

# Weak Saturation of the Cube in the Clique: Exact Values for $n \leq 13$ and Linear Bounds

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## Abstract

For graphs  $G$  and  $H$ ,  $\text{wsat}(G, H)$  is the least number of edges of a spanning subgraph  $F$  of  $G$  such that the edges of  $G$  missing from  $F$  can be added one at a time, each completing a new copy of  $H$ . Morrison and Noel asked (Open Problem Garden, 2016) to determine  $\text{wsat}(K_n, Q_3)$ , where  $Q_3$  is the cube. We give a partial answer. For  $n \leq 7$  no edge can be added and the value is  $\binom{n}{2}$ . By exhaustive computer searches, each carried out with at least two independent programs,  $\text{wsat}(K_n, Q_3) = 15, 16, 18, 19, 21, 23$  for  $n = 8, 9, 10, 11, 12, 13$ ; extremal graphs are given explicitly, with certificates that can be checked without a computer. The values for  $n \leq 11$  and the upper bound  $\text{wsat}(K_n, Q_3) \leq \lfloor (7n + 2)/4 \rfloor$  for  $n \geq 9$ , which comes from a gadget with four vertices and seven edges, were obtained earlier in two unrefereed preprints (Carptopus, 2026; V. Saha and N. Saha, 2026); the values for  $n = 12, 13$  are new. We prove  $\text{wsat}(K_n, Q_3) \geq (5n - 7)/3$  for  $n \geq 8$  by the cluster method of Terekhov and Zhukovskii, with clusters that may share a vertex. Hence the limit of  $\text{wsat}(K_n, Q_3)/n$ , which exists by a result of Alon, lies in  $[5/3, 7/4]$ ; the lower end improves the value  $11/7$  given by a general theorem of Terekhov and Zhukovskii. We conjecture, on computational evidence only, that  $\text{wsat}(K_n, Q_3) = \lfloor (7n + 2)/4 \rfloor$  for all  $n \geq 9$ . The exact value for  $n \geq 14$  remains open. This is an unrefereed note.

## 1 Introduction

### 1.1 The problem

Let  $G$  and  $H$  be graphs. A spanning subgraph  $F$  of  $G$  is *weakly  $H$ -saturated* in  $G$  if the edges of  $E(G) \setminus E(F)$  can be added to  $F$  one at a time so that each added edge completes a new copy of  $H$ , that is, a copy of  $H$  which contains the added edge. The *weak saturation number*  $\text{wsat}(G, H)$  is the least number of edges of a weakly  $H$ -saturated subgraph of  $G$ . The cube  $Q_3$  has vertex set  $\{0, 1\}^3$ , two strings being adjacent when they differ in one coordinate; it has 8 vertices and 12 edges, and it is 3-regular and bipartite.

**Problem 1.1** (Morrison and Noel [MN16]). *Determine  $\text{wsat}(K_n, Q_3)$ .*

The problem page adds that a natural next step would be  $\text{wsat}(K_n, Q_m)$  for all  $m$ , and that Morrison, Noel and Scott [MNS17] determined  $\text{wsat}(Q_d, Q_m)$  for all  $d \geq m$ , where the host graph is the hypercube  $Q_d$ . Here the host graph is the complete graph, and we write

$$\text{wsat}(n) := \text{wsat}(K_n, Q_3).$$

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**Conventions.** (a) Copies of  $Q_3$  are subgraphs, not necessarily induced. (b) Some authors also require that  $F$  contains no copy of  $H$ . This does not change the minimum. If  $F$  is weakly  $Q_3$ -saturated and contains a copy  $C_1$  of  $Q_3$ , delete an edge  $e_1$  of  $C_1$ ; if  $F - e_1$  still contains a copy  $C_2$ , delete an edge  $e_2$  of  $C_2$ , and so on, until the graph  $F' = F - e_1 - \dots - e_m$  contains no copy of  $Q_3$ . Adding  $e_m, e_{m-1}, \dots, e_1$  back in this order, each  $e_j$  completes the copy  $C_j$ , all of whose other edges are present at that moment; after that the steps for  $F$  can be made. So  $F'$  is weakly saturated and has fewer edges; in particular a weakly saturated graph with  $\text{wsat}(n)$  edges contains no copy of  $Q_3$ . (c) The order of the additions does not matter (Lemma 2.1). (d) For  $n \leq 7$  the graph  $K_n$  contains no copy of  $Q_3$ , no edge can ever be added, and  $\text{wsat}(n) = \binom{n}{2}$ . The question therefore concerns  $n \geq 8$ .

## 1.2 Results

**Theorem 1.2** (computer-assisted). *For  $n = 8, 9, 10, 11, 12, 13$  one has  $\text{wsat}(n) = 15, 16, 18, 19, 21, 23$ , respectively.*

The upper bounds are given by explicit graphs (Section 3); for  $n = 8, 9, 11$  they are certified by lists of steps printed in this note, and for  $n = 10, 12, 13$  they follow from these by Lemmas 2.2 and 2.5. The lower bounds rest on exhaustive computer searches, described in Section 5; this is the only part of the note that depends on computation. The values for  $n \leq 11$  were found earlier:  $\text{wsat}(8) = 15$  in [TXZ25, Car26],  $\text{wsat}(9) = 16$  and  $\text{wsat}(10) = 18$  in [Car26, SS26], and  $\text{wsat}(11) = 19$  in [SS26] (Section 1.3).

**Theorem 1.3** (upper bounds; Carptopus [Car26]). (a)  $\text{wsat}(n + 4) \leq \text{wsat}(n) + 7$  for every  $n \geq 8$ . (b)  $\text{wsat}(n) \leq \lfloor (7n + 2)/4 \rfloor$  for every  $n \geq 9$ .

**Theorem 1.4** (lower bound).  $\text{wsat}(n) \geq (5n - 7)/3$  for every  $n \geq 8$ .

Theorem 1.3 was proved earlier by Carptopus [Car26], with the same gadget (Lemma 2.5); we include our proof, which was obtained without knowledge of [Car26], because the values for  $n = 12, 13$  use it. By a result of Alon [Alo85], the limit of  $\text{wsat}(K_n, H)/n$  exists for every graph  $H$ ; this statement is taken from [TZ25a, Sections 1 and 2], where it is derived from the subadditivity of  $\text{wsat}(K_n, H) + (|V(H)| - 2)^2$ , and we did not read [Alo85].

**Corollary 1.5.**  $\frac{5}{3} \leq \lim_{n \rightarrow \infty} \text{wsat}(n)/n \leq \frac{7}{4}$ .

The upper end is due to [Car26]; the lower end improves the value  $\frac{11}{7}$  which follows from [TZ25a, Theorem 2.1] (Remark 4.2).

**Conjecture 1.6** (supported by computation only).  $\text{wsat}(n) = \lfloor (7n + 2)/4 \rfloor$  for every  $n \geq 9$ .

Theorem 1.2 proves Conjecture 1.6 for  $9 \leq n \leq 13$ ; the further evidence (a local search for  $n \leq 18$  and searches over small gadgets) is described in Section 6. Table 1 lists the bounds for small  $n$ .

| $n$                                        | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|--------------------------------------------|----|----|----|----|----|----|----|----|----|----|----|
| $\lceil (5n - 7)/3 \rceil$ (Theorem 1.4)   | 11 | 13 | 15 | 16 | 18 | 20 | 21 | 23 | 25 | 26 | 28 |
| $\text{wsat}(n)$ (Theorem 1.2)             | 15 | 16 | 18 | 19 | 21 | 23 |    |    |    |    |    |
| $\lfloor (7n + 2)/4 \rfloor$ (Theorem 1.3) | 14 | 16 | 18 | 19 | 21 | 23 | 25 | 26 | 28 | 30 | 32 |

Table 1: Lower bound, exact value and upper bound of  $\text{wsat}(n)$ . The upper bound of Theorem 1.3 holds for  $n \geq 9$ ;  $\text{wsat}(8) = 15$ .

### 1.3 Known results and what is new

Terekhov and Zhukovskii [TZ25a] developed a combinatorial method for lower bounds, in which the edges of a weakly saturated graph are distributed among growing vertex sets (“clusters”). For  $Q_3$  their Theorem 1.1 gives  $\text{wsat}(n) \geq \frac{3}{2}(n-8) + 11$ , and their Theorem 2.1, with pairwise disjoint clusters, a bound with slope  $\frac{11}{7}$ . In [TZ25a, Section 5.3] the clusters may share one vertex, under a hypothesis that does not hold for  $Q_3$  (Remark 4.2). The proof of Theorem 1.4 uses the same device, for  $Q_3$  and without that hypothesis; we claim no novelty for the method. The limits of the linear algebraic approach to lower bounds are studied in [TZ25b]. Kronenberg, Martins and Morrison [KMM21] proved  $\text{wsat}(K_n, K_{t,t}) = (t-1)(n+1-t/2)$  for  $n \geq 3t-3$ ; since every edge of  $K_{4,4}$  lies in a copy of  $Q_3$  inside  $K_{4,4}$ , a weakly  $K_{4,4}$ -saturated graph is weakly  $Q_3$ -saturated, and so  $\text{wsat}(n) \leq 3(n-1)$  for  $n \geq 9$ .

We found three sources specific to Problem 1.1. The slides of a 2025 student project [TXZ25], with the same definition, report  $\text{wsat}(8) = 15$  and  $\text{wsat}(9) = 17$  from a computer search, state  $\frac{3}{2}n - 1 \leq \text{wsat}(n) \leq 2n - 1$  for  $n \geq 8$  (the lower bound is quoted from [TZ25a]), and conjecture  $\text{wsat}(n) = 2n - 1$ . Two unrefereed preprints on Zenodo contain part of our results. Carptopus [Car26] (2 September 2026) determined  $\text{wsat}(8), \text{wsat}(9), \text{wsat}(10) = 15, 16, 18$  by exhaustive search, which corrects the value 17 and disproves the conjecture of [TXZ25], and gave a weakly saturated graph on 11 vertices consisting of a complete set of seven vertices and the gadget of Lemma 2.5; with a gluing lemma this gives Theorem 1.3, and with [TZ25a, Theorem 2.1] the bounds  $\frac{11}{7} \leq \lim \text{wsat}(n)/n \leq \frac{7}{4}$ . V. Saha and N. Saha [SS26] (3 October 2026) determined  $\text{wsat}(9), \text{wsat}(10), \text{wsat}(11) = 16, 18, 19$  by exhaustive searches with a symmetry reduction, with certificates, and proved  $\text{wsat}(n) \leq 2n - 3$  for  $n \geq 11$ ; their Proposition 7.1 is Corollary 2.3. These two preprints were found by the third verification run (Section 7), after the note had been written; the results of Sections 2 and 3 had been obtained without them.

New in this note, as far as we know, are the values  $\text{wsat}(12) = 21$  and  $\text{wsat}(13) = 23$ , the lower bound of Theorem 1.4, and hence the lower end  $\frac{5}{3}$  in Corollary 1.5, and Conjecture 1.6. The searches are described in the paragraph “Scope and priority” in Section 7.

## 2 Preliminaries

**Notation.** We identify  $\{0, 1\}^3$  with  $\{0, 1, \dots, 7\}$  by binary expansion; then  $x$  and  $y$  are adjacent in  $Q_3$  if and only if  $x \oplus y \in \{1, 2, 4\}$  (bitwise sum modulo 2), and the twelve edges of  $Q_3$  are

$$01, 02, 04, 13, 15, 23, 26, 37, 45, 46, 57, 67. \quad (1)$$

A copy of  $Q_3$  in a graph  $G$  is written as a tuple  $(\varphi_0, \dots, \varphi_7)$  of distinct vertices such that  $\varphi_x \varphi_y \in E(G)$  for the twelve pairs  $xy$  of (1). A *step* adds to a graph  $G$  an edge  $uv \notin E(G)$  such that  $G + uv$  contains a copy of  $Q_3$  through  $uv$ ; then  $uv$  is *addable* in  $G$ , and such a copy is a *certificate* of the step. Checking a certificate means checking that the eight vertices are distinct, that  $uv = \varphi_x \varphi_y$  for one of the pairs in (1), and that the other eleven pairs are edges of  $G$ . Throughout,  $Q$  denotes the copy  $(0, 1, \dots, 7)$  on the vertex set  $\{0, \dots, 7\}$ , with the edges (1), and  $Q^- := Q - \{0, 1\}$ ; the sixteen pairs inside  $\{0, \dots, 7\}$  that are not edges of  $Q$  are its *chords*.

**Lemma 2.1** (closure). *If  $uv$  is addable in  $G$ , and  $G' \supseteq G$  is a graph on the same vertex set with  $uv \notin E(G')$ , then  $uv$  is addable in  $G'$ . Consequently all maximal sequences of steps starting from  $G$  end with the same graph  $\text{cl}(G)$ ; moreover  $G \subseteq G'$  implies  $\text{cl}(G) \subseteq \text{cl}(G')$ ,  $\text{cl}(\text{cl}(G)) = \text{cl}(G)$ , and a spanning subgraph  $F$  of  $K_n$  is weakly  $Q_3$ -saturated if and only if  $\text{cl}(F) = K_n$ .*

*Proof.* The same copy of  $Q_3$  works in  $G' + uv$ . Let  $G \subseteq G'$ , let  $f_1, \dots, f_q$  be any sequence of steps starting from  $G$ , and let  $A$  be the end of a maximal sequence of steps starting from  $G'$ . If

$f_1, \dots, f_{j-1} \in E(A)$  and  $f_j \notin E(A)$ , then  $f_j$  is addable in  $G + f_1 + \dots + f_{j-1} \subseteq A$ , hence in  $A$ , contradicting maximality; so all  $f_j$  lie in  $A$ . With  $G = G'$  this shows that any two maximal sequences have the same end, and in general it gives the monotonicity. Idempotence and the last statement are clear.  $\square$

**Lemma 2.2** (absorption). *Let  $K$  be a set of at least 7 vertices which is complete in  $G$ , and let  $z \notin K$  have two neighbours  $a, b \in K$ . Then  $\text{cl}(G)$  contains all edges from  $z$  to  $K$ , so  $K \cup \{z\}$  is complete in  $\text{cl}(G)$ .*

*Proof.* Let  $c \in K \setminus \{a, b\}$  with  $zc \notin E(G)$ , and choose distinct  $p, q, r, x \in K \setminus \{a, b, c\}$ . The tuple  $(z, a, b, p, c, q, r, x)$  certifies the step  $zc$ : its pairs (1) are  $za, zb, zc, ap, aq, bp, br, px, cq, cr, qx, rx$ , and all except  $zc$  are edges of  $G$ . Repeat for every such  $c$ .  $\square$

**Corollary 2.3** ([TXZ25], [SS26, Proposition 7.1]).  $\text{wsat}(n+1) \leq \text{wsat}(n) + 2$  for every  $n \geq 7$ .

*Proof.* Let  $F$  be weakly saturated on  $[n]$  and join a new vertex to two vertices of  $[n]$ . By Lemma 2.1 the closure contains  $\text{cl}(F) = K_n$ , and Lemma 2.2 with  $K = [n]$  completes it.  $\square$

This is the argument of [TZ25a, (1)], where vertices of degree  $\delta - 1$  are added to a weakly saturated graph on  $v$  vertices.

**Lemma 2.4** (deadlock). (a) *If  $v$  is an endpoint of a step in  $G$ , then  $v$  has at least two neighbours in  $G$ . Hence a vertex of degree at most 1 in  $G$  has the same neighbourhood in  $\text{cl}(G)$ .* (b) *Let  $n \geq 4$ . If a graph  $G$  on  $n$  vertices has two adjacent vertices  $v, w$  of degree 2, then  $\text{cl}(G) \neq K_n$ .*

*Proof.* (a) In a copy of  $Q_3$  through the new edge  $vc$  the vertex  $v$  has degree 3, and its two other edges are edges of  $G$ . (b) Suppose  $\text{cl}(G) = K_n$ . Since  $n \geq 4$ , both  $v$  and  $w$  gain edges. Consider the first step at  $v$  or  $w$ , say  $vc$  (so  $c \neq w$ , as  $vw \in E(G)$ ). Just before it,  $v$  and  $w$  still have degree 2. In its certificate the three neighbours of  $v$  are  $c$  and the two neighbours of  $v$ , one of which is  $w$ ; so  $w$  is a vertex of the copy, with three edges of the copy at  $w$ , none of which is  $vc$ . These are edges of the current graph, so  $w$  has degree at least 3, a contradiction.  $\square$

**Lemma 2.5** (face gadget). *Let  $K$  be a set of at least 7 vertices which is complete in  $G$ , let  $s_0, s_1, s_2, s_3$  be four further vertices, and let  $k_0, k_1, k_2, k_3 \in K$  be distinct. If  $G$  contains the seven edges*

$$s_0s_2, \quad s_2s_1, \quad s_1s_3, \quad s_0k_0, \quad s_1k_1, \quad s_2k_2, \quad s_3k_3,$$

*then  $K \cup \{s_0, s_1, s_2, s_3\}$  is complete in  $\text{cl}(G)$ .*

*Proof.* The tuple  $(s_0, s_2, s_3, s_1, k_0, k_2, k_3, k_1)$  certifies the step  $s_0s_3$ : its pairs are  $s_0s_2, s_0s_3, s_0k_0, s_2s_1, s_2k_2, s_3s_1, s_3k_3, s_1k_1$  and four pairs inside  $K$ . (The two 4-cycles  $s_0s_2s_1s_3$  and  $k_0k_2k_1k_3$  are opposite faces, joined by the edges  $s_ik_i$ .) Next choose  $z \in K \setminus \{k_0, \dots, k_3\}$ ; the tuple  $(s_0, s_2, s_3, s_1, z, k_2, k_3, k_1)$  certifies the step  $s_0z$ , its pairs being  $s_0s_2, s_0s_3, s_0z, s_2s_1, s_2k_2, s_3s_1, s_3k_3, s_1k_1$  and four pairs inside  $K$ . Now  $s_0$  has the two neighbours  $k_0, z$  in  $K$ , and Lemma 2.2 makes  $K_1 = K \cup \{s_0\}$  complete. Then  $s_2$  (neighbours  $s_0, k_2 \in K_1$ ) and  $s_3$  (neighbours  $s_0, k_3$ ) are absorbed in the same way, and finally  $s_1$  (neighbours  $s_2, s_3$ ).  $\square$

The same gadget, on a complete set of seven vertices, appears in [Car26, Section 3], with a certificate found by computer and a gluing lemma that gives Theorem 1.3(a).

**Lemma 2.6** (the first copy). *Let  $n \geq 8$ . Then*

$$\text{wsat}(n) = 11 + \min\{|R| : R \subseteq E(K_n) \setminus E(Q), \text{cl}(Q \cup R) = K_n\}.$$

*If no such  $R$  with  $|R| = k$  exists, where  $k \leq \binom{n}{2} - 12$ , then no such  $R$  with  $|R| \leq k$  exists.*

*Proof.* Let  $F$  be weakly saturated with  $\text{wsat}(n)$  edges. Then  $F \neq K_n$ , since  $K_n - e$  is weakly saturated for every edge  $e$ . Let  $e$  be the first added edge and  $(\varphi_0, \dots, \varphi_7)$  a certificate. Relabelling the vertices by a permutation that maps  $\varphi_x$  to  $x$ , we may assume that the certificate is  $Q$ . Then  $R := E(F+e) \setminus E(Q)$  has  $|F| - 11$  elements and  $\text{cl}(Q \cup R) = \text{cl}(F+e) = \text{cl}(F) = K_n$ . Conversely, if  $\text{cl}(Q \cup R) = K_n$ , let  $e \in E(Q)$  and  $F := (Q \cup R) - e$ ; then  $e$  is addable in  $F$ , so  $\text{cl}(F) = \text{cl}(F+e) = K_n$  by Lemma 2.1, and  $F$  has  $11 + |R|$  edges. The last statement follows from the monotonicity in Lemma 2.1, by enlarging  $R$ .  $\square$

### 3 Upper bounds

The upper bounds rest on three explicit graphs, the *seeds*  $S_8, S_9, S_{11}$  of Table 2. Each consists of  $Q^-$ , the cube  $Q$  on  $\{0, \dots, 7\}$  minus the edge 01 (11 edges), and a few further edges; we write  $a-b$  for the edge  $ab$ .

| graph    | edges added to $Q^-$                     | edges |
|----------|------------------------------------------|-------|
| $S_8$    | 0-3, 0-5, 0-6, 1-2                       | 15    |
| $S_9$    | 0-3, 0-5, 1-2, 0-8, 4-8                  | 16    |
| $S_{11}$ | 0-3, 0-5, 1-8, 2-9, 2-10, 3-9, 5-10, 8-9 | 19    |

Table 2: The seeds, on the vertex sets  $\{0, \dots, n-1\}$ ; the edges of  $Q^-$  are 0-2, 0-4, 1-3, 1-5, 2-3, 2-6, 3-7, 4-5, 4-6, 5-7, 6-7.

In  $S_9$  the vertex 8 has the two neighbours 0 and 4. The subgraph of  $S_9$  on  $\{0, \dots, 7\}$  has  $14 < \text{wsat}(8)$  edges, so it is not weakly saturated in  $K_8$ ; the vertex 8 is used in the steps before  $\{0, \dots, 7\}$  becomes complete.

**Proposition 3.1.**  $S_8, S_9$  and  $S_{11}$  are weakly  $Q_3$ -saturated in  $K_8, K_9$  and  $K_{11}$ .

*Proof.* Table 3 lists steps with certificates, to be made in the order given; each certificate is checked as explained in Section 2. After the listed steps, the first seven vertices of the given absorption order form a complete set, and each further vertex of that order has two neighbours among the vertices before it; so Lemma 2.2, applied repeatedly, completes the graph.  $\square$

*Proof of Theorem 1.3.* (a) Let  $F$  be weakly saturated on  $[n]$  with  $\text{wsat}(n)$  edges,  $n \geq 8$ . Add four new vertices  $s_0, \dots, s_3$  and the seven edges of Lemma 2.5, with any four distinct vertices  $k_0, \dots, k_3 \in [n]$ . By Lemma 2.1 the closure contains  $\text{cl}(F) = K_n$ , and Lemma 2.5 with  $K = [n]$  completes it. (b) By Proposition 3.1 and Corollary 2.3,  $\text{wsat}(9) \leq 16$ ,  $\text{wsat}(10) \leq 18$ ,  $\text{wsat}(11) \leq 19$  and  $\text{wsat}(12) \leq 21$ , which are the values of  $\lfloor (7n+2)/4 \rfloor$  for  $n = 9, \dots, 12$ . Since  $\lfloor (7(n+4)+2)/4 \rfloor = \lfloor (7n+2)/4 \rfloor + 7$ , part (a) and induction on  $n$  give (b).  $\square$

The upper bounds of Theorem 1.2 are  $\text{wsat}(8) \leq 15$  ( $S_8$ ), the bounds for  $9 \leq n \leq 12$  just proved, and  $\text{wsat}(13) \leq \text{wsat}(9) + 7 = 23$ . An explicit extremal graph for  $n = 13$  is  $S_9$  together with the vertices  $9, \dots, 12$  and the edges 9-11, 11-10, 10-12, 9-0, 10-1, 11-2, 12-3. Not needed for the proofs: the package also certifies other seeds for  $n = 10, 12$  (18 and 21 edges) and the graphs with gadgets for  $13 \leq n \leq 24$  (exactly  $\lfloor (7n+2)/4 \rfloor$  edges); for the latter the certification consists of the program's assertions at every step, and the step lists are recorded by `check_note.py` (Section 7).

|                                                               |                                   |                                  |
|---------------------------------------------------------------|-----------------------------------|----------------------------------|
| $S_8$ ; absorption order 0, 1, 2, 3, 4, 5, 7   6              |                                   |                                  |
| 1. 0-1 (0, 2, 4, 6, 1, 3, 5, 7)                               | 2. 1-4 (1, 0, 2, 3, 4, 5, 6, 7)   | 3. 1-7 (1, 2, 4, 6, 7, 3, 5, 0)  |
| 4. 2-4 (2, 0, 1, 3, 4, 6, 5, 7)                               | 5. 1-6 (1, 2, 3, 0, 6, 4, 7, 5)   | 6. 0-7 (0, 2, 4, 6, 7, 3, 5, 1)  |
| 7. 2-5 (2, 0, 3, 1, 5, 4, 7, 6)                               | 8. 2-7 (2, 0, 4, 5, 7, 3, 6, 1)   | 9. 3-4 (3, 0, 1, 2, 4, 5, 6, 7)  |
| 10. 3-5 (3, 0, 1, 2, 5, 4, 7, 6)                              | 11. 4-7 (4, 0, 1, 3, 7, 5, 6, 2)  |                                  |
| $S_9$ ; absorption order 0, 1, 2, 3, 4, 5, 7   6, 8           |                                   |                                  |
| 1. 0-1 (0, 2, 4, 6, 1, 3, 5, 7)                               | 2. 1-4 (1, 0, 2, 3, 4, 5, 6, 7)   | 3. 2-8 (2, 3, 6, 7, 8, 0, 4, 5)  |
| 4. 1-6 (1, 3, 5, 0, 6, 2, 4, 8)                               | 5. 0-7 (0, 2, 4, 6, 7, 3, 5, 1)   | 6. 1-7 (1, 0, 2, 8, 7, 5, 6, 4)  |
| 7. 2-4 (2, 0, 1, 3, 4, 5, 6, 7)                               | 8. 2-5 (2, 0, 3, 1, 5, 4, 7, 6)   | 9. 2-7 (2, 0, 4, 5, 7, 3, 6, 1)  |
| 10. 3-4 (3, 0, 1, 2, 4, 5, 6, 7)                              | 11. 3-5 (3, 0, 1, 2, 5, 4, 7, 6)  | 12. 4-7 (4, 0, 1, 3, 7, 5, 6, 2) |
| $S_{11}$ ; absorption order 0, 1, 2, 3, 4, 6, 7   5, 8, 9, 10 |                                   |                                  |
| 1. 0-1 (0, 2, 4, 6, 1, 3, 5, 7)                               | 2. 8-10 (8, 1, 9, 3, 10, 5, 2, 0) | 3. 2-7 (2, 9, 10, 8, 7, 3, 5, 1) |
| 4. 0-8 (0, 2, 3, 7, 8, 10, 1, 5)                              | 5. 1-2 (1, 3, 5, 0, 2, 9, 10, 8)  | 6. 1-4 (1, 0, 2, 3, 4, 5, 6, 7)  |
| 7. 3-6 (3, 0, 2, 1, 6, 4, 7, 5)                               | 8. 0-10 (0, 3, 4, 6, 10, 2, 5, 7) | 9. 0-6 (0, 1, 10, 5, 6, 3, 2, 7) |
| 10. 0-7 (0, 1, 10, 8, 7, 3, 2, 9)                             | 11. 1-6 (1, 0, 2, 10, 6, 4, 7, 5) | 12. 1-7 (1, 2, 4, 6, 7, 3, 5, 0) |
| 13. 2-4 (2, 0, 1, 3, 4, 5, 6, 7)                              | 14. 3-4 (3, 0, 1, 2, 4, 5, 6, 7)  | 15. 4-7 (4, 0, 1, 8, 7, 2, 3, 9) |

Table 3: Steps for the seeds, to be made in the order of their numbers, with certificates  $(\varphi_0, \dots, \varphi_7)$ . After the listed steps the seven vertices before the bar are pairwise adjacent, and the remaining vertices are absorbed in the given order by Lemma 2.2.

## 4 The lower bound

For a set  $W$  of vertices of a copy  $C$  of  $Q_3$  let  $e_C(W)$  be the number of edges of  $C$  with both ends in  $W$ .

**Lemma 4.1.** *If  $|W| = w$ , then  $e_C(W) \leq \text{ex}(w)$ , where  $\text{ex}(2), \dots, \text{ex}(7) = 1, 2, 4, 5, 7, 9$ .*

*Proof.*  $Q_3$  is  $K_{4,4}$  minus a perfect matching: the classes are the strings of even and of odd weight, and  $x$  is adjacent to every string of the other class except its antipode  $x \oplus 7$ . Let  $W$  have  $i$  vertices in one class and  $j$  in the other. The  $i$  antipodes of the first kind lie in a class of size 4, so at least  $i + j - 4$  of them belong to  $W$ , and  $e_C(W) \leq ij - \max(0, i + j - 4)$ . Maximising over  $i + j = w$  gives 1 ( $w = 2$ ), 2 ( $w = 3$ ), 4 ( $i = j = 2$ ), 5 ( $i = 3, j = 2$ ), 7 ( $i = j = 3$ ) and 9 ( $i = 4, j = 3$ ).  $\square$

Let  $\alpha = \frac{5}{3}$ ,  $c = \frac{7}{3}$  and  $f(x) = \alpha x - c$ , so that  $f(8) = 11$  and  $f(2) = 1$ . The following identities are checked directly:

$$11 - 8\alpha + c = 0, \quad 2\alpha - c = 1, \quad \alpha w - c - \text{ex}(w) = 0, \frac{2}{3}, \frac{1}{3}, 1, \frac{2}{3}, \frac{1}{3} \quad (w = 2, \dots, 7). \quad (2)$$

*Proof of Theorem 1.4.* We follow the proof of [TZ25a, Theorem 2.1], with clusters that may share one vertex as in [TZ25a, Section 5.3]. Let  $F$  be weakly saturated on  $[n]$ ,  $n \geq 8$ . If  $F = K_n$  then  $|E(F)| = \binom{n}{2} \geq f(n)$ . Otherwise fix a sequence of steps  $e_1, \dots, e_N$  and certificates  $Q_1, \dots, Q_N$ ; so  $Q_i$  is a copy of  $Q_3$  through  $e_i$  in  $G_i := F + e_1 + \dots + e_i$ , and  $G_N = K_n$ . We maintain a family  $\mathcal{B}$  of *clusters*, subsets of  $[n]$  with at least 8 elements, each cluster  $B$  with a *tally*  $T(B)$ , a set of edges of  $F$  inside  $B$ , such that

- (I1) two distinct clusters share at most one vertex;
- (I2) the tallies of distinct clusters are disjoint;
- (I3)  $|T(B)| \geq f(|B|)$  for every cluster  $B$ ;

(I4) after step  $i$  has been processed, each of  $e_1, \dots, e_i$  lies inside some cluster.

By (I1), an edge lies inside at most one cluster. Start with  $\mathcal{B} = \emptyset$ . Process step  $i$  as follows. If  $e_i$  lies inside a cluster, do nothing. Otherwise let  $V = V(Q_i)$ , let  $\mathcal{I}$  be the set of clusters  $B$  with  $|B \cap V| \geq 2$ , and put  $W_B = B \cap V$  and  $w_B = |W_B|$  for  $B \in \mathcal{I}$ . Then  $2 \leq w_B \leq 7$ , since  $W_B = V$  would put  $e_i$  inside  $B$ .

(a) *Free edges.* Every edge of  $Q_i$  other than  $e_i$  is an edge of  $G_{i-1}$ , hence an edge of  $F$  or some  $e_j$  with  $j < i$ ; in the second case it lies inside a cluster, by (I4). An edge of  $Q_i$  inside a cluster  $B$  has both ends in  $W_B$ , so  $B \in \mathcal{I}$ . Let  $Fr$  be the set of edges of  $Q_i - e_i$  that lie inside no cluster. They are edges of  $F$ , and by Lemma 4.1

$$|Fr| \geq 11 - \sum_{B \in \mathcal{I}} e_{Q_i}(W_B) \geq 11 - \sum_{B \in \mathcal{I}} \text{ex}(w_B).$$

(b) *A new cluster.* Replace the clusters of  $\mathcal{I}$  by  $B^* = V \cup \bigcup_{B \in \mathcal{I}} B$ , with  $T(B^*) = Fr \cup \bigcup_{B \in \mathcal{I}} T(B)$ . This union is disjoint, and  $T(B^*)$  is disjoint from the tallies of the other clusters: an edge of  $Fr$  lies inside no cluster, a tally lies inside its cluster, and (I2) holds. Since  $|B^*| \leq 8 + \sum_{B \in \mathcal{I}} (|B| - w_B)$  and  $f$  is increasing, (I3) for the clusters in  $\mathcal{I}$  gives

$$\begin{aligned} |T(B^*)| - f(|B^*|) &\geq \left(11 - \sum \text{ex}(w_B) + \sum (\alpha|B| - c)\right) - \left(8\alpha - c + \sum \alpha(|B| - w_B)\right) \\ &= (11 - 8\alpha + c) + \sum_{B \in \mathcal{I}} (\alpha w_B - c - \text{ex}(w_B)), \end{aligned}$$

which is  $\geq 0$  by (2). (If  $\mathcal{I} = \emptyset$  this says  $|Fr| = 11 = f(8)$ .)

(c) *Restoring (I1).* While some other cluster  $D$  shares at least two vertices with  $B^*$ , replace  $B^*$  and  $D$  by  $B^* \cup D$  with tally  $T(B^*) \cup T(D)$ . By (I2) and (I3),

$$|T(B^*) \cup T(D)| \geq \alpha(|B^*| + |D|) - 2c \geq \alpha(|B^* \cup D| + 2) - 2c = f(|B^* \cup D|) + (2\alpha - c) \geq f(|B^* \cup D|).$$

Each merge lowers the number of clusters, so this ends, and then (I1) holds. (I2) and (I3) have been kept;  $e_i$  lies inside  $V \subseteq B^*$ , and every earlier cluster is contained in a current one, so (I4) holds.

*The end.* After step  $N$ , every pair  $xy$  of vertices is an edge of  $G_N = K_n$ , so it is some  $e_i$ , which lies inside a cluster, or an edge of  $F$ . For every edge  $xy$  of  $F$  inside no cluster, add the *small block*  $\{x, y\}$  with tally  $\{xy\}$ ; note  $f(2) = 1$ . Now every pair of vertices lies inside a block (a cluster or a small block), two blocks share at most one vertex, so every pair lies inside exactly one block, and the tallies are pairwise disjoint subsets of  $E(F)$ . Hence  $|E(F)| \geq \sum_B f(|B|)$ , the sum over all blocks. If  $[n]$  is a block,  $|E(F)| \geq f(n)$ . Otherwise write

$$\sum_B f(|B|) = \sum_{x \in [n]} w(x), \quad w(x) = \sum_{B \ni x} \left( \alpha - \frac{c}{|B|} \right).$$

The sets  $B \setminus \{x\}$ , for the blocks  $B$  through  $x$ , partition  $[n] \setminus \{x\}$ ; let  $b_1, \dots, b_r$  be the sizes of these blocks. Then  $\sum (b_i - 1) = n - 1$ ,  $r \geq 2$ , and every  $b_i$  is 2 or at least 8, so each term  $\alpha - c/b_i$  is  $\frac{1}{2}$  or at least  $\alpha - \frac{c}{8} = \frac{11}{8}$ . If  $r \geq 4$ , then  $w(x) \geq 2$ . If  $r = 3$ , not all  $b_i$  are 2 (that would give  $n = 4$ ), so  $w(x) \geq \frac{1}{2} + \frac{1}{2} + \frac{11}{8}$ . If  $r = 2$ , the case  $b_1 = b_2 = 2$  would give  $n = 3$ , so  $w(x) \geq \frac{1}{2} + \frac{11}{8}$ . Thus  $w(x) \geq \frac{15}{8}$  for every  $x$ , and  $|E(F)| \geq \frac{15}{8}n > f(n)$ .  $\square$

*Remark 4.2* (comparison with [TZ25a]). For a graph with  $v$  vertices and  $\ell$  edges, Terekhov and Zhukovskii put  $e_i = \min_{|S|=i} |\{\text{edges meeting } S\}| - 1$ . For  $Q_3$  one has  $e_i = 11 - \text{ex}(8 - i)$ , that is  $e_1, \dots, e_7 = 2, 4, 6, 7, 9, 10, 11$ . Their Theorem 2.1 gives the bound  $g^*(n - 8) + 11$ , where  $g^*(i)$  is the least value of  $e_{i_1} + \dots + e_{i_s}$  over  $i_1 + \dots + i_s = i$  with parts at most  $v - 1 = 7$ , whose slope

| $n$ | $k$ | conclusion                | P1        | P2          | P3      | VA    | VB             | VC    |
|-----|-----|---------------------------|-----------|-------------|---------|-------|----------------|-------|
| 8   | 3   | $\text{wsat}(8) \geq 15$  | 560       | 560         | 560     | 560   | 560            | as VB |
| 9   | 4   | $\text{wsat}(9) \geq 16$  | 4,326     | 4,326       | 4,326   | 4,326 | 4,326          | as VB |
| 10  | 6   | $\text{wsat}(10) \geq 18$ | 272,048   | 172,984     | 272,048 | both  | 236,208        | as VB |
| 11  | 7   | $\text{wsat}(11) \geq 19$ | 3,116,702 | 884,272     | –       | both  | 1,942,704      | as VB |
| 12  | 9   | $\text{wsat}(12) \geq 21$ | –         | 51,502,128  | –       | as P2 | 225,604,296    | as VB |
| 13  | 11  | $\text{wsat}(13) \geq 23$ | –         | –           | –       | –     | 35,276,745,720 | as VB |
| 13  | 10  | $\text{wsat}(13) \geq 22$ | –         | 270,778,851 | –       | as P2 | 2,424,592,215  | as VB |

Table 4: The decisive runs: numbers of sets  $R$  tested, none of which works. The counts depend on the enumeration (P1, P3 and the first mode of VA: all  $R$  with (R1); P2 and the second mode of VA: the structured enumeration; VB and VC: weighted numbers of labelled  $R$  with (R1) and (R2)). “both”: VA was run in both modes and gave the counts of P1 and P2; “as P2”: structured mode, same count as P2; “as VB”: VC, with a different reduction, gave the same weighted count as VB. The last row, which gives only  $\text{wsat}(13) \geq 22$ , is included as a consistency check.

is  $\min_{i \leq 7} e_i/i = \frac{11}{7}$ . In [TZ25a, Theorem 5.3] (stated in Section 5.2 and proved in Section 5.3) the function  $g_2^*$  has parts at most  $v - 2 = 6$ , with slope  $\min_{i \leq 6} e_i/i = e_6/6 = \frac{5}{3}$ , and the conclusion is the exact value  $\text{wsat}(K_n, F) = g_2^*(n - v) + \ell - 1$  under the hypothesis that  $\text{wsat}(K_n, F) \leq g_2^*(n - v) + \ell - 1$  for all  $n \geq v$ . For  $Q_3$  this hypothesis fails already for  $n = v = 8$ : no chord of  $Q$  is addable in  $Q$  (a copy of  $Q_3$  in  $Q + uv$  would have to omit one edge at  $u$  and one at  $v$ , hence  $uv$  itself), so by Lemma 2.6  $\text{wsat}(8) \geq 12 > 11 = g_2^*(0) + 11$ . Theorem 1.4 is the bound  $\text{wsat}(n) \geq \frac{5}{3}(n - 8) + 11$ , proved without this hypothesis. In the scheme of the proof,  $f(8) = 11$  and the conditions  $\alpha w - c \geq \text{ex}(w)$ ,  $2 \leq w \leq 7$ , are equivalent to  $\alpha \leq e_i/i$  for  $1 \leq i \leq 6$ ; so no slope larger than  $\frac{5}{3}$  can be obtained in this way. The binding case  $w = 2$  is a copy of  $Q_3$  that meets a cluster in two vertices which are adjacent in the copy.

## 5 Exact values for $n \leq 13$

The upper bounds of Theorem 1.2 are proved in Section 3. By Lemma 2.6,  $\text{wsat}(n) \geq 12 + k$  as soon as no set  $R$  of  $k$  pairs outside  $Q$  satisfies  $\text{cl}(Q \cup R) = K_n$ . The lower bounds of Theorem 1.2 therefore follow from the following computational result, with  $k = 3, 4, 6, 7, 9, 11$  for  $n = 8, 9, 10, 11, 12, 13$ .

**Proposition 5.1** (computer-assisted). *For  $(n, k) = (8, 3), (9, 4), (10, 6), (11, 7), (12, 9), (13, 11)$  there is no set  $R$  of  $k$  pairs of vertices of  $K_n$  outside  $E(Q)$  with  $\text{cl}(Q \cup R) = K_n$ .*

**Reductions.** Call the vertices  $8, \dots, n - 1$  *outside vertices*. If  $\text{cl}(Q \cup R) = K_n$ , then (R1) every outside vertex has degree at least 2 in  $Q \cup R$ , and (R2) no two adjacent vertices both have degree 2, by Lemma 2.4; (R3) every permutation of the outside vertices fixes  $Q$ , so it suffices to test one  $R$  in each orbit; and (R4) if  $R_o$  is the set of pairs of  $R$  that meet an outside vertex, then  $\text{cl}(K \cup R_o) = K_n$ , where  $K$  is the complete graph on  $\{0, \dots, 7\}$  (monotonicity). The remaining pairs of  $R$  are chords of  $Q$ .

**Programs.** Each program decides  $\text{cl}(G) = K_n$  by scanning the non-edges repeatedly and testing for a copy of  $Q_3$  through each of them, until nothing changes; it accepts a non-edge  $uv$  only after finding eight distinct vertices with the eleven required edges. The programs differ in the cube test and in the enumeration.

P1–P3 P1, `exhaust.c`: all  $k$ -subsets of the  $\binom{n}{2} - 12$  pairs outside  $Q$ , with (R1); cube test: the other neighbours  $a, b$  of  $u$  and  $c, d$  of  $v$  with edges  $ac, bd$ , then the vertices 011 and 111. P2, `exhaust2.c`: the outside vertices choose their neighbourhoods in order, with non-increasing degrees (a form of (R3)), with (R1), (R2), and then all sets of chords of the right size; same cube test. P3, `indep_exhaust.py`: as P1, in pure Python, with its own cube test and no shared code.

VA `vsearch.c`, written in the first independent verification run with a different labelling of the cube: modes as P1 and as P2.

VB `own_wsats.c`, written in the second independent verification run: the graph induced on the outside vertices runs over representatives of its isomorphism classes, the tuple of neighbourhoods of the outside vertices in  $\{0, \dots, 7\}$  over representatives under the automorphisms of that graph (R3), each representative weighted by the size of its orbit, so that weighted counts are numbers of labelled sets  $R$ ; (R1), (R2), (R4); a face-first cube test ( $a, b \in N(u)$ , a common neighbour  $x$  of  $a, b$ , a neighbour  $y$  of  $x$ , then  $c \in N(v) \cap N(a) \cap N(y)$  and  $d \in N(v) \cap N(b) \cap N(y)$ ). The closure is started from a partial closure of  $Q \cup R_o$  together with the closure of  $Q \cup R_c$  on  $\{0, \dots, 7\}$ , where  $R_c = R \setminus R_o$ ; both lie in  $\text{cl}(Q \cup R)$ , so the result is  $\text{cl}(Q \cup R)$ .

VC `r2search.c`, written in the third independent verification run: a generic cube test, which embeds the eight vertices of  $Q_3$  one by one along the twelve pattern edges, starting with  $\varphi_0 = u$ ,  $\varphi_1 = v$  ( $Q_3$  is arc-transitive); the graph induced on the outside vertices over all labelled graphs, the neighbourhoods of the outside vertices in  $\{0, \dots, 7\}$  in non-decreasing order of their codes (R3), and the chords over representatives of the orbits of  $j$ -sets of chords under the 48 automorphisms of  $Q$ , each weighted so that weighted counts are numbers of labelled sets  $R$ ; (R1), (R2), (R4); the closure started from  $\text{cl}(Q \cup R_o) \cup R_c$ .

**The runs.** Table 4 lists the decisive runs. For  $n \leq 12$  each lower bound was obtained by programs among P1–P3, with which the results were first found, and recomputed in the independent verification runs with programs of their own (VA, VB and VC). For  $n = 13$  the lower bound  $\text{wsat}(13) \geq 23$  rests on two independent programs: the run of VB with  $k = 11$ , in three parts of about 860 seconds each, tested 35,276,745,720 labelled sets  $R$  (as weighted count) and found none that works, and the run of VC with  $k = 11$ , in four parts of 685 seconds each, enumerated the same number of labelled sets with a different reduction and found none that works. VB was also run again with  $k = 11$  when this note was written, from the copy in the reproducibility package, and reproduced every recorded number (Section 7). The run of P2 and VA with  $k = 10$  proves only  $\text{wsat}(13) \geq 22$ .

**Checks of the programs.** For  $k = \text{wsat}(n) - 11$  the programs find working sets, and their numbers agree: 768 for  $n = 8$ ,  $k = 4$  (P1, P2, P3, VA, VB, VC); 6,864 for  $n = 9$ ,  $k = 5$  (all six); 554,712 for  $n = 10$ ,  $k = 7$  (P1, VA, VB without symmetry, and VC with and without symmetry); and 428,832 for  $n = 11$ ,  $k = 8$  (P1, VB with (R2) switched off, and VC with and without (R2)). VC also finds 111,021,696 working labelled sets for  $n = 12$ ,  $k = 10$ . Since the weighted counts of VC agree with these labelled counts, its weights and orbit representatives are validated. Mutated versions of VB and of VC whose cube test omits one of the eleven edges reported working sets for  $(n, k) = (8, 3), (9, 4), (10, 6)$ , so the zero counts are not vacuous. The first two verification runs also reviewed the cube test of P1 and P2 and the reductions of P2 line by line; every copy of  $Q_3$  through  $uv$  is found.

## 6 Evidence for the conjecture

Nothing in this section is proved. Theorem 1.2 proves Conjecture 1.6 for  $9 \leq n \leq 13$ ; the exhaustive search for  $n = 14$  is out of reach of the programs of Section 5 within the time we allowed.

**Local search.** A simulated annealing program (`anneal.c`) varies a set  $R$  of pairs outside  $Q$ , with the number of edges of  $\text{cl}(Q \cup R)$  as objective. For  $12 \leq n \leq 18$  it found weakly saturated graphs with  $\lfloor (7n + 2)/4 \rfloor$  edges in 20 of 20 restarts, and none with one edge fewer in 40 of 40 restarts of 60,000 moves each.

**Gadgets on a complete core.** Let  $K$  be complete, and add  $t$  new vertices and  $E$  edges, inside the new set or between the new set and  $K$ . By Lemma 2.2, a new vertex with two neighbours in  $K$  is absorbed at the cost of two edges; so the search was restricted to gadgets in which every new vertex has at most one neighbour in  $K$ . The program `gadget.c` enumerates these exhaustively, with the new vertices labelled and the neighbours in  $K$  described by a partition of the new vertices (vertices with the same neighbour in  $K$ ), on a core of  $t+8$  vertices (and also of 8 vertices for  $t = 4$ ). No gadget percolates for  $(t, E) = (3, 5), (3, 6), (4, 6), (5, 8), (6, 10), (7, 12)$ ; the last case has 80,646,629 configurations. For  $(t, E) = (4, 7)$  exactly 24 labelled configurations percolate: the 12 Hamiltonian paths on the four new vertices with all four vertices matched into  $K$  (the gadget of Lemma 2.5), and the 12 labelled 4-cycles with three vertices matched into  $K$ . Up to isomorphism these are 2 configurations; together with a suitable 4-cycle in  $K$ , each forms a copy of  $Q_3$  minus one edge. Thus, on the cores examined, no gadget of this kind with at most seven new vertices has fewer than  $\frac{7}{4}$  edges per vertex (for  $t \leq 2$  this follows from Lemma 2.4). Larger cores, and  $t \geq 8$ , were not examined.

**A heuristic remark.** Let  $B$  be a complete cluster and  $C$  a copy of  $Q_3$  that meets  $B$  in exactly two adjacent vertices. Its six other vertices carry 10 edges of  $C$  minus the new edge, a rate of  $\frac{10}{6} = \frac{5}{3}$ ; this is the case  $w = 2$  of (2), with slack 0. This suggests that an argument which only counts edges of  $F$  inside vertex sets cannot give a slope above  $\frac{5}{3}$  (for the scheme of Section 4 this is the exact statement of Remark 4.2). On the other hand, the configuration does not suffice alone: it is a gadget with  $t = 6$  and  $E = 10$  in the sense above, and no such gadget percolates on the core of 14 vertices that was examined. A proof of the slope  $\frac{7}{4}$  would have to use obstructions of this kind. This remark is a heuristic, not a theorem.

## 7 Computations and reproducibility

Everything in Sections 2 and 4, Theorem 1.3(a) and Corollary 1.5 is proved without computation; Theorem 1.3(b) and the upper bounds of Theorem 1.2 also use the certificates of Table 3, found by computer and checkable directly; the lower bounds of Theorem 1.2 rest on the exhaustive searches of Proposition 5.1; Section 6 proves nothing. The reproducibility package has four parts.

*The programs with which the results were first obtained (original/):* P1–P3 of Section 5 with the closure header `q3lib.h`; `certify_upper.py`, which runs the process on the seeds found by the search, on the face gadget on  $K_8$  and on the graphs with gadgets for  $13 \leq n \leq 24$ , and asserts all twelve edges of an explicit copy of  $Q_3$  at every step; `gadget.c`, `anneal.c` and `q3_induced.py` (the table of Lemma 4.1); and their recorded outputs.

*The programs of the first two independent verification runs*, with their recorded outputs, among them the program VB and its run for  $n = 13$ ,  $k = 11$ , are in the folders `verification_run_A/` and `verification_run_B/`.

*The programs of this note* (`writing_stage/`). The program `check_note.py` (Python standard library only) checks the tuples in the proofs of Lemmas 2.2 and 2.5; a copy of the certificates of Table 3; Lemma 4.1, the numbers  $e_i$  of Remark 4.2 and (2); the minimum  $\frac{15}{8}$  at the end of the proof of Theorem 1.4 for  $8 \leq n \leq 80$ ; Table 1 and the arithmetic of Theorem 1.3; the searches of Proposition 5.1 for  $n = 8, 9$  (optionally  $n = 10$ ) with the positive controls; and the graphs with gadgets for  $13 \leq n \leq 24$ , whose step lists it prints on request. The folder also holds its recorded output and that of the new runs of VB ( $n = 13, k = 11$ : 799 seconds per part; the smaller cases of Table 4; the positive controls).

*The programs of the third verification run* (`independent_run_2/`): VC (`r2search.c`) with its recorded runs (Table 4, the positive controls, the mutated cube test), and `check_run2.py` (standard library only), which reads Tables 1 to 3 and the graph for  $n = 13$  from the source file of this note and checks them as printed, together with Lemmas 2.2, 2.5 and 4.1 and Corollary 2.3, (2), Remark 4.2, the graphs with gadgets for  $13 \leq n \leq 28$  and the end of the proof of Theorem 1.4 for  $8 \leq n \leq 120$ .

**Verification.** The results were first obtained with the programs in `original/`. Three independent verification runs, all AI-assisted, followed; each wrote its own programs. The first two examined the results in the form in which they were first written down; the third examined the text of this note. The first run re-derived the lemmas and the proofs of Theorems 1.3 and 1.4 line by line, checked the seeds and the graphs with gadgets up to  $n = 24$  with its own closure program, replayed the 172 recorded certificates of `certify_upper.py`, and recomputed the lower bounds for  $n \leq 12$ , and the run with  $k = 10$  for  $n = 13$ , with program VA; all counts agree with those of P1 and P2. The second run re-derived the same proofs, recomputed the lower bounds with its own program VB, including the run with  $k = 11$  for  $n = 13$ , which the first computation lacked (it had used  $k = 10$ , which proves only  $\text{wsat}(13) \geq 22$ ), re-ran the original programs and reproduced their recorded outputs, compared the statements about the literature with the sources, and found the device of clusters sharing one vertex in [TZ25a, Section 5.3]. All corrections required by these two runs were made; besides precision edits, the lower bound for  $n = 13$  was based on the run of VB with  $k = 11$ , [TZ25a, Section 5.3] is cited for the method of Theorem 1.4, and the remark on counting arguments is labelled as a heuristic. When this note was written, VB was run again from the copy in the package, with all three recorded outputs for  $n = 13, k = 11$  and all its recorded smaller counts reproduced exactly, and the program `check_note.py` was written and run.

The third run re-derived all proofs line by line, first the parts written or changed when the note was written (Corollary 2.3 for  $n = 10, 12$ , the step lists of Table 3, Lemma 2.5 and the induction of Theorem 1.3, the proof of Theorem 1.4 and the provenance of each entry of Table 4); it compared the statements about the problem page, the slides, [TZ25a] (Sections 1, 2, 5.2, 5.3 and (1)), [MNS17] (definitions) and [KMM21] (abstract) with these sources, and repeated the literature search, in which it found the preprints [Car26, SS26]. With programs of its own it checked Tables 2 and 3 as printed and the other finite statements of Sections 2 to 4, and it recomputed every case of Proposition 5.1 with program VC, including  $n = 13, k = 11$ , with the same weighted counts as VB; it also re-ran the package from the archive and reproduced the recorded outputs. It found no mathematical error. The corrections it required were made: the credit to [Car26, SS26] (abstract, Sections 1 and 2, Theorem 1.3 and this section), the second independent program for  $n = 13$ , the precise citations of [TZ25a, (1)] and [TZ25a, Theorem 5.3], the failure of the hypothesis of the latter for  $n = 8$  without computation (Remark 4.2), the ranges of the bounds in the abstract and below, and qualifications in Section 6.

**Scope and priority.** The note determines  $\text{wsat}(K_n, Q_3)$  for  $n \leq 13$ , with a computer-assisted proof of the lower bounds, and proves  $\text{wsat}(K_n, Q_3) \geq (5n-7)/3$  for  $n \geq 8$  and  $\text{wsat}(K_n, Q_3) \leq \lfloor (7n+2)/4 \rfloor$  for  $n \geq 9$ ; it does not determine  $\text{wsat}(K_n, Q_3)$  for  $n \geq 14$ , nor the limit of  $\text{wsat}(K_n, Q_3)/n$ , and

Conjecture 1.6 is supported by computation only. Problem 1.1 is due to Morrison and Noel [MN16]; the version with the hypercube as host graph is due to Morrison, Noel and Scott [MNS17]. The values for  $n \leq 11$  were obtained earlier, in [TXZ25] ( $n = 8$ ), [Car26] ( $n = 8, 9, 10$ ) and [SS26] ( $n = 9, 10, 11$ ), and Theorem 1.3, with the gadget of Lemma 2.5, and the upper end of Corollary 1.5 in [Car26]; these sources are unrefereed. Corollary 2.3 is the argument of [TZ25a, (1)], a lemma of [TXZ25] and [SS26, Proposition 7.1]. The method of the proof of Theorem 1.4 is the cluster method of Terekhov and Zhukovskii [TZ25a], including clusters that share one vertex [TZ25a, Section 5.3]; we claim no novelty for the method. The existence of the limit is due to Alon [Alo85], as reported in [TZ25a]. Only the values for  $n = 12, 13$ , the bound of Theorem 1.4 for  $Q_3$  in  $K_n$ , the resulting lower end  $\frac{5}{3}$  of Corollary 1.5 and Conjecture 1.6 are, to our knowledge, not in print.

What was read: the problem page [MN16]; the slides [TXZ25] and the preprints [Car26] (version 0.1-beta, with its source file) and [SS26] (version 1.0.0) in full; of [TZ25a], in the arXiv version 2305.11043v2, whose numbering we use, Sections 1, 2, 5.2 and 5.3, the journal version not being compared; of [MNS17], in the arXiv version 1408.5488v2, the abstract and the definitions of Section 1; of [TZ25b, KMM21] the abstracts. We did not read [Alo85]; what we say about it is taken from [TZ25a]. The searches were made on 19 October 2026: queries to arXiv (titles and abstracts with “weak saturation” or “weakly saturated”, alone and with “cube” or “hypercube”), to Crossref, DataCite and zbMATH, the lists of works citing [MNS17] in OpenAlex and Semantic Scholar, a search of OpenAlex for recent works on weak saturation, three web searches, and a search of the full texts of ten arXiv papers on weak saturation (listed in the verification report) for “cube”, “ $Q_3$ ” and “hypercube”. The searches of the first two verification runs found [TXZ25] as the only source specific to Problem 1.1; those of the third run found [Car26, SS26], through OpenAlex and DataCite. No other source specific to Problem 1.1 was found; a search that finds nothing is not a proof of novelty, and no priority is claimed.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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