

On the Ohba Number of $K_{3,b}$: Exact Values for $b \leq 8$ by a Certified Exhaustive Search

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Abstract

For integers $a, b \geq 2$ let $\varphi(a, b)$ be the least $t \geq 0$ such that the join $K_{a,b} + K_t$ of the complete bipartite graph $K_{a,b}$ with the complete graph K_t is chromatic-choosable, that is, $(t+2)$ -choosable. This is the Ohba number $\tau_0(a, b)$ of $K_{a,b}$; to determine it is a question of the Open Problem Garden, attributed there to J. Allagan and posted by J. Noel. The case $a = 2$ was settled by Cano, Gutknecht, Kappaganthula, Miller, Mudrock and Thornburgh, who also showed by a construction that $K_{3,b} + K_t$ is not $(t+2)$ -choosable when $t \geq 1$ and $b \geq 1 + (t+2)\lceil(t+2)/2\rceil$; a theorem of J. Zhu and X. Zhu implies that $K_{3,b} + K_t$ is $(t+2)$ -choosable when $t \geq 1$ and $b \leq t+3$. Let $N_3(t)$ be the least b for which $K_{3,b} + K_t$ is not $(t+2)$ -choosable. We prove, by a computer-assisted exhaustive search with exact certificates, that $K_{3,6} + K_1$ is 3-choosable and that $K_{3,8} + K_2$ is 4-choosable. With the construction this gives $N_3(1) = 7$ and $N_3(2) = 9$, hence $\varphi(3, b) = \varphi(b, 3) = 1$ for $3 \leq b \leq 6$, $\varphi(3, b) = \varphi(b, 3) = 2$ for $b \in \{7, 8\}$, and $\varphi(3, b) \geq 3$ for $b \geq 9$. The search reduces the question to a covering problem for the lists of the $t+3$ vertices outside the part of size b , runs, up to renaming of colours, through all these lists in which no colour is common to the three vertices of the part of size 3 (the others are settled at once; 551 cases for $t = 1$, 297,955 for $t = 2$), and settles every case either by counting configurations or by one of 18, respectively 435, dual solutions of a linear program, which are verified in exact rational arithmetic; the certificates and a program that checks them in less than a minute are provided. No novelty is claimed for the values $\varphi(3, b)$, $5 \leq b \leq 8$: they were not found in the sources read (the problem page, the two papers named above, a 2026 survey and two zbMATH entries), but Allagan's 2009 thesis, which according to its abstract treats the Ohba numbers of the graphs $K(m, n, 1, \dots, 1) = K_{m,n} + K_t$, could not be consulted and may contain some of them; neither could two 2015 papers of Allagan with coauthors and a 2007 paper of G. L. Chia and V. L. Chia. We conjecture that $N_3(t) = 1 + (t+2)\lceil(t+2)/2\rceil$ for all t ; for $t = 3$ this is supported only by searches without certificates. The general question remains open. This is an unrefereed note.

1 Introduction

1.1 The problem

All graphs are finite and simple. A *list assignment* L of a graph G gives every vertex v a finite set $L(v)$ of colours, and it is a *k-assignment* if $|L(v)| = k$ for every v . An *L-colouring* is a proper colouring f of G with $f(v) \in L(v)$ for every v . The graph G is *k-choosable* if it has an L -colouring for every k -assignment L , and the *list chromatic number* $\chi_\ell(G)$ is the least such k . Always $\chi(G) \leq \chi_\ell(G)$, and G is *chromatic-choosable* if equality holds. The *join* $G + H$ of two graphs is their disjoint union together with all edges between them; K_t is the complete graph on t vertices, and $G + K_0 = G$. Thus

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$K_{a,b} + K_t$ is the complete $(t+2)$ -partite graph with one part of size a , one part of size b and t parts of size one, and $\chi(K_{a,b} + K_t) = t+2$. Ohba [Ohb02] proved that for every graph G the join $G + K_t$ is chromatic-choosable when t is large. For complete bipartite graphs the Open Problem Garden asks the following.

Problem 1.1 ([Ope14]). *Given $a, b \geq 2$, what is the smallest integer $t \geq 0$ such that $\chi_\ell(K_{a,b} + K_t) = \chi(K_{a,b} + K_t)$?*

The problem page attributes the question, for the time being, to Julian Allagan, since it seems to have been studied first in his thesis [All09], although the page adds that this is unclear; it was posted there by Jon Noel on 12 April 2014. As on the problem page we write $\varphi(a, b)$ for the answer. Cano, Gutknecht, Kappaganthula, Miller, Mudrock and Thornburgh [CGK⁺26] call the same number the *Ohba number* $\tau_0(a, b)$ of $K_{a,b}$ (a notion that, according to [CGK⁺26], was introduced by Allagan and Johnson [AJ15]); they write it for $a \leq b$, and they describe it as the smallest N such that $K_{a,b} + K_t$ is chromatic-choosable for every $t \geq N$. By Lemma 2.1 below the two definitions agree. Since $K_{a,b} \cong K_{b,a}$, we have $\varphi(a, b) = \varphi(b, a)$. For $a \geq 2$ and $t \geq 0$ we put

$$N_a(t) := \min\{b \geq 1 : K_{a,b} + K_t \text{ is not } (t+2)\text{-choosable}\}. \quad (1)$$

By Corollary 2.3, $\varphi(a, b) = \min\{t \geq 0 : b < N_a(t)\}$, so that determining $\varphi(a, \cdot)$ amounts to determining the nondecreasing sequence $N_a(0), N_a(1), \dots$.

1.2 Known results

The problem page lists $\varphi(a, b) = 0$ if $a = 1$ or $b = 1$, $\varphi(2, 2) = \varphi(2, 3) = 0$ and $\varphi(3, 3) = \varphi(2, 4) = 1$; the bound $\varphi(a, b) \leq a + b - 5$ for $a + b \geq 5$, which follows from the theorem of Noel, Reed and Wu [NRW15] (Ohba's conjecture) that every graph G with $|V(G)| \leq 2\chi(G) + 1$ is chromatic-choosable; the bound $\varphi(a, b) \geq \chi_\ell(K_{a,b}) - 2$; and bounds of Allagan [All09] for $\varphi(a, 2)$. Cano et al. proved $\tau_0(2, b) = \lfloor \sqrt{b} \rfloor - 1$ for every $b \geq 2$ [CGK⁺26, Corollary 11], which settles the problem for $a = 2$; in the notation (1), $N_2(t) = (t+2)^2$. For $a \geq 3$ they proved lower bounds for $\tau_0(a, b)$ by explicit list assignments [CGK⁺26, Theorems 8 and 9]; for $a = 3$ these give $N_3(t) \leq 1 + (t+2)\lceil (t+2)/2 \rceil$ when $t \geq 1$ (Proposition 3.1). They ask [CGK⁺26, Question 14] whether $\tau_0(a, b) = \Theta(\sqrt{b})$ for fixed a ; for $a \geq 3$ this needs upper bounds for $\tau_0(a, b)$, that is, lower bounds for $N_a(t)$.

Zhu and Zhu [ZZ26, Theorem 1.2 and Corollary 1.3] proved that a complete k -partite graph with at most $2k+2$ vertices is k -choosable unless k is even and the graph is $K_{4,2\star(k-1)}$ or $K_{3\star(k/2+1),1\star(k/2-1)}$ (here $K_{m\star r, \dots}$ has r parts of size m); see also [ZZ23, Theorem 1.1], where it is stated in this form. For $t \geq 1$ the graph $K_{3,t+3} + K_t$ is a complete $(t+2)$ -partite graph with $2(t+2) + 2$ vertices; it has a part of size $t+3 \geq 4$ and parts of size one, so it is neither of the two exceptions. Hence

$$N_3(t) \geq t+4 \quad (t \geq 1), \quad (2)$$

and in the same way $K_{4,3} + K_1$ is 3-choosable and $K_{4,4} + K_2$ is 4-choosable, that is, $N_4(1) \geq 4$ and $N_4(2) \geq 5$. The bound (2) is the best lower bound for $N_3(t)$ that we found in print; it is linear in t , while the upper bound is quadratic.

1.3 Results

Theorem 1.2. *$N_3(1) = 7$ and $N_3(2) = 9$. That is, $K_{3,b} + K_1$ is 3-choosable for $b \leq 6$ and not for $b = 7$, and $K_{3,b} + K_2$ is 4-choosable for $b \leq 8$ and not for $b = 9$.*

By the construction of Cano et al. (Proposition 3.1), $K_{3,7} + K_1$ is not 3-choosable and $K_{3,9} + K_2$ is not 4-choosable; by (2), $K_{3,b} + K_1$ is 3-choosable for $b \leq 4$ and $K_{3,b} + K_2$ is 4-choosable for $b \leq 5$. The part of Theorem 1.2 that was not found in the sources read (Section 1.4) is that $K_{3,5} + K_1$ and $K_{3,6} + K_1$ are 3-choosable and that $K_{3,6} + K_2$, $K_{3,7} + K_2$ and $K_{3,8} + K_2$ are 4-choosable; it suffices to prove this for $K_{3,6} + K_1$ and $K_{3,8} + K_2$. Allagan’s 2009 thesis [All09], which treats Ohba numbers of the graphs $K(m, n, 1, \dots, 1) = K_{m,n} + K_t$, could not be consulted and may contain some of these statements; no novelty is claimed for them. The proof is computer-assisted (Section 4): an exhaustive search over the lists of the vertices outside the part of size b , in which every case is closed by an exact certificate. The certificates can be checked independently of the programs that found them (Section 7). Together with the classical value $N_3(0) = 3$ (Section 5), Theorem 1.2 and Corollary 2.3 give the following.

Corollary 1.3. $\varphi(3, 2) = \varphi(2, 3) = 0$; $\varphi(3, b) = \varphi(b, 3) = 1$ for $3 \leq b \leq 6$; $\varphi(3, b) = \varphi(b, 3) = 2$ for $b \in \{7, 8\}$; and $\varphi(3, b) = \varphi(b, 3) \geq 3$ for $b \geq 9$.

For $a = 4$ we record what follows from the results in print (Proposition 5.1): $N_4(0) = 2$, $N_4(1) = 4$ and $5 \leq N_4(2) \leq 9$. All exact values found so far for $a = 3$ agree with the upper bound of Cano et al., and we conjecture that it is the truth.

Conjecture 1.4. $N_3(t) = 1 + (t + 2)\lceil(t + 2)/2\rceil$ for every $t \geq 0$; equivalently, $\varphi(3, b) = \min\{t \geq 0 : 1 + (t + 2)\lceil(t + 2)/2\rceil > b\}$ for every $b \geq 2$.

b	2	3–6	7–8	9–15	16–18	19–28
$\varphi(3, b)$, proved	0	1	2	$[3, b - 3]$	$[4, b - 3]$	$[5, b - 3]$
$\varphi(3, b)$, Conjecture 1.4	0	1	2	3	4	5

Table 1: The values of $\varphi(3, b) = \varphi(b, 3)$. The lower bounds for $b \geq 9$ come from the construction of Cano et al. (Proposition 3.1), the upper bound $b - 3$ from (2); the first three columns are Corollary 1.3.

Conjecture 1.4 is proved for $t \leq 2$; for $t = 3$ it asserts $N_3(3) = 16$, while we can only show $9 \leq N_3(3) \leq 16$, and the evidence for $t = 3$ (Section 6) comes from searches without certificates. If it holds, then $\tau_0(3, b) = \Theta(\sqrt{b})$, which answers the case $a = 3$ of Question 14 of [CGK⁺26] for the Ohba number.

1.4 What is known, what was not found in the sources read, and what stays open

The upper bounds in Theorem 1.2 are the cases $t = 1, 2$ of the construction of Cano et al., and the lower bounds $N_3(1) \geq 5$ and $N_3(2) \geq 6$ follow from the theorem of Zhu and Zhu. The value $N_3(0) = 3$ is classical, $\varphi(3, 3) = 1$ is on the problem page, and $\varphi(3, 4) = 1$ follows at once from (2) and $\chi_\ell(K_{3,3}) = 3$. The values for $a = 4$ in Proposition 5.1 are consequences of [ZZ26] and [CGK⁺26]. The reductions of Section 2 are elementary, and Lemma 2.1 is [CGK⁺26, Lemma 15] together with, for $H = K_{a,b}$, [CGK⁺26, Lemma 16]. What was not found in the sources read is the choosability of $K_{3,5} + K_1$, $K_{3,6} + K_1$, $K_{3,6} + K_2$, $K_{3,7} + K_2$ and $K_{3,8} + K_2$, that is, the bounds $N_3(1) \geq 7$ and $N_3(2) \geq 9$, and with them the exact values of $\varphi(3, b)$ and $\varphi(b, 3)$ for $5 \leq b \leq 8$. The sources read are the problem page [Ope14], the papers [CGK⁺26] and [ZZ26], the survey [VSN26], and the zbMATH review of [AJ15] and summary of [ABJ15]; the thesis [All09], the papers [AJ15, ABJ15] themselves and the paper [CC07] of G. L. Chia and V. L. Chia could not be read, and they may contain some of these values (see “Scope and priority” in Section 8). According to its abstract, which is known to us only from secondary sources, the thesis estimates the choice numbers and Ohba numbers of

the complete multipartite graphs $K(m, n, 1, \dots, 1)$ and $K(m, n, 2, \dots, 2)$, that is, among others, of the graphs $K_{m,n} + K_t$ studied here, and the problem page cites it for bounds on $\varphi(a, 2)$, $a \geq 5$. We therefore claim neither novelty nor priority for these values.

Open are $\varphi(a, b)$ for $a, b \geq 3$ in general; for $a = 3$ the values for $b \geq 9$, where the gap between the quadratic upper bound and the lower bound $N_3(t) \geq \max\{t + 4, 9\}$ ($t \geq 2$) remains; Conjecture 1.4; and Question 14 of [CGK⁺26].

2 Reductions

Lemma 2.1 ([CGK⁺26, Lemma 15]). *For every graph H , $\chi_\ell(H + K_1) \leq \chi_\ell(H) + 1$. Consequently, if $H + K_t$ is chromatic-choosable, then so is $H + K_{t'}$ for every $t' \geq t$.*

Proof. Let w be the vertex of K_1 and L a $(\chi_\ell(H) + 1)$ -assignment of $H + K_1$. Give w a colour $c \in L(w)$. The lists $L(v) \setminus \{c\}$, $v \in V(H)$, have at least $\chi_\ell(H)$ elements, so H has a proper colouring from them; it avoids c , and together with w it is an L -colouring. If $\chi_\ell(H + K_t) = \chi(H + K_t)$, then $\chi_\ell(H + K_{t+1}) \leq \chi(H + K_t) + 1 = \chi(H + K_{t+1}) \leq \chi_\ell(H + K_{t+1})$, and induction on t' gives the second statement. $\square$

For $H = K_{a,b}$ the second statement is [CGK⁺26, Lemma 16].

Lemma 2.2. *Let $2 \leq a \leq a'$, $1 \leq b \leq b'$ and $t \geq 0$. If $K_{a',b'} + K_t$ is $(t + 2)$ -choosable, then so is $K_{a,b} + K_t$. If $K_{a,b} + K_t$ is $(t + 2)$ -choosable, then $K_{a,b} + K_{t+1}$ is $(t + 3)$ -choosable.*

Proof. $K_{a,b} + K_t$ is an induced subgraph of $K_{a',b'} + K_t$. A $(t + 2)$ -assignment of the smaller graph extends to one of the larger graph by arbitrary lists, and an L -colouring of the larger graph restricts to one of the smaller graph. The second statement is Lemma 2.1, since both graphs are chromatic-choosable exactly when they are $(t + 2)$ -, respectively $(t + 3)$ -choosable. $\square$

For $a \geq 2$ the number $N_a(t)$ of (1) is finite: for $a = 2$ by [CGK⁺26, Corollary 11], and for $a \geq 3$ by Proposition 3.1 and Lemma 2.2.

Corollary 2.3. *Let $a \geq 2$ and $t \geq 0$.*

- (a) $K_{a,b} + K_t$ is $(t + 2)$ -choosable if and only if $b < N_a(t)$.
- (b) $N_a(t) \leq N_a(t + 1)$, and $N_{a'}(t) \leq N_a(t)$ for $a' \geq a$.
- (c) $\varphi(a, b) = \min\{t \geq 0 : b < N_a(t)\}$ for $b \geq 2$, and $\varphi(a, b) = \varphi(b, a)$.

Proof. (a) and (b) are Lemma 2.2. For (c): $\varphi(a, b) \leq t$ means that $K_{a,b} + K_{t'}$ is chromatic-choosable for some $t' \leq t$, which by Lemma 2.1 is equivalent to the chromatic-choosability of $K_{a,b} + K_t$, that is, to $b < N_a(t)$ by (a). Symmetry holds because $K_{a,b} \cong K_{b,a}$. $\square$

Notation. For the rest of the note $a \geq 2$, $t \geq 0$, $k := t + 2$, and $G := K_{a,b} + K_t$ has the vertex set $S \cup X \cup Y$, where $S = \{s_1, \dots, s_t\}$ is the clique, $X = \{x_1, \dots, x_a\}$ and $Y = \{y_1, \dots, y_b\}$ are the parts of sizes a and b . The *inner graph* is $H := G[S \cup X]$, the clique S joined to the independent set X . An *inner assignment* is a k -assignment L' of H . A *configuration* of L' is a set $f(S \cup X)$ of colours, where f is an L' -colouring of H , and $\mathcal{R}(L')$ is the set of all configurations. A colour is *common to X* if it lies in $L'(x)$ for every $x \in X$.

Lemma 2.4. *Let L be a k -assignment of G and L' its restriction to $S \cup X$. Then G has no L -colouring if and only if every configuration $U \in \mathcal{R}(L')$ contains $L(y)$ for some $y \in Y$.*

Proof. Every $y \in Y$ is adjacent to all of $S \cup X$, and Y is independent. If f is an L -colouring of G , then $U = f(S \cup X) \in \mathcal{R}(L')$ and $f(y) \in L(y) \setminus U$, so $L(y) \not\subseteq U$ for every y . Conversely, if $U = f(S \cup X)$ is a configuration with $L(y) \not\subseteq U$ for every y , then f extends by giving each y a colour of $L(y) \setminus U$. $\square$

Lemma 2.5. *If a colour is common to X , then L' has a configuration with $k - 1$ colours, and G is L -colourable for every k -assignment L of G that extends L' . If no colour is common to X , every configuration of L' has at least k colours, and $\mathcal{R}(L') \neq \emptyset$.*

Proof. If c is common to X , colour X with c , and then $s_1, \dots, s_t$ in turn with distinct colours from $L'(s_i) \setminus \{c\}$, which has at least $t + 1$ elements; the configuration has $t + 1 = k - 1$ colours, so it contains no k -set, and Lemma 2.4 applies. Every L' -colouring gives S t distinct colours, and X colours outside $f(S)$; without a common colour X uses at least two colours, so $|f(S \cup X)| \geq k$. An L' -colouring exists: colour S greedily with distinct colours, then each $x \in X$ with one of the $k - t = 2$ or more colours of $L'(x) \setminus f(S)$. $\square$

A family $\mathcal{F}$ of k -sets of colours *blocks* L' if every configuration of L' contains a member of $\mathcal{F}$. Let $\nu(L')$ be the least size of a blocking family. If no colour is common to X , the k -subsets of the configurations form a finite blocking family, so $1 \leq \nu(L') < \infty$ by Lemma 2.5.

Proposition 2.6. $N_a(t) = \min \nu(L')$, the minimum over all inner assignments L' without a colour common to X .

Proof. Let L' attain the minimum m , with a blocking family $\mathcal{F}$ of size m . Extend L' to $K_{a,m} + K_t$ by giving the members of $\mathcal{F}$ to the m vertices of Y ; by Lemma 2.4 there is no colouring, so $N_a(t) \leq m$. Conversely, let $b = N_a(t)$ and let L be a k -assignment of G without L -colouring. By Lemma 2.5 no colour is common to X in $L' = L|_{S \cup X}$, and by Lemma 2.4 the lists $L(y)$, $y \in Y$, form a blocking family of size at most b ; so $m \leq \nu(L') \leq N_a(t)$. $\square$

Lemma 2.7. *Let L' be an inner assignment without a colour common to X , and $\mathcal{R} = \mathcal{R}(L')$.*

(a) $\nu(L')$ is at least the number of configurations with exactly k colours.

(b) Let $y : \mathcal{R} \rightarrow \mathbb{Q}_{\geq 0}$ satisfy

$$\sum_{U \in \mathcal{R}, U \supseteq F} y_U \leq 1 \quad \text{for every } k\text{-set } F. \quad (3)$$

Then $\nu(L') \geq \sum_{U \in \mathcal{R}} y_U$. In particular, if $\sum_U y_U > N - 1$ for an integer N , then $\nu(L') \geq N$.

Proof. (a) A k -set contained in a configuration U with $|U| = k$ equals U , so distinct configurations of size k need distinct members of a blocking family. (b) Let $\mathcal{F}$ block L' . Every $U \in \mathcal{R}$ contains a member of $\mathcal{F}$, so by (3)

$$|\mathcal{F}| \geq \sum_{F \in \mathcal{F}} \sum_{U \supseteq F} y_U = \sum_{U \in \mathcal{R}} y_U |\{F \in \mathcal{F} : F \subseteq U\}| \geq \sum_{U \in \mathcal{R}} y_U.$$

Since $\nu(L')$ is an integer, $\sum_U y_U > N - 1$ implies $\nu(L') \geq N$. $\square$

Part (b) is weak duality for the linear relaxation of the covering problem ($\min \sum_F z_F$, $\sum_{F \subseteq U} z_F \geq 1$ for $U \in \mathcal{R}$, $z \geq 0$). In (3) only the k -sets F inside some U with $y_U > 0$ need to be checked.

3 The upper bounds

The following construction is, for $t \geq 1$, the one in the proofs of [CGK⁺26, Theorems 8 and 9] with their parameters $s = 0$, $a = 3$ and $l = k - 1$ (Theorem 8 for even k , where $Q = Q'$ below, and Theorem 9 for odd k), written in our notation. Those theorems assume $l \geq 2$, that is, $t \geq 1$; the argument below covers $t = 0$ as well, where it gives the classical bad 2-assignment of $K_{3,3}$.

Proposition 3.1 (Cano et al.). *Let $k \geq 2$, $t = k - 2$ and $[k] = \{1, \dots, k\}$. Split $[k] = P_1 \sqcup P_2$ with $|P_1| = \lceil k/2 \rceil$ and $|P_2| = \lfloor k/2 \rfloor$, let Q' be a set of $\lceil k/2 \rceil$ further colours and $Q \subseteq Q'$ with $|Q| = \lfloor k/2 \rfloor$. Put*

$$L'(s_i) = L'(x_3) = [k] \quad (1 \leq i \leq t), \quad L'(x_1) = ([k] \setminus P_1) \cup Q', \quad L'(x_2) = ([k] \setminus P_2) \cup Q,$$

and let $\mathcal{F}$ consist of $[k]$ and the $k\lceil k/2 \rceil$ sets $I \cup \{q\}$ with $I \subset [k]$, $|I| = k - 1$, and $q \in Q'$. Then $\mathcal{F}$ blocks L' . Hence $K_{3,b} + K_t$ is not k -choosable for $b \geq 1 + k\lceil k/2 \rceil$, that is, $N_3(t) \leq 1 + k\lceil k/2 \rceil$.

Proof. All lists have k elements. Let f be an L' -colouring of H . The set $S \cup \{x_3\}$ is a clique of size $k - 1$ whose lists equal $[k]$, so $I := f(S \cup \{x_3\})$ is a $(k - 1)$ -subset of $[k]$. The vertices x_1 and x_2 are adjacent to S and avoid $f(S) = I \setminus \{f(x_3)\}$. If $f(x_1)$ and $f(x_2)$ both lay in I , both would equal $c := f(x_3)$; but $c \notin L'(x_1)$ if $c \in P_1$, and $c \notin L'(x_2)$ if $c \in P_2$. So some x_j has $f(x_j) \notin I$. Either $f(x_j)$ is the colour of $[k] \setminus I$, and then $[k] \subseteq f(S \cup X)$, or $f(x_j) \in Q'$ (recall $Q \subseteq Q'$), and then $I \cup \{f(x_j)\} \subseteq f(S \cup X)$. In both cases $f(S \cup X)$ contains a member of $\mathcal{F}$. For $b \geq |\mathcal{F}|$ give the members of $\mathcal{F}$ to vertices of Y , and one of them to the remaining vertices; Lemma 2.4 shows that there is no colouring. $\square$

For $k = 2, 3, 4$ the construction reads as follows (with P_1 the first $\lceil k/2 \rceil$ colours of $[k]$, $Q' = \{k + 1, \dots, k + \lceil k/2 \rceil\}$ and $Q = \{k + 1, \dots, k + \lfloor k/2 \rfloor\}$):

k	$L'(s_i) = L'(x_3)$	$L'(x_1)$	$L'(x_2)$	$\mathcal{F}$ (lists of Y)
2	$\{1, 2\}$	$\{2, 3\}$	$\{1, 3\}$	$\{1, 2\}, \{1, 3\}, \{2, 3\}$
3	$\{1, 2, 3\}$	$\{3, 4, 5\}$	$\{1, 2, 4\}$	$\{1, 2, 3\}$ and $I \cup \{q\}$, $q \in \{4, 5\}$ (7 sets)
4	$\{1, 2, 3, 4\}$	$\{3, 4, 5, 6\}$	$\{1, 2, 5, 6\}$	$\{1, 2, 3, 4\}$ and $I \cup \{q\}$, $q \in \{5, 6\}$ (9 sets)

These are the witnesses for $N_3(0) \leq 3$, $N_3(1) \leq 7$ and $N_3(2) \leq 9$. The program `check_note.py` confirms by a backtracking search over the whole graph, which does not use Lemma 2.4, that the graphs $K_{3,1+k\lceil k/2 \rceil} + K_{k-2}$ with these lists have no proper colouring for $k = 2, \dots, 6$ ($b = 3, 7, 9, 16, 19$), and that every member of $\mathcal{F}$ is needed: after the removal of any one vertex of Y the graph is colourable.

4 The lower bounds: proof of Theorem 1.2

From now on $a = 3$, and the inner vertices are $v_1, \dots, v_{t+3}$ with $S = \{v_1, \dots, v_t\}$ and $X = \{v_{t+1}, v_{t+2}, v_{t+3}\}$.

4.1 Type vectors

For an inner assignment L' and a colour c let $T(c) := \{v : c \in L'(v)\}$. The *type vector* of L' is $m = (m_T)$, indexed by the nonempty subsets T of $S \cup X$, with $m_T := |\{c : T(c) = T\}|$. It satisfies

$$\sum_{T \ni v} m_T = k \quad \text{for every inner vertex } v, \quad m_T = 0 \quad \text{whenever } T \supseteq X, \quad (4)$$

where the second condition says that no colour is common to X . Conversely, every vector of non-negative integers satisfying (4) is the type vector of an inner assignment without a colour common to X (take m_T new colours for each T), and two inner assignments have the same type vector if and only if one arises from the other by renaming colours. Renaming colours maps configurations to configurations of the same size, so ν is a function $\nu(m)$ of the type vector; it is also invariant under the permutations of S and of X , which are automorphisms of H . By Proposition 2.6,

$$N_3(t) = \min\{\nu(m) : m \text{ satisfies (4)}\}, \quad (5)$$

and to prove $N_3(t) \geq N$ it suffices to show $\nu(m) \geq N$ for one vector m in each orbit of $\text{Sym}(S) \times \text{Sym}(X)$. The numbers of types, of solutions of (4) and of orbits are given in Table 2; for $t = 3$ there are 55 types and 2,586,506,581 solutions.

t	k	N	types	vectors	orbits	by Lemma 2.7(a)	certificates	smallest $\sum_U y_U$
0	2	3	6	11	5	3	2	3
1	3	7	13	551	125	107	18	15/2
2	4	9	27	297,955	27,871	27,436	435	16

Table 2: The exhaustive search for $N_3(t) \geq N$. “Vectors” are the solutions of (4), “orbits” their orbits under $\text{Sym}(S) \times \text{Sym}(X)$; every orbit is closed either by Lemma 2.7(a) or by an exact certificate for Lemma 2.7(b).

4.2 The search

The search was made with the program `enum_N3.py` (Python, with NumPy, and the linear programming solver HiGHS through SciPy). For $(t, N) = (0, 3), (1, 7), (2, 9)$ it generates all solutions of (4), by a recursion that gives each type T with $|T| \geq 2$ a multiplicity between 0 and the remaining capacity of its vertices while the singletons take the rest, and it prints their number next to a count by dynamic programming; the two agree. It keeps the lexicographically smallest vector of every orbit, builds the lists, and computes all configurations by running through the colourings of S with distinct colours and the colourings of X that avoid them. Then:

- (a) if at least N configurations have exactly k colours, then $\nu(m) \geq N$ by Lemma 2.7(a);
- (b) otherwise it solves the linear relaxation of the covering problem numerically, converts the dual solution to rational numbers, divides it by the largest left side of (3) if that exceeds 1, so that (3) holds exactly, and checks $\sum_U y_U > N - 1$ in rational arithmetic. If this holds, $\nu(m) \geq N$ by Lemma 2.7(b), and the rational vector y is stored as a certificate.

Had neither rule applied, the program would have computed $\nu(m)$ by integer programming and reported it; this did not happen for any orbit. The numerical solver only proposes the dual vectors, so the certificates do not depend on its tolerances. The smallest certified totals, 3, 15/2 and 16, are well above $N - 1 = 2, 6, 8$.

Proof of Theorem 1.2. By Proposition 3.1, $N_3(1) \leq 7$ and $N_3(2) \leq 9$. By the search, every orbit of solutions of (4) with $t = 1$ has $\nu \geq 7$ and every orbit with $t = 2$ has $\nu \geq 9$: for the orbits closed by rule (a) by Lemma 2.7(a), and for the 18 and 435 others by their certificates and Lemma 2.7(b). By (5), $N_3(1) \geq 7$ and $N_3(2) \geq 9$. $\square$

The proof rests on three finite computations: the list of solutions of (4), the configurations of the orbit representatives that are closed by rule (a), and the exact verification of the 455 certificates. All three were repeated independently (Section 4.4), and the program `check_note.py` written with this note repeats them in exact arithmetic (Section 7).

4.3 The extremal assignments

For $t \leq 1$ the program has a second mode, which computes the minimum of $\nu(m)$ over the orbits by integer programming, skipping the orbits with more configurations of size k than the current minimum. The minimum $\nu(m) = N_3(t)$ is attained by exactly 3 orbits for $t = 0$ and 2 orbits for $t = 1$; the first two verification runs of Section 4.4, with exact branch and bound instead of integer programming, found the same orbits, and for $t = 2$ both found exactly one optimal orbit, that of Proposition 3.1; the third run found the same numbers 3, 2 and 1. For $t = 1$ one optimal orbit is that of Proposition 3.1. The other is represented by $L'(s_1) = L'(x_3) = \{0, 3, 4\}$, $L'(x_1) = \{1, 2, 3\}$, $L'(x_2) = \{1, 2, 4\}$, and it is blocked by $\{0, 3, 4\}$ and the six sets $I \cup \{q\}$ with $I \subset \{0, 3, 4\}$, $|I| = 2$, and $q \in \{1, 2\}$. It is not an instance of Proposition 3.1: the lists of x_1 and x_2 meet $\{0, 3, 4\}$ in the disjoint sets $\{3\}$ and $\{4\}$, which do not cover $\{0, 3, 4\}$, and they share both new colours. These statements are computational observations and are not used in the proofs.

4.4 Independent implementations

The original computation contains a second decision procedure, `sat_N3.py`, without type vectors, orbits or linear programming: a counterexample-guided loop over a satisfiability model (python-sat with CaDiCaL 1.5.3) searches for an inner assignment with a blocking family of size b , and every bad assignment found is re-checked. It reports $K_{3,6} + K_1$ 3-choosable and $K_{3,8} + K_2$ 4-choosable and finds bad assignments for $K_{3,3}$, $K_{3,7} + K_1$, $K_{3,9} + K_2$ and $K_{3,16} + K_3$. Its “choosable” answers are solver results without a proof certificate; they agree with the search but are not used in the proof.

Three independent verification runs (see “Verification” in Section 8) wrote programs of their own. Those of the first two use no linear or integer programming, no SAT solver and no reduction to orbits. The first generates the inner assignments vertex by vertex, each new list taking colours from the classes of earlier colours with equal sets of lists and new colours, which gives every inner assignment up to renaming once; it computes ν exactly by a branch and bound in which the configurations of size k are forced and the others are covered by a depth-first search with a packing bound. It obtained $N_3(0), N_3(1), N_3(2) = 3, 7, 9$ and, as a test of the pipeline, $N_2(0), N_2(1), N_2(2) = 4, 9, 16$, in agreement with [CGK⁺26, Corollary 11]. The second generates the assignments by refining the Venn regions of the lists vertex by vertex and certifies $\nu \geq N$ by an integral argument: the configurations of size k , together with a family of further configurations that pairwise share fewer than k colours, need distinct blockers. For $t = 2$ this closed 294,180 vectors with the configurations of size k alone and 3,775 with the additional family, so that its exact branch-and-bound search for small blocking families was never needed. Both runs checked the 455 certificates in exact arithmetic with their own enumerators. The third run generates the solutions of (4) vertex by vertex, each new vertex taking colours from the existing colour classes and new colours, reduces them to orbits by a canonical form, and closes each of the 2, 18 and 435 orbits with fewer than N configurations of size k by an integral certificate of the same kind as the second run, found by a randomised greedy search: at least N configurations that pairwise share at most $k - 1$ colours (the smallest such families found have 3, 7 and 12 members). Like those of the second run, these certificates prove the lower bounds of Theorem 1.2 without linear programming. This run, too, checked the 455 certificates and their coverage of the orbits in exact arithmetic with its own enumerator.

5 Consequences

The value $N_3(0) = 3$ is classical: $K_{3,2}$ has $5 = 2\chi + 1$ vertices and is 2-choosable by [NRW15], and $K_{3,3}$ is not 2-choosable, as noted on the problem page; the case $k = 2$ of Proposition 3.1 is the usual bad assignment.

Proof of Corollary 1.3. By Corollary 2.3(c), $\varphi(3, b) = \min\{t : b < N_3(t)\}$, where $N_3(0) = 3$, $N_3(1) = 7$, $N_3(2) = 9$ (Theorem 1.2) and $N_3(t) \geq 9$ for $t \geq 2$ (Corollary 2.3(b)). This gives $\varphi(3, 2) = 0$, $\varphi(3, b) = 1$ for $3 \leq b \leq 6$, $\varphi(3, b) = 2$ for $b \in \{7, 8\}$ and $\varphi(3, b) \geq 3$ for $b \geq 9$, and symmetry gives the same for $\varphi(b, 3)$. $\square$

In Corollary 1.3 the lower bounds $\varphi(3, b) \geq 1, 2, 3$ for $b \geq 3, 7, 9$ come from Proposition 3.1, and $\varphi(3, b) \leq 1$ for $b \leq 4$ from (2); Theorem 1.2 gives $\varphi(3, b) \leq 1$ for $b \in \{5, 6\}$ and $\varphi(3, b) \leq 2$ for $b \in \{7, 8\}$.

Proposition 5.1. $N_4(0) = 2$, $N_4(1) = 4$ and $5 \leq N_4(2) \leq 9$.

Proof. $K_{4,1}$ is a star, and $K_{4,2} \cong K_{2,4}$ is not 2-choosable ([Ope14]; [CGK⁺26, Corollary 11] gives $\tau_0(2, 4) = 1$); so $N_4(0) = 2$. The graphs $K_{4,3} + K_1$ and $K_{4,4} + K_2$ are complete k -partite graphs with $2k + 2$ vertices for $k = 3$ and $k = 4$, with parts of sizes 4, 3, 1 and 4, 4, 1, 1; for the odd $k = 3$ there is no exception, and $K_{4,4} + K_2$ is neither $K_{4,2,2,2}$ nor $K_{3,3,3,1}$, so they are 3- and 4-choosable by [ZZ26, Theorem 1.2], and $N_4(1) \geq 4$, $N_4(2) \geq 5$. (Since $K_{4,3} + K_1 \cong K_{3,4} + K_1$, the bound $N_4(1) \geq 4$ also follows from (2).) The construction in the proof of [CGK⁺26, Theorem 8] with $s = 0$, $a = 4$, $l = 2$ shows $N_4(1) \leq 4$; in our notation it is $L'(s_1) = L'(x_4) = \{1, 2, 3\}$ and $L'(x_j) = \{1, 2, 3, 4\} \setminus \{j\}$ for $j = 1, 2, 3$, blocked by the four 3-subsets of $\{1, 2, 3, 4\}$: no colour is common to X , so every configuration consists of at least three colours of $\{1, 2, 3, 4\}$ (Lemma 2.5). Finally $N_4(2) \leq N_3(2) = 9$ by Corollary 2.3(b); this also follows from [CGK⁺26, Theorem 9] with $s = 0$, $a = 4$, $l = 3$. $\square$

The first two verification runs computed $N_4(1) = 4$ directly, over all 31,321 inner 3-assignments of $K_{4,b} + K_1$ without a colour common to X , up to renaming, and the SAT procedure of Section 4.4 reported $K_{4,4} + K_2$ to be 4-choosable (without a certificate); these computations agree with Proposition 5.1 and are not needed for it.

6 A conjecture, and the case $t = 3$

Conjecture 1.4 says that the construction of Proposition 3.1 is optimal. It holds for $t \leq 2$. For $t = 3$ it asserts $N_3(3) = 16$, that is, that $K_{3,15} + K_3$ is 5-choosable; rigorously we only know $9 \leq N_3(3) \leq 16$ (Corollary 2.3(b) and Proposition 3.1). As implemented, the method of Section 4 does not reach $t = 3$: (4) has 2,586,506,581 solutions. The following computations are evidence for the case $t = 3$ and nothing more; none of them covers the whole search space with certificates.

- The SAT procedure of Section 4.4 for $K_{3,15} + K_3$ was stopped undecided after about 27 minutes, when the process reached its memory limit of 4 GB; it had completed 53 iterations and collected 334,280 nogood clauses. For $K_{3,16} + K_3$ it found the assignment of Proposition 3.1 in 18 seconds.
- The second verification run proved $\nu \geq 16$, with its integral certificates, for two subfamilies: all 7,490 solutions of (4) in which the three lists of S are equal, and all 690 inner assignments in which every list of S is a copy of a list of X .
- A randomised local search of the same run (two runs of eight minutes, about 5.7 million proposed moves among inner 5-assignments with 8 to 13 colours) met no assignment with a certified lower bound below 16.

All optimal assignments found for $t = 1, 2$ (Section 4) have the lists of S equal to the list of a vertex of X . A proof that it suffices to consider such assignments would shrink the search for $t = 3$ considerably, but we do not have one.

7 What a reader can check

The computer-assisted part of Theorem 1.2 consists of finitely many exact statements, which can be checked without the programs that found them.

1. The reductions of Section 2 and Proposition 3.1 have short proofs.
2. The certificates are in the folder `certificates/` of the reproducibility package: the 2 certificates for $(t, N) = (0, 3)$ in the file `enum_N3_t0_prove3_certified.json`, and the 18 and 435 certificates for $(1, 7)$ and $(2, 9)$ in the files with `t1_prove7` and `t2_prove9` in place of `t0_prove3`; the SHA-256 sums of the three files are listed in the `README.md` of the package. A certificate gives the $t+3$ lists (first S , then X) under `lists`, and the weights y_U under `dual`, as a map from a configuration U (its colours, separated by spaces) to a fraction. To check it: the lists have k elements and no colour is common to the three lists of X ; every weighted U is a configuration, and $y_U \geq 0$; for every k -subset F of a weighted U the sum of $y_{U'}$ over the weighted $U' \supseteq F$ is at most 1; and $\sum_U y_U > N - 1$. Then $\nu \geq N$ for the instance by Lemma 2.7(b).
3. Coverage: every solution of (4) with fewer than N configurations of size k is equivalent under $\text{Sym}(S) \times \text{Sym}(X)$ and renaming of colours to the instance of a certificate. With items 1 and 2 this proves $N_3(t) \geq N$.

The program `reproducibility/check_note.py`, written with this note, does exactly this. It uses only the Python standard library and integer and rational arithmetic. Part A verifies the 455 certificates with its own enumerator of configurations (smallest totals 3, 15/2, 16); Part B generates the solutions of (4) again (11, 551 and 297,955, also counted by dynamic programming), counts their configurations of size k , compares this count with a full enumeration on 12,480 of them, and checks that the 4, 65 and 3,775 solutions with fewer than N such configurations fall into exactly the 2, 18 and 435 orbits of the certificates, among 5, 125 and 27,871 orbits in all; Part C checks the assignments of Proposition 3.1 for $k \leq 6$ and of Proposition 5.1 for $K_{4,4} + K_1$ by a backtracking search over the whole graph; Part D derives Corollary 1.3 from the values of N_3 . It runs in about 20 seconds on one core, and its recorded output is `check_note_output.txt`. Changing one weight of one certificate, or deleting one certificate, makes it fail.

8 Computations and reproducibility

All statements of Sections 2 and 3 and Proposition 5.1 are proved without computation; the lower bounds of Theorem 1.2 are proved by the certificates of Section 7. The reproducibility package (`README.md`: command lines, running times, what each program checks) contains the program `check_note.py` written with this note and the certificates (Section 7); the programs with which the results were first obtained, in `original/` with their recorded outputs: `enum_N3.py` (Section 4), `sat_N3.py` (Section 4.4), `verify_certs.py` (an exact check of the certificates with the enumerator of `sat_N3.py`) and `crosscheck_configs.py` (the two enumerators agree on 550 random inner assignments, and the integer programming value of ν agrees with an exact cover computed by SAT on 21 small instances); and the programs and recorded outputs of the three independent verification runs, in `verification_run_A/`, `verification_run_B/` and `independent_run_2/` (the third run, made on the written note in a second round of verification).

Verification. The results were first obtained with the programs in `original/`. Three independent verification runs, all AI-assisted, followed; each wrote its own programs. The first two examined the first account of the results, and neither read the other’s report. The first run re-derived the reductions of Section 2, the construction of Proposition 3.1 and the rules of the search, read the problem page, [CGK⁺26] and the survey [VSN26], and recomputed $N_2(t)$ and $N_3(t)$ for $t \leq 2$, $N_4(0)$ and $N_4(1)$ with its own generator, enumerator and exact branch and bound (Section 4.4); it tested the branch and bound against integer programming on 14,186 instances and its enumerator against brute force, checked the 455 certificates and their coverage of the orbits in exact arithmetic, checked the bad assignments by backtracking over the whole graph, and counted the solutions of (4) by a generating function. The second run wrote its programs from the definitions before reading those of the original computation; it recomputed $N_3(t)$ for $t \leq 2$ and $N_4(1)$ with its own generator and integral certificates, tested its programs on deliberately false bounds and on altered certificates, re-ran the original programs from a copy and reproduced their recorded outputs, checked the certificates exactly, made the searches for $t = 3$ of Section 6, and checked the bibliography. Both runs confirmed every value; neither found a mathematical error. Both found that the theorem of Zhu and Zhu [ZZ26] had been overlooked in the first account of the results, so that (2), $N_4(1) \geq 4$ and $N_4(2) \geq 5$ are in print and $N_3(0) = 3$ and $N_4(1) = 4$ are not new. They required, besides these corrections of the novelty statements, that the theorems of [CGK⁺26] behind Proposition 3.1 be quoted with their assumption $t \geq 1$, that the sources which could not be read be named, corrections of bibliographic data, the numbers of the stopped SAT run of Section 6, and three corrections of descriptions (the second optimal assignment for $t = 1$ in Section 4, the formulation of the survey, and that the statement for $\varphi(b, 3)$ needs only symmetry). All of these were made, and `check_note.py` was written to repeat the decisive checks exactly.

The third run examined the final text of this note, in which the proofs had been written anew. It re-derived every proof of Sections 2 to 5 line by line; compared the statements quoted from [CGK⁺26], [ZZ26], [ZZ23] and [VSN26] with these texts; read the problem page and the record of the problem corpus again; checked the bibliographic data with Crossref, zbMATH and arXiv; and repeated the literature search. Before reading the programs of the package it wrote its own (`independent_run_2/`): the generator, orbits and integral certificates of Section 4.4; an exact check of the 455 certificates and of their coverage; the bad assignments of Proposition 3.1 for $k \leq 6$, of Section 4 and of Proposition 5.1 by backtracking over the whole graph; the optimal orbits of Section 4; and the numbers of Table 2, of Section 7, of solutions for $t = 3$ and of the 31,321 assignments for $a = 4$. It then re-ran from the archive `check_note.py`, both modes of `enum_N3.py`, `verify_certs.py`, seven cases of `sat_N3.py` and five programs of the first two runs; all recorded outputs were reproduced. It found no mathematical error. It required that [ZZ26, Theorem 1.2] be quoted with its exceptions for even k only (the conclusions drawn from it were not affected), two precisions of counts (the assignments without a colour common to X , and the 18 and 435 dual solutions for $t = 1, 2$ in the abstract), the precise citation of [CGK⁺26, Lemmas 15 and 16], the reservation of the problem page about the origin of the problem, a remark in the `README.md` on the recorded outputs of the full mode of `enum_N3.py`, and the date. All of these were made.

Scope and priority. The note determines $N_3(1)$ and $N_3(2)$, and hence $\varphi(3, b)$ and $\varphi(b, 3)$ for $b \leq 8$; Problem 1.1 remains open for $a, b \geq 3$ in general, and for $a = 3$ when $b \geq 9$. The problem is from the Open Problem Garden [Ope14], where it is attributed to J. Allagan and was posted by J. Noel; it concerns the joins of Ohba’s theorem [Ohb02], and the bound $\varphi(a, b) \leq a + b - 5$ comes from the theorem of Noel, Reed and Wu [NRW15] on Ohba’s conjecture. The case $a = 2$ and the constructions behind all upper bounds for $N_3(t)$ and $N_4(t)$ used here are due to Cano, Gutknecht, Kappaganthula, Miller, Mudrock and Thornburgh [CGK⁺26], from which Lemma 2.1 is also taken; the lower bounds

(2), $N_4(1) \geq 4$ and $N_4(2) \geq 5$ follow from the theorem of J. Zhu and X. Zhu [ZZ26, Theorem 1.2] (see also [ZZ23]). The choosability of $K_{3,5} + K_1$, $K_{3,6} + K_1$ and $K_{3,b} + K_2$ for $6 \leq b \leq 8$, and the exact values $\varphi(3, b) = \varphi(b, 3)$ for $5 \leq b \leq 8$, were not found in the sources read; Allagan’s 2009 thesis, which treats Ohba numbers of $K(m, n, 1, \dots, 1)$, could not be consulted and may contain some of them, and no novelty is claimed.

What was read: the problem page, on 15 and 16 October 2026; [CGK⁺26] in the arXiv version v3, Sections 1 and 2, the proofs of Theorems 6, 8 and 9 and the statements of Section 4, the journal version not being compared; [ZZ26] in the arXiv version v2, the introduction with Theorem 1.2 and its corollaries, whose proofs were not checked; of [ZZ23], in the arXiv version v1, the abstract and the introduction with its Theorem 1.1; the survey [VSN26], whose Section 11.1 states Problem 1.1 as open, with “ $\leq$ ” in place of “ $=$ ” (equivalent, as $\chi_\ell \geq \chi$), and which does not cite [CGK⁺26]; the zbMATH review of [AJ15] and the zbMATH summary of [ABJ15]; and the abstracts of [ABS⁺26], which concerns $\chi_\ell(K_{m,n})$ only, and of [KMM26, BCE⁺25], the two works citing [CGK⁺26] that Semantic Scholar lists, neither of which concerns $\tau_0(a, b)$ for $a \geq 3$. Not read: Allagan’s thesis [All09], whose repository could not be reached, and whose abstract, known to us only from secondary sources, concerns the Ohba numbers of $K(m, n, 1, \dots, 1)$ and $K(m, n, 2, \dots, 2)$; the full texts of [AJ15] and [ABJ15]; and [CC07], for which neither zbMATH nor Crossref has an abstract and whose publisher’s page was not accessible. According to zbMATH, [AJ15] estimates the choice numbers and Ohba numbers of $K(m, n, 1, \dots, 1)$ and $K(m, n, 2, \dots, 2)$, and [ABJ15] presents chromatic-choosable graphs beyond the theorem of Noel, Reed and Wu. Each of these four works may contain some of the values above. [CGK⁺26], which cites [All09] and [AJ15] for the bounds for $a = 2$ and the notion of Ohba number, proposes upper bounds for $\tau_0(a, b)$, $a \geq 3$, as future research.

The searches were made on 15 and 16 October 2026 and repeated on 16 October 2026 by the third verification run: queries to the arXiv interface (for “Ohba number” and for chromatic-choosability of joins and of complete bipartite and multipartite graphs), to zbMATH (for “Ohba number”, for titles with “Ohba” or “chromatic-choosable”, and for the works of Allagan), to Crossref and OpenAlex (which lists no work citing [CGK⁺26]), the lists of citing works in Semantic Scholar, and four web searches. No determination of $\tau_0(3, b)$ for $b \geq 5$ was found. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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