. If at least one of A, B is finite, at most two typed maps occur.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing).

In particular, when A, B are nonempty, the missing map is not constant top: its value at bottom is $\Omega_A \times^{\text{FCD}} \Omega_B$, which is strictly below top. The formulation with a product of Fréchet filters specifies the element explicitly, without assigning a meaning to the symbol Ω^{FCD} used without a definition in the cited draft chapter.

The Stone-space representation used below is due to Trimble [4]; it is also recorded in Porton's second volume [5, Chapter 8]. Our argument derives the missing map and a finite transition system from this representation. We do not attribute the representation or ordinary adjoint formulas to this paper.

2 Closed sets and the three original maps

For a discrete set E , let βE be its Stone space of ultrafilters. A subset $S \subseteq E$ determines the clopen set

$$\widehat{S} = \{p \in \beta E : S \in p\}.$$

Principal ultrafilters identify E with a dense discrete subspace of βE . Let $\mathcal{C}(W)$ be the complete lattice of closed subsets of a compact Hausdorff space W , ordered by inclusion. Meets are intersections and joins are closures of unions.

Set

$$X = \beta(A \times B), \quad Y = \beta A \times \beta B,$$

and let $\pi : X \rightarrow Y$ send an ultrafilter to its two projections. The map π is a continuous surjection: for ultrafilters p, q , the rectangles $S \times T$, $S \in p$, $T \in q$, generate a proper filter, which extends to an ultrafilter. We write

$$X_y = \pi^{-1}\{y\}.$$

Both spaces contain $A \times B$ as a dense subspace of isolated points, and π is the identity on this subspace.

A reloid is a filter on $A \times B$, including the improper filter. The correspondence

$$\mathcal{F} \longmapsto \{u \in X : \mathcal{F} \subseteq u\}$$

is an order isomorphism from the reloid lattice to $\mathcal{C}(X)$. The funcoid lattice is order isomorphic to $\mathcal{C}(Y)$ [4, 5].

For completeness, the latter correspondence can be expressed without the funcoid terminology. A funcoid is equivalently specified by a relation δ on $\mathcal{P}(A) \times \mathcal{P}(B)$ that is false if either entry is empty and distributes over finite unions in each variable; the extension to filters is obtained by requiring $\delta(S, T)$ for every pair of filter members [3, Chapter 26]. The closed set representing it is

$$C_\delta = \{(p, q) : \delta(S, T) \text{ for every } S \in p, T \in q\}. \quad (3)$$

Conversely a closed $C \subseteq Y$ gives

$$\delta_C(S, T) \iff C \cap (\widehat{S} \times \widehat{T}) \neq \emptyset.$$

These maps are inverse. To see the nontrivial implication, suppose $\delta(S, T)$ but C_δ misses $\widehat{S} \times \widehat{T}$. Compactness gives a finite cover of this rectangle by clopen rectangles corresponding to pairs on which δ is false. Refine all the subsets involved into finite partitions of S and T . Finite additivity yields a pair of cells on which δ is true. Their product is contained in one of the bad rectangles, contradicting monotonicity of δ . The other direction follows by separating a point outside a closed set with a basic clopen rectangle. This also verifies the order correspondence.

Under these identifications put, for $r \in \mathcal{C}(X)$ and $C \in \mathcal{C}(Y)$,

$$P(r) = \pi(r), \quad I(C) = \pi^{-1}(C), \quad (4)$$

$$Q(C) = \bigcap \{W : W \subseteq X \text{ clopen, } C \subseteq \pi(W)\}. \quad (5)$$

Then $P = (\text{FCD})$, $I = (\text{RLD})_{\text{in}}$, and $Q = (\text{RLD})_{\text{out}}$. Indeed, a principal relation $R \subseteq A \times B$ corresponds to $\widehat{R}$ on X and to $\pi(\widehat{R})$ on Y . The induced funcoid of a filter is the meet of these principal funcoids; compactness gives the closed image $\pi(r)$. The outward map is exactly the meet (5). The inward product of two ultrafilters is represented by the whole fiber $X_{(p,q)}$; taking the join over $(p,q) \in C$ gives $I(C)$. These agree with the definitions in [3, Chapter 28]. In particular,

$$P(r) \subseteq C \iff r \subseteq I(C), \quad \Phi_* P = I, \quad \Phi^* I = P. \quad (6)$$

3 Two density facts about the fibers

Assume for now that A, B are infinite. Define the closed sets

$$K = (\beta A \setminus A) \times (\beta B \setminus B), \quad Z = \pi^{-1}(K),$$

and let

$$S = \{y \in Y : |X_y| > 1\}.$$

Outside K every fiber is a singleton. For example, if the first projection is the principal ultrafilter at a , the ultrafilter on $A \times B$ is uniquely determined by its restriction to $\{a\} \times B$ and its second projection.

Lemma 2. *The following equalities hold:*

$$\text{cl}_Y S = K, \quad \text{cl}_X \pi^{-1}(S) = Z.$$

Moreover K and Z each contain at least two points.

Proof. We already have $S \subseteq K$. A basic clopen rectangle meeting K is $\widehat{U} \times \widehat{V}$ for infinite $U \subseteq A$, $V \subseteq B$. Choose distinct sequences (a_n) in U and (b_n) in V , and a free ultrafilter p on $\mathbb{N}$. Its two pushforwards give p_A, p_B in this rectangle. Every product of a member of p_A and a member of p_B meets each of the disjoint sets

$$R_+ = \{(a_i, b_j) : i < j\}, \quad R_- = \{(a_i, b_j) : i > j\}.$$

Indeed the corresponding two sets of indices are infinite and therefore unbounded. The rectangle filter can be extended after adjoining either R_+ or R_- . The resulting ultrafilters have the same projections and are distinct. Thus $(p_A, p_B) \in S$, proving the first density assertion.

For the second, let a clopen $\widehat{R} \subseteq X$ meet Z . The relation R is not contained in finitely many rows and columns. Otherwise an ultrafilter in $\widehat{R} \cap Z$, whose projections are both free, would contain both R and the complement of such a cover. We can consequently select an infinite matching $(a_n, b_n) \in R$, with both coordinate sequences injective. Push a free ultrafilter on $\mathbb{N}$ forward along this matching. The resulting point $u \in \widehat{R}$ has projections (p_A, p_B) of the preceding kind, so $\pi(u) \in S$. Every clopen set meeting Z therefore meets $\pi^{-1}(S)$. The reverse inclusion after closure holds since Z is closed. Finally an infinite set has at least two free ultrafilters, by extending the cofinite filters on two disjoint infinite subsets. This proves the last assertion. $\square$

We require only these density statements. In particular, the proof does not assume that every pair of free ultrafilters has more than one joint extension.

4 The missing map

Two elementary facts will also be used for the transition table. For a monotone $f : \mathcal{C}(V) \rightarrow \mathcal{C}(W)$,

$$(\Phi_* f)(D) = \text{cl}_V \{v \in V : f(\{v\}) \subseteq D\}. \quad (7)$$

Indeed every closed set in the defining sublevel family is a union of its singletons, which also belong to that family; conversely each permitted singleton belongs to it.

Second, from (5),

$$Q(\{y\}) = \begin{cases} X_y, & |X_y| = 1, \\ \emptyset, & |X_y| > 1. \end{cases} \quad (8)$$

For a nonsingleton fiber, any given point of it can be excluded by a clopen set containing another point of the fiber; points outside the fiber can also be excluded. For a singleton fiber, every clopen set meeting it must contain its unique point.

Define

$$H(r) = K \cup P(r). \quad (9)$$

By (7) and (8), $\Phi_* Q(r)$ is the closure of S together with those singleton-fiber points whose lifts lie in r . This set before closure is precisely $S \cup P(r)$: a point of $P(r)$ with nonsingleton fiber already lies in S . Lemma 2 therefore gives

$$\Phi_* Q = H. \quad (10)$$

The closed set of ultrafilters extending Ω_A is $\beta A \setminus A$, and a funcoidal product corresponds to a product of closed sets. Thus (9) proves (2) in the infinite case.

5 A closed transition system

Here are the remaining maps. All closures carry the displayed ambient subscript. In addition to P, Q, I, H , put

$$U(r) = \text{cl}_Y \{y : X_y \subseteq r\}, \quad V(r) = \text{cl}_Y \pi(r \setminus Z), \quad (11)$$

$$B(C) = \text{cl}_X \pi^{-1}(C \setminus K), \quad E(C) = Z \cup I(C), \quad (12)$$

$$D(C) = \begin{cases} I(C), & K \subseteq C, \\ \emptyset, & K \not\subseteq C, \end{cases} \quad F(r) = \begin{cases} U(r), & Z \subseteq r, \\ \emptyset, & Z \not\subseteq r, \end{cases} \quad (13)$$

$$H_0(r) = \begin{cases} H(r), & r \neq \emptyset, \\ \emptyset, & r = \emptyset, \end{cases} \quad B_1(C) = \begin{cases} Z \cup B(C), & C \neq \emptyset, \\ \emptyset, & C = \emptyset. \end{cases} \quad (14)$$

Each of these maps is monotone. The types and all transforms are listed in Table 1. An XY map has domain $\mathcal{C}(X)$ and codomain $\mathcal{C}(Y)$; YX means the reverse.

For either ordered pair (L, M) of our two lattices, let T_{LM} and 0_{LM} be the constant top and bottom maps, and put

$$a_{LM}(x) = \begin{cases} \top_M, & x = \top_L, \\ \perp_M, & x \neq \top_L, \end{cases} \quad c_{LM}(x) = \begin{cases} \perp_M, & x = \perp_L, \\ \top_M, & x \neq \perp_L. \end{cases}$$

We write XY and YX in these subscripts to denote $(\mathcal{C}(X), \mathcal{C}(Y))$ and its reverse.

We prove every entry; the finite table alone is not used as evidence for the identities.

Map	Type	Φ^*	Φ_*
P	XY	Q	I
Q	YX	P	H
I	YX	P	U
U	XY	Q	E
H	XY	B	D
B	YX	P	H
D	YX	H_0	T_{XY}
H_0	XY	B	D
E	YX	V	F
V	XY	Q	E
F	XY	B_1	T_{YX}
B_1	YX	V	F

Table 1: The twelve principal labels. Boundary maps are defined below.

Map	Φ^*	Φ_*
T_{LM}	0_{ML}	a_{ML}
0_{LM}	c_{ML}	T_{ML}
a_{LM}	c_{ML}	T_{ML}
c_{LM}	0_{ML}	a_{ML}

Table 2: The eight boundary labels: four for each orientation.

5.1 Clopen tests and left transforms

For any closed $r \subseteq X$, compactness and the clopen basis give

$$\bigcap_{\substack{W \supseteq r \\ W \text{ clopen}}} \pi(W) = \pi(r). \quad (15)$$

For example, a fiber disjoint from r can be separated from r by a clopen set. It follows directly from (5) that

$$\Phi^*P = Q. \quad (16)$$

To justify the use of clopens here, if $C \subseteq P(r)$, every clopen $W \supseteq r$ satisfies $C \subseteq P(W)$. Their intersection is r , so restricting the infimum to clopens does not change it.

For clopen $W \subseteq X$ the following identities hold:

$$Q(P(W)) = W, \quad B(P(W)) = W, \quad U(W) = P(W), \quad V(W) = P(W). \quad (17)$$

For the first equality, W is one of the sets intersected in $Q(P(W))$, and every principal point of W must belong to $Q(P(W))$. Principal points are dense in W , proving equality. For the second, every point of $P(W) \setminus K$ has a unique lift, and that lift is in W . The same dense principal points give the opposite inclusion. For the last two equalities, the defining sets are contained in $P(W)$ and contain the images of all principal points of W ; these images are dense in $P(W)$.

We always have $Q(C), B(C) \subseteq I(C)$. Thus $r \subseteq Q(C)$ or $r \subseteq B(C)$ implies $P(r) \subseteq C$. Conversely apply the first two equalities in (17) to every clopen $W \supseteq r$, then use (15). This proves

$$\Phi^*Q = P, \quad \Phi^*B = P. \quad (18)$$

Similarly $U(r), V(r) \subseteq P(r)$, while U, V agree with P on clopens. Consequently their super-level infima coincide with that for P : every clopen superset of a feasible argument for P is feasible for U and for V . Hence

$$\Phi^*U = Q, \quad \Phi^*V = Q. \quad (19)$$

5.2 The two additional adjunctions

Uniqueness of fibers outside K gives

$$B(C) \subseteq r \iff C \subseteq K \cup P(r) = H(r), \quad (20)$$

$$V(r) \subseteq C \iff r \subseteq Z \cup I(C) = E(C). \quad (21)$$

For (20), only points of $C \setminus K$ impose a condition; their unique lifts must lie in the closed set r . For (21), only points of $r \setminus Z$ impose a condition; their images must lie in the closed set C . It follows that

$$\Phi^*H = B, \quad \Phi_*B = H, \quad \Phi^*E = V, \quad \Phi_*V = E.$$

The condition $H(r) \subseteq C$ is impossible unless $K \subseteq C$. In that case it is equivalent to $r \subseteq I(C)$, proving $\Phi_*H = D$. Likewise $E(C) \subseteq r$ is impossible unless $Z \subseteq r$. In that case its supremum is $U(r)$, proving $\Phi_*E = F$.

5.3 The second fiber transform

Formula (7) applied to I gives $\Phi_*I = U$. For a point $x \in X$,

$$U(\{x\}) = \begin{cases} \{\pi(x)\}, & |X_{\pi(x)}| = 1, \\ \emptyset, & |X_{\pi(x)}| > 1. \end{cases}$$

Another use of (7), now with the second density statement of Lemma 2, yields

$$\Phi_*U(C) = Z \cup I(C) = E(C).$$

Indeed every point above S is permitted, regardless of C , and these points are dense in Z . Outside Z the permitted points are precisely the unique lifts of $C \setminus K$. The containing set $Z \cup I(C)$ is closed. This proves both inclusions.

5.4 The zero-sensitive maps

If $r \neq \emptyset$, then $r \subseteq D(C)$ is equivalent to $K \subseteq C$ and $P(r) \subseteq C$. Its least solution is $H(r)$. If $r = \emptyset$, every C is a solution and their intersection is empty. Thus $\Phi^*D = H_0$. Every singleton C fails to contain K , because K has at least two points. It is therefore in every sublevel family for D . Formula (7) gives $\Phi_*D = T_{XY}$.

For $H_0(r) \subseteq C$, the empty argument is always allowed. A nonempty argument is allowed exactly when $K \subseteq C$ and $r \subseteq I(C)$. Hence $\Phi_*H_0 = D$. For its left transform, if $C \setminus K$ is nonempty, the least admissible argument is $B(C)$ by (20), and it is nonempty. If $\emptyset \neq C \subseteq K$, all nonempty arguments are admissible and their intersection is empty, again $B(C)$. At $C = \emptyset$ the answer is also empty. Thus $\Phi^*H_0 = B$.

We will use the identity

$$Z \subseteq r \implies U(r) = H(r). \quad (22)$$

All fibers over K are then contained in r . Outside K , the fiber-containment condition is simply membership in $P(r)$. Since $P(r)$ is closed and contains K , this proves (22).

If $C \neq \emptyset$, the condition $C \subseteq F(r)$ forces $Z \subseteq r$. By (22) and (20), its least solution is $Z \cup B(C)$. At $C = \emptyset$ its infimum is empty. Therefore $\Phi^*F = B_1$. Every singleton r fails to contain Z , so $\Phi_*F = T_{YX}$.

For the right transform of B_1 , nonempty admissible arguments exist only if $Z \subseteq r$. In that case $B(C) \subseteq r$ is equivalent to $C \subseteq H(r)$, whose greatest solution is $H(r) = U(r)$. Thus $\Phi_*B_1 = F$. For its left transform, if $r \setminus Z \neq \emptyset$, the least C with $r \subseteq B_1(C)$ is $V(r)$: necessity follows from $B(C) \subseteq I(C)$, and sufficiency from the unique lifts of points outside K . If $r \subseteq Z$, either r is empty or every nonempty C is admissible. The intersection is empty, again $V(r)$. This proves $\Phi^*B_1 = V$ and completes Table 1.

5.5 Boundary maps and finiteness

Both spaces have at least two points. In each closed-set lattice, the join of all proper closed sets is top, and the meet of all nonempty closed sets is bottom. These statements follow already from singletons. Substituting the definitions of $T, 0, a, c$ into (1) gives Table 2, including its bottom and top inputs.

All twenty labels are therefore closed under both transforms. Starting at H , every finite word remains in these tables. This proves the asserted upper bound; possible coincidences among the labels only decrease the orbit size.

If one of A, B is finite, every fiber of π is a singleton. Thus π is a homeomorphism, P and I are inverse order isomorphisms, $Q = I$, and $\Phi_*Q = P$. Both transforms alternate between P and I . Here $K = \emptyset$, so (2) still holds. If either set is empty, both closed-set lattices have one element and the assertion is immediate. This finishes the proof of Theorem 1.

6 An arrow that must be removed from the earlier diagram

The draft [2, Lemma 2199], also in the source file [6], asserts $\Phi_*I = P$. The correct general entry is $\Phi_*I = U$, and it can differ from P even on $\mathbb{N} \times \mathbb{N}$.

Let p be a free ultrafilter on $\mathbb{N}$ and let u be its pushforward along $n \mapsto (n, n)$. Then $\pi(u) = (p, p)$. The triangular construction in Lemma 2 shows that $X_{(p,p)}$ contains other points. No fiber is contained in $\{u\}$. Consequently

$$(\Phi_*I)(\{u\}) = U(\{u\}) = \emptyset, \quad P(\{u\}) = \{(p, p)\}.$$

The equivalence $I(C) \subseteq r \iff C \subseteq P(r)$ used in that draft is therefore not valid. The actual adjunction is (6), with its inequalities in the opposite positions. The transition proof above uses the definitions and does not assume the erroneous arrow.

7 Scope and reproducibility

The theorem settles the missing-node and finiteness questions for fixed arbitrary sets in ZFC. It does not determine the smallest universal orbit bound, and it does not assert that the twenty maps are always distinct. The Stone representation, filter-extension principle, and ordinary adjunction identities are prior inputs.

The accompanying diagnostic script checks the type consistency and reachability of the twenty labels. It also checks 7,392 exact values for four finite surjection models. In those models it tests all right-transform identities and the left-transform identities that use only fiber algebra. It deliberately does not transfer the four Stone cofinality identities $\Phi^*Q = P$, $\Phi^*B = P$, $\Phi^*U = Q$, $\Phi^*V = Q$ to noninjective finite discrete maps: their proofs here use the dense isolated subspace. These diagnostics are not a substitute for the infinite-space proofs.

AI assistance was used for literature retrieval, development of the argument, exact diagnostic calculations, and preparation of the manuscript. This preprint is self-audited and unrefereed; no independent review, proof-assistant certification, or absolute priority claim is made.

References

- [1] V. Porton, *A diagram about funcoids and reloids*, Open Problem Garden, posted 26 November 2016, revised 29 November 2016. [Problem page](#).
- [2] V. Porton, *Algebraic General Topology. Volume 1 addons*, Chapter 7, pp. 25–26, Definitions 2194–Question 2204, online draft accessed 25 October 2026. [Author’s PDF](#).
- [3] V. Porton, *General Topology as Ordered Semigroup Actions. Book 1*, Chapters 26 and 28, online version accessed 25 October 2026. [Author’s PDF](#).
- [4] T. Trimble, *Topogeny*, online mathematical notes, in particular the closed-subset representation of funcoids and staroids, accessed 25 October 2026. [Author’s notes](#).
- [5] V. Porton, *Algebraic General Topology. Volume 2 partial draft*, Chapter 8, *Funcoids as closed sets*, p. 32, online version accessed 25 October 2026. [Author’s PDF](#).
- [6] V. Porton, *Algebraic General Topology*, source file `chap-fcd-rld-gal.tex`, Git blob `e0dc75e6b72547be4384b2adbff33f97754d317c`, retrieved 25 October 2026. [Source repository](#).