

Two Problems of Porton on Reloids: Metamonovalued Reloids Are Monovalued, and $S(S(f)) = S(f)$ Fails

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Abstract

A reloid from a set A to a set B is a filter on $A \times B$. Reloids were introduced by V. Porton as a common generalisation of binary relations and uniformities. We settle two problems on reloids from Porton's book *General Topology as Ordered Semigroup Actions* which are also listed in the Open Problem Garden. First, every metamonovalued reloid is monovalued: if $(g \sqcap h) \circ f = (g \circ f) \sqcap (h \circ f)$ holds for the identity reloid g of B and the principal reloid h of the complement of the diagonal of B , then some member of the filter f is the graph of a partial function. Together with a theorem of Porton this shows that, for nonempty families, monovalued, metamonovalued and weakly metamonovalued reloids are the same. Second, for a reloid f from a set to itself let $S(f)$ be the join of the powers f^n , $n \geq 0$. We give such an f on a countable set with $S(f) \circ S(f) \neq S(f)$ and $S(S(f)) \neq S(f)$. This answers Porton's question negatively and refutes two conjectures of the book. The identities do hold for principal reloids, in particular on finite sets, and for Porton's operator S^* . A funcoïd on the same set shows that the corresponding two conjectures for funcoïds fail as well. This is an unrefereed note.

1 Introduction

Funcoïds and reloids are the basic objects of a theory of generalised spaces developed by V. Porton in the self-published book [Por26], cited below as [P] (earlier title: *Algebraic General Topology. Volume 1*). A *reloid* from a set A to a set B is a filter on $A \times B$. Principal filters correspond to binary relations, the entourages of a uniform space form a reloid, and reloids are composed by composing the members of the filters. The book leaves many statements as conjectures. This note settles four of them:

- (I) [P, Conjecture 1363, p. 282]: every metamonovalued reloid is monovalued;
- (II) [P, Conjecture 1253, p. 261]: $S(S(f)) = S(f)$ for (1°) every endoreloid f , (2°) every endofuncoïd f ;
- (III) [P, Conjectures 1254 and 1256, p. 261]: $S(f) \circ S(f) = S(f)$ for every endoreloid f , and for every endofuncoïd f .

Here $S(f) = 1 \sqcup f \sqcup f^2 \sqcup \dots$. Statement (I) and part (1°) of (II) are also listed in the Open Problem Garden [Por13, Por08c], and they are the records OPG-57403 and OPG-751 of the UnsolvedMath dataset [ula26]. Part (2°) of (II) is [Por08b]. The definitions are recalled in Sections 2 and 6.

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Theorem 1.1. *Let f be a relop from a set A to a set B . Let $g = 1_B$ be the identity relop of B , and let h be the principal relop of the relation $\{(y, z) \in B \times B : y \neq z\}$. Then $(g \sqcap h) \circ f = (g \circ f) \sqcap (h \circ f)$ if and only if f is monovalued. Consequently every weakly metamonovalued relop, and in particular every metamonovalued relop, is monovalued.*

So (I) is true. Porton proved the converse [P, Thm. 1362, p. 282]; see Proposition 4.2. Hence monovalued, metamonovalued and weakly metamonovalued relops are the same, if the families in the definition of “metamonovalued” are nonempty. The definition in [P] literally admits the empty family; then the metamonovalued relops are the principal relops of maps $A \rightarrow B$ (Remark 4.4). Theorem 1.1 holds under both readings.

Theorem 1.2. *There is an endorelop f on a countably infinite set such that $S(f) \circ S(f) \neq S(f)$ and $S(S(f)) \neq S(f)$. Moreover, for every endorelop f the equations $S(S(f)) = S(f)$ and $S(f) \circ S(f) = S(f)$ are equivalent.*

So part (1°) of (II) and the relop part of (III) are false, and the answer to the question of [Por08c] is negative. The reason is that composition of relops does not distribute over infinite joins: a member of $S(f)$ contains, for every n , the n -th power of a member F_n of f that may shrink as n grows, whereas a member of $S(f) \circ S(f)$ contains one fixed member of f followed by arbitrarily long powers. The identities hold for principal relops, and with S^* in place of S they are theorems of [P] (Proposition 5.2).

Theorem 1.3. *There is an endofunctor φ on a countably infinite set such that $S(\varphi) \circ S(\varphi) \neq S(\varphi)$ and $S(S(\varphi)) \neq S(\varphi)$.*

So part (2°) of (II) and the functor part of (III) are false as well. All proofs are short and elementary. We write them out in full, with the conventions made explicit, because the answers depend on them.

2 Definitions and conventions

Relations. A relation from A to B is a subset of $A \times B$. For $F \subseteq A \times B$ and $G \subseteq B \times C$ put $G \circ F = \{(x, z) : (x, y) \in F \text{ and } (y, z) \in G \text{ for some } y\}$ and $F^{-1} = \{(y, x) : (x, y) \in F\}$. So F acts first, as in [P] (see the proof of Prop. 1360, p. 281). We use $\text{id}_A = \{(x, x) : x \in A\}$, the domain $\text{dom } F$, the image $\text{im } F$, and $F[X] = \{y : (x, y) \in F \text{ for some } x \in X\}$. Composition is monotone in both arguments, and $(G \cap G') \circ (F \cap F') \subseteq (G \circ F) \cap (G' \circ F')$. A relation F is *single valued* (“monovalued” in [P]) if $(x, y), (x, y') \in F$ implies $y = y'$, that is, if F is the graph of a partial function. This holds if and only if $F \circ F^{-1} \subseteq \text{id}_B$, and then $F \circ F^{-1} = \text{id}_{\text{im } F}$. For a relation F on a set U let $F^0 = \text{id}_U$, $F^{n+1} = F \circ F^n$ and $S(F) = \bigcup_{n \geq 0} F^n$, the reflexive and transitive closure. Then $S(F) \circ S(F) = S(F) = S(S(F))$, and S is monotone.

Filters and relops. A filter on a set X is a set $\mathcal{F}$ of subsets of X with $X \in \mathcal{F}$ that is closed under supersets and under finite intersections [P, Defs. 405, 413, 419, p. 73; Prop. 434, p. 75]. The *improper*

filter $\mathcal{P}(X)$, the only filter containing $\emptyset$, is allowed [P, Def. 408, Remark 410, p. 73]. A *reloid* from A to B is a triple $f = (A, B, \mathcal{F})$ with $\mathcal{F}$ a filter on $A \times B$ [P, Defs. 1010, 1012, p. 210]; formally [P] uses a poset isomorphic to the poset of these filters, which makes no difference. As in [P] we write $\text{up } f$ for $\mathcal{F}$, so $\text{up } f$ is a set of relations from A to B . $\text{RLD}(A, B)$ is the set of reloids from A to B , and an *endoreloid* on U is an element of $\text{RLD}(U, U)$. The *principal* reloid $\uparrow F$ of a relation $F \subseteq A \times B$ has $\text{up } \uparrow F = \{K \subseteq A \times B : K \supseteq F\}$ [P, Defs. 1014–1016, p. 210].

Order, joins and meets. Reloids are ordered by *reverse* inclusion of filters: $f \sqsubseteq g$ if and only if $\text{up } f \supseteq \text{up } g$ [P, Def. 1021, p. 210; Section 5.3.3, p. 75; Obvious 459, p. 77]. So $f \sqsubseteq \uparrow F$ if and only if $F \in \text{up } f$. The least reloid is $\perp = \uparrow \emptyset$, whose filter is the improper one, and the greatest is $\top = \uparrow(A \times B)$. $\text{RLD}(A, B)$ is a complete lattice. For a nonempty family $(f_i)_{i \in I}$ and a nonempty set $\mathcal{G}$ of reloids from A to B ,

$$\text{up } \bigsqcup_{i \in I} f_i = \bigcap_{i \in I} \text{up } f_i, \quad (1)$$

$$\text{up } \sqcap \mathcal{G} = \{K : K \supseteq \Gamma_1 \cap \cdots \cap \Gamma_n \text{ for some } n \geq 1, g_1, \dots, g_n \in \mathcal{G}, \Gamma_i \in \text{up } g_i\} \quad (2)$$

[P, Cor. 517, 518, p. 86; Cor. 521, p. 87]. Indeed, the right-hand side of (1) is the largest filter contained in all $\text{up } f_i$, and that of (2) is the smallest filter containing all $\text{up } g$. In particular

$$\text{up}(g \sqcap h) = \{K : K \supseteq G \cap H \text{ for some } G \in \text{up } g, H \in \text{up } h\}, \quad (3)$$

so $g \sqcap h = \perp$ if and only if $G \cap H = \emptyset$ for some $G \in \text{up } g$ and $H \in \text{up } h$. The meet of the empty family is $\top$. (With the inclusion order of filters, $\sqcap$ and $\sqcup$ would be exchanged, and (I)–(III) would be different statements.)

Composition, reverse, identity. For $f \in \text{RLD}(A, B)$ and $g \in \text{RLD}(B, C)$,

$$g \circ f = \sqcap \{\uparrow(G \circ F) : F \in \text{up } f, G \in \text{up } g\}, \quad \text{up } f^{-1} = \{F^{-1} : F \in \text{up } f\}, \quad 1_A = \uparrow \text{id}_A$$

[P, Def. 1024, p. 211 (online: Def. 1030, p. 215); Defs. 1017, 1019, p. 210; Def. 1042, p. 215]. With sets as objects and $\text{RLD}(A, B)$ as the set of morphisms from A to B , reloids form a category [P, Section 26.5, p. 217; Thm. 1027, p. 211], partially ordered in the sense of [P, Def. 248, p. 46] by Lemma 3.2(b).

Monovalued and metamonovalued. A reloid $f \in \text{RLD}(A, B)$ is *monovalued* if $f \circ f^{-1} \sqsubseteq 1_B$ [P, Def. 256, p. 47; Obvious 1058, p. 218]; see also Remark 3.5. It is *metamonovalued* if $(\sqcap \mathcal{G}) \circ f = \sqcap_{g \in \mathcal{G}} (g \circ f)$ for every set $\mathcal{G}$ of reloids from B to a set C , and *weakly metamonovalued* if $(g \sqcap h) \circ f = (g \circ f) \sqcap (h \circ f)$ for all sets C and all $g, h \in \text{RLD}(B, C)$ [P, Defs. 265, 270, p. 48]. Metamonovalued reloids are weakly metamonovalued: take $\mathcal{G} = \{g, h\}$ [P, Obvious 274, p. 49]. Def. 265 does not exclude $\mathcal{G} = \emptyset$; see Remark 4.4.

Powers and the series S . For an endoreloid f on U let $f^0 = 1_U$ and $f^{n+1} = f \circ f^n$. Then

$$S_1(f) = \bigsqcup_{n \geq 1} f^n, \quad S(f) = \bigsqcup_{n \geq 0} f^n, \quad S^*(f) = \sqcap \{\uparrow S(F) : F \in \text{up } f\}$$

[P, Defs. 1219, 1220, p. 256; Def. 1235, p. 258; online: Defs. 1225, 1226, p. 260, and 1241, p. 262]. So S contains the identity term f^0 and S_1 does not, and S^* is a different operator.

3 Bases, composition and powers

We say that a reloid r from A to B has the base $\mathcal{B}$ if $\mathcal{B} \subseteq \text{up } r$ and every member of $\text{up } r$ contains a member of $\mathcal{B}$. Then $\text{up } r$ is the set of all relations from A to B that contain a member of $\mathcal{B}$, and $r = \perp$ if and only if $\emptyset \in \mathcal{B}$.

Lemma 3.1. *Let $\mathcal{B}$ be a nonempty set of relations from A to B that is directed downwards: for all $X, Y \in \mathcal{B}$ there is $Z \in \mathcal{B}$ with $Z \subseteq X \cap Y$. Then $\sqcap\{\uparrow X : X \in \mathcal{B}\}$ has the base $\mathcal{B}$.*

Proof. By (2), a relation K belongs to the filter of the meet if and only if $K \supseteq X_1 \cap \dots \cap X_n$ for some $n \geq 1$ and $X_i \in \mathcal{B}$. By directedness and induction on n , this holds if and only if $K \supseteq Z$ for some $Z \in \mathcal{B}$. (This is a case of [P, Thm. 572, p. 98].) $\square$

Lemma 3.2. *Let $f \in \text{RLD}(A, B)$ and $g \in \text{RLD}(B, C)$.*

- (a) $g \circ f$ has the base $\{G \circ F : F \in \text{up } f, G \in \text{up } g\}$.
- (b) Composition is monotone: if $f \sqsubseteq f'$ and $g \sqsubseteq g'$, then $g \circ f \sqsubseteq g' \circ f'$.
- (c) If $g = \uparrow G$, then $g \circ f$ has the base $\{G \circ F : F \in \text{up } f\}$. If $f = \uparrow F$, then $g \circ f$ has the base $\{G \circ F : G \in \text{up } g\}$.
- (d) $\uparrow G \circ \uparrow F = \uparrow(G \circ F)$, $1_B \circ f = f = f \circ 1_A$ and $\perp \circ f = \perp$.

Proof. (a) The family is nonempty and directed downwards, because $(G \cap G') \circ (F \cap F') \subseteq (G \circ F) \cap (G' \circ F')$. Apply Lemma 3.1. (This is [P, Lemma 1090, p. 225].) (b) Here $\text{up } f' \subseteq \text{up } f$ and $\text{up } g' \subseteq \text{up } g$, so by (a) $\text{up}(g' \circ f') \subseteq \text{up}(g \circ f)$. (c) If $G' \supseteq G$, then $G' \circ F \supseteq G \circ F$, and $G \in \text{up } \uparrow G$. So the smaller family is still a base. The second statement is symmetric. (d) The first equation follows from (c). For $g = 1_B = \uparrow \text{id}_B$ the base in (c) is $\{\text{id}_B \circ F : F \in \text{up } f\} = \text{up } f$, so $1_B \circ f = f$, and likewise $f \circ 1_A = f$. Finally $\emptyset \circ F = \emptyset$ belongs to $\text{up}(\perp \circ f)$. $\square$

Lemma 3.3. *Let f be an endoreloid on U .*

- (a) For $n \geq 1$, f^n has the base $\{F^n : F \in \text{up } f\}$.
- (b) $\text{up } S(f) = \{K \subseteq U \times U : \text{id}_U \subseteq K \text{ and for every } n \geq 1 \text{ there is } F \in \text{up } f \text{ with } F^n \subseteq K\}$.
- (c) $1_U \sqsubseteq S(f)$, $f \sqsubseteq S(f)$ and $S(f) \sqsubseteq S(f) \circ S(f) \sqsubseteq S(S(f))$.
- (d) $S(S(f)) = S(f)$ if and only if $S(f) \circ S(f) = S(f)$, if and only if $S(f) \circ S(f) \sqsubseteq S(f)$.

Proof. (a) Induction on n ; the case $n = 1$ holds because $f^1 = f \circ 1_U = f$. Let $K \in \text{up } f^{n+1}$. By Lemma 3.2(a), $K \supseteq G \circ H$ with $G \in \text{up } f$ and $H \in \text{up } f^n$, and $H \supseteq F'^n$ with $F' \in \text{up } f$. Put $F = G \cap F' \in \text{up } f$. Then $K \supseteq F \circ F^n = F^{n+1}$. Conversely $F^{n+1} = F \circ F^n \in \text{up } f^{n+1}$ for $F \in \text{up } f$. (This is [P, Exercise 1030, p. 212], where “endofunctor” evidently means endoreloid.)

(b) By (1), $\text{up } S(f) = \bigcap_{n \geq 0} \text{up } f^n$. Now use (a) and $\text{up } f^0 = \{K : K \supseteq \text{id}_U\}$.

(c) $f^0 = 1_U$ and $f^1 = f$ are among the reloids whose join is $S(f)$. By Lemma 3.2(b),(d), $S(f) = S(f) \circ 1_U \sqsubseteq S(f) \circ S(f)$, and $S(f) \circ S(f) = S(f)^2 \sqsubseteq S(S(f))$.

(d) If $S(S(f)) = S(f)$, then by (c) $S(f) \sqsubseteq S(f) \circ S(f) \sqsubseteq S(S(f)) = S(f)$. If $S(f) \circ S(f) \sqsubseteq S(f)$, then $S(f)^n \sqsubseteq S(f)$ for all $n \geq 1$ by induction, since $S(f)^{n+1} = S(f) \circ S(f)^n \sqsubseteq S(f) \circ S(f)$. Also $S(f)^0 = 1_U \sqsubseteq S(f)$. So $S(S(f)) \sqsubseteq S(f) \sqsubseteq S(S(f))$. $\square$

Lemma 3.4 ([P, Thm. 1060, 1°, p. 218]). *A reloid f is monovalued if and only if some member of $\text{up } f$ is single valued.*

Proof. By Lemma 3.2(a), $f \circ f^{-1}$ has the base $\{F_1 \circ F_2^{-1} : F_1, F_2 \in \text{up } f\}$. Now $f \circ f^{-1} \sqsubseteq \uparrow \text{id}_B$ means $\text{id}_B \in \text{up}(f \circ f^{-1})$, that is, $F_1 \circ F_2^{-1} \subseteq \text{id}_B$ for some $F_1, F_2 \in \text{up } f$. Then $F = F_1 \cap F_2 \in \text{up } f$ satisfies $F \circ F^{-1} \subseteq \text{id}_B$, so F is single valued. Conversely, if $F \in \text{up } f$ is single valued, then $F \circ F^{-1} \subseteq \text{id}_B$ belongs to the base. $\square$

Remark 3.5. [P] first calls f monovalued if $f \circ f^{-1} \sqsubseteq \text{id}_{\text{im } f}^{\text{RLD}}$ (p. 218) and states the equivalence with $f \circ f^{-1} \sqsubseteq 1_B$ as Obvious 1058. Here $\text{id}_{\text{im } f}^{\text{RLD}} \sqsubseteq 1_B$ has the base $\{\text{id}_{\text{im } F} : F \in \text{up } f\}$ [P, Defs. 1045, 1047, Thm. 1048, p. 215]. The equivalence holds: if $F_0 \in \text{up } f$ is single valued and $F \in \text{up } f$, then $F_1 = F \cap F_0$ satisfies $F_1 \circ F_1^{-1} = \text{id}_{\text{im } F_1} \subseteq \text{id}_{\text{im } F}$, so $\text{id}_{\text{im } F} \in \text{up}(f \circ f^{-1})$.

4 Metamonovalued reloids

Proof of Theorem 1.1. Let $N = \{(y, z) \in B \times B : y \neq z\}$, so $g = \uparrow \text{id}_B$ and $h = \uparrow N$ are reloids from B to B .

Left side. Since $\text{id}_B \cap N = \emptyset$, (3) gives $g \sqcap h = \perp$, and $(g \sqcap h) \circ f = \perp$ by Lemma 3.2(d).

Right side. By Lemma 3.2(d), $g \circ f = f$. By Lemma 3.2(c), $h \circ f$ has the base $\{N \circ F : F \in \text{up } f\}$. By (3), every member of $\text{up}((g \circ f) \sqcap (h \circ f))$ contains a set $F_1 \cap (N \circ F_2)$ with $F_1, F_2 \in \text{up } f$. Hence it contains $F \cap (N \circ F)$ for $F = F_1 \cap F_2 \in \text{up } f$. Each set $F \cap (N \circ F)$ with $F \in \text{up } f$ is itself a member. So $(g \circ f) \sqcap (h \circ f)$ has the base $\{F \cap (N \circ F) : F \in \text{up } f\}$, and it equals $\perp$ if and only if $F \cap (N \circ F) = \emptyset$ for some $F \in \text{up } f$.

The sets $F \cap (N \circ F)$. We have $N \circ F = \{(x, z) : (x, y) \in F \text{ for some } y \neq z\}$. So $F \cap (N \circ F)$ is the set of all $(x, z) \in F$ such that $(x, y) \in F$ for some $y \neq z$. It is empty if and only if F is single valued.

Together: the identity $(g \sqcap h) \circ f = (g \circ f) \sqcap (h \circ f)$ holds if and only if some $F \in \text{up } f$ is single valued. By Lemma 3.4 this means that f is monovalued. A weakly metamonovalued reloid satisfies the identity for all pairs in $\text{RLD}(B, C)$, in particular for this pair with $C = B$. A metamonovalued reloid is weakly metamonovalued. $\square$

No hypothesis on A and B is needed; if $B = \emptyset$, both sides are $\perp$ and the only reloid from A to B is monovalued.

Corollary 4.1. *Conjecture 1363 of [P] (online: Conjecture 1369) is true: every metamonovalued reloid is monovalued. This also settles [Por13].*

Proposition 4.2 (Porton, [P, Thm. 1362, 1°, p. 282; online: Thm. 1368, p. 286]). *Let $f \in \text{RLD}(A, B)$ be monovalued, let C be a set and let $\mathcal{G}$ be a nonempty set of reloids from B to C . Then $(\sqcap_{g \in \mathcal{G}} g) \circ f = \sqcap_{g \in \mathcal{G}} (g \circ f)$.*

Proof. “ $\sqsubseteq$ ”: $\sqcap \mathcal{G} \sqsubseteq g$ for every $g \in \mathcal{G}$, so $(\sqcap \mathcal{G}) \circ f \sqsubseteq g \circ f$ by Lemma 3.2(b).

“ $\supseteq$ ”: let $K \in \text{up}((\sqcap \mathcal{G}) \circ f)$. By Lemma 3.2(a) and (2), $K \supseteq (\Gamma_1 \cap \dots \cap \Gamma_n) \circ F$ with $F \in \text{up } f$, $n \geq 1$, $g_i \in \mathcal{G}$ and $\Gamma_i \in \text{up } g_i$. By Lemma 3.4 there is a single valued $F_0 \in \text{up } f$. Put $F' = F \cap F_0 \in \text{up } f$. Then

$$K \supseteq (\Gamma_1 \cap \dots \cap \Gamma_n) \circ F' = (\Gamma_1 \circ F') \cap \dots \cap (\Gamma_n \circ F').$$

The equality holds because F' is single valued and $n \geq 1$: if (x, z) lies in every $\Gamma_i \circ F'$, with $(x, y_i) \in F'$ and $(y_i, z) \in \Gamma_i$, then all y_i are equal. Each $\Gamma_i \circ F'$ belongs to $\text{up}(g_i \circ f)$, and this filter is contained in the filter $\text{up } \sqcap_{g \in \mathcal{G}} (g \circ f)$. So K belongs to it as well. $\square$

In [P] the theorem is stated for “a set of reloids”, and it is proved through the principal case [P, Lemma 1361, p. 282] and a restriction argument. Theorem 1.1 and Proposition 4.2 give:

Corollary 4.3. *For a reloid f from A to B the following are equivalent: (i) f is monovalued; (ii) $(\sqcap \mathcal{G}) \circ f = \sqcap_{g \in \mathcal{G}} (g \circ f)$ for every set C and every nonempty $\mathcal{G} \subseteq \text{RLD}(B, C)$; (iii) f is weakly metamonovalued; (iv) the identity of Theorem 1.1 holds.*

Remark 4.4 (The empty family). Let $\mathcal{G} = \emptyset \subseteq \text{RLD}(B, C)$. Both meets in Def. 265 are then greatest elements, and the condition reads $\top \circ f = \top$. By Lemma 3.2(c), $\top \circ f$ has the base $\{\text{dom } F \times C : F \in \text{up } f\}$. So for $C \neq \emptyset$ the condition says that $\text{dom } F = A$ for all $F \in \text{up } f$. If this holds and f is monovalued, let $\phi \in \text{up } f$ be single valued. Then ϕ is a map $A \rightarrow B$. For every $F \in \text{up } f$ the relation $F \cap \phi$ has domain A , so $F \cap \phi = \phi$ and $F \supseteq \phi$. Hence $f = \uparrow \phi$. Conversely, $\uparrow \phi$ satisfies the condition for every map $\phi: A \rightarrow B$. With Corollary 4.3 we get: if the empty family is admitted, a reloid from A to B is metamonovalued if and only if it is the principal reloid of a map $A \rightarrow B$. For example, the least reloid from a nonempty set is monovalued but not metamonovalued in this literal sense. So Proposition 4.2 needs nonempty families, and so do the corresponding implications for relations and funcoids in Prop. 388 and Thm. 989 of [P]; this is presumably the intended reading. Theorem 1.1 and Corollary 4.1 do not depend on it, since Theorem 1.1 uses a family with two members. For ordered categories in general, the accepted answer to a question of Porton on Mathematics Stack Exchange [Mat13] already states, without an example, that “monovalued implies metamonovalued” has counterexamples in the category of relations, and it proves the implication for monovalued morphisms that are in addition entirely defined. This agrees with the description above. A reloid f from A to B is entirely defined, that is, $f^{-1} \circ f \supseteq 1_A$ [P, Def. 257, p. 47], if and only if $\text{dom } F = A$ for all $F \in \text{up } f$. Indeed, by Lemma 3.2(a) $f^{-1} \circ f$ has the base $\{F_1^{-1} \circ F_2 : F_1, F_2 \in \text{up } f\}$, each of these sets contains $F^{-1} \circ F$ for $F = F_1 \cap F_2$, and $\text{id}_A \subseteq F^{-1} \circ F$ means $\text{dom } F = A$.

Remark 4.5 (Test pairs). For relations and for funcoids the statements corresponding to (I) are theorems of [P] (Prop. 388, p. 69, and Thm. 989, pp. 205–207). Their proofs test with $g = \{(a, y)\}$ and $h = \{(b, y)\}$ for two points $a \neq b$, and for funcoids with the analogous products of two ultrafilters. For reloids, pairs of points do not suffice. Let $A = \{0\}$, $B = \mathbb{N}$, and let f have the base $\{\{0\} \times \{k, k+1, \dots\} : k \in \mathbb{N}\}$. Then f is not monovalued, but for all $a \neq b$ in $\mathbb{N}$ and every point y of a set C the reloids $g = \uparrow \{(a, y)\}$ and $h = \uparrow \{(b, y)\}$ satisfy $g \circ f = \perp = h \circ f$ by Lemma 3.2(c), so both sides of the identity are $\perp$. Products of ultrafilters as test pairs for reloids are not discussed here. The pair of Theorem 1.1 deals with all members of $\text{up } f$ at once. Our first proof of Theorem 1.1 used another such pair: $\uparrow G$ and $\uparrow H$, where $G = \{(y, (y, y')) : (y, y') \in N\}$ and $H = \{(y', (y, y')) : (y, y') \in N\}$ are relations from B to the set N of the proof above. Here $G \cap H = \emptyset$, and $(G \circ F) \cap (H \circ F)$ contains $(x, (y, y'))$ whenever $(x, y), (x, y') \in F$ and $y \neq y'$.

5 The endomorphism series

Proof of Theorem 1.2. The last sentence is Lemma 3.3(d). For the example let $\mathbb{N} = \{0, 1, 2, \dots\}$, let $c_1, c_2, \dots$ be distinct objects that are not natural numbers, and let

$$U = \mathbb{N} \cup \{c_n : n \geq 1\}, \quad P = \{(i, i+1) : i \in \mathbb{N}\}, \quad F_j = P \cup \{(n, c_n) : n \geq j\} \quad (j \geq 1).$$

Since $F_1 \supseteq F_2 \supseteq \dots$, the supersets of the sets F_j form a filter on $U \times U$. Let f be the endoreloid on U with this filter, so f has the base $\{F_j : j \geq 1\}$. Put $K = \text{id}_U \cup \bigcup_{m \geq 1} (F_m)^m$.

Step 1: $K \in \text{up } S(f)$. This follows from Lemma 3.3(b), because $\text{id}_U \subseteq K$ and, for every $n \geq 1$, $F_n \in \text{up } f$ and $(F_n)^n \subseteq K$.

Step 2: $(0, c_n) \notin K$ for every $n \geq 1$. Clearly $(0, c_n) \notin \text{id}_U$. Let $m \geq 1$ and suppose $(0, c_n) \in (F_m)^m$. Then there are $u_0 = 0, u_1, \dots, u_m = c_n$ with $(u_t, u_{t+1}) \in F_m$ for $0 \leq t < m$. No pair in F_m has first entry in $\{c_1, c_2, \dots\}$. So $u_0, \dots, u_{m-1} \in \mathbb{N}$, the pairs (u_t, u_{t+1}) with $t+1 < m$ lie in P , and $u_t = t$ for $t \leq m-1$. The last pair is $(m-1, c_n) \in F_m$, which requires $n = m-1$ and $n \geq m$. This is impossible.

Step 3: $K \notin \text{up}(S(f) \circ S(f))$. Suppose the contrary. By Lemma 3.2(a) there are $K_1, K_2 \in \text{up } S(f)$ with $K_2 \circ K_1 \subseteq K$. Since $f \sqsubseteq S(f)$, we have $K_2 \in \text{up } f$, so $K_2 \supseteq F_j$ for some $j \geq 1$, and $(j, c_j) \in K_2$. By Lemma 3.3(b), $K_1 \supseteq F^j$ for some $F \in \text{up } f$. Every member of $\text{up } f$ contains P , so $K_1 \supseteq P^j \ni (0, j)$. Hence $(0, c_j) \in K_2 \circ K_1 \subseteq K$, which contradicts Step 2.

Conclusion. By Lemma 3.3(c), $\text{up } S(S(f)) \subseteq \text{up}(S(f) \circ S(f)) \subseteq \text{up } S(f)$. The relation K lies in the last set and not in the middle one. So $S(f) \circ S(f) \neq S(f)$ and $S(S(f)) \neq S(f)$. $\square$

Corollary 5.1. *Part 1° of Conjecture 1253 and Conjecture 1254 of [P] (online: Conjectures 1259 and 1260) are false, and by Lemma 3.3(d) they are the same statement. The answer to the question of [Por08c] is negative.*

Proposition 5.2 (What remains true). *Let f be an endoreloid on U .*

- (a) *If $f = \uparrow F$ is principal, then $S(f) = \uparrow S(F)$ and $S(f) \circ S(f) = S(S(f)) = S(f)$. This applies to every endoreloid on a finite set.*
- (b) *$S^*(f) \circ S^*(f) = S^*(f)$ and $S(S^*(f)) = S^*(f)$ [P, Thm. 1255 and Cor. 1252, p. 261].*
- (c) *$S(f) \sqsubseteq S(f) \circ S(f) \sqsubseteq S(S(f)) \sqsubseteq S^*(f)$.*

Proof. (a) By Lemma 3.2(d), $f^n = \uparrow(F^n)$ for all $n \geq 0$. By (1), $\text{up } S(f)$ consists of the relations containing all F^n , so $S(f) = \uparrow S(F)$. Then $S(f) \circ S(f) = \uparrow(S(F) \circ S(F)) = \uparrow S(F)$ and $S(S(f)) = \uparrow S(S(F)) = \uparrow S(F)$. A filter on a finite set contains the intersection of its members, so it is principal.

(b) The family $\{S(F) : F \in \text{up } f\}$ is directed downwards, because $S(F \cap F') \subseteq S(F) \cap S(F')$. So $S^*(f)$ has this family as a base (Lemma 3.1). If $K \in \text{up}(S^*(f) \circ S^*(f))$, then $K \supseteq S(F_2) \circ S(F_1)$ with $F_1, F_2 \in \text{up } f$, and with $F = F_1 \cap F_2$ we get $K \supseteq S(F) \circ S(F) = S(F)$. So $K \in \text{up } S^*(f)$. Conversely $S(F) = S(F) \circ S(F) \in \text{up}(S^*(f) \circ S^*(f))$ for $F \in \text{up } f$. This proves the first equation. It gives $S^*(f)^n = S^*(f)$ for $n \geq 1$. Every member of $\text{up } S^*(f)$ contains id_U , so $1_U \sqsubseteq S^*(f)$. Hence $S(S^*(f)) = 1_U \sqcup S^*(f) = S^*(f)$.

(c) The first two relations are Lemma 3.3(c). For $F \in \text{up } f$ the relation $S(F)$ contains id_U and all F^n , so $S(F) \in \text{up } S(f)$ by Lemma 3.3(b). Hence $\text{up } S^*(f) \subseteq \text{up } S(f)$, that is, $S(f) \sqsubseteq S^*(f)$. The operator S is monotone by Lemma 3.2(b). With (b) we get $S(S(f)) \sqsubseteq S(S^*(f)) = S^*(f)$. $\square$

[P] also proves $S^*(S^*(f)) = S^*(f)$ and $S^*(S(f)) = S^*(f)$ [P, Thm. 1251, Cor. 1252, pp. 260–261]. Theorem 1.2 does not affect these theorems.

Remark 5.3 (The example more closely). For the reloid f of the proof of Theorem 1.2 we have $S(f) \neq S(f) \circ S(f) = S(S(f)) = S^*(f)$. Indeed, let $K' \in \text{up}(S(f) \circ S(f))$. By Lemma 3.2(a), $K' \supseteq K_2 \circ K_1$ with $K_1, K_2 \in \text{up } S(f)$. By Lemma 3.3(b), K_1 contains id_U and, for every $n \geq 1$, a set F^n with $F \in \text{up } f$, and $F^n \supseteq P^n$. So $K_1 \supseteq Q$, where $Q = \text{id}_U \cup \bigcup_{n \geq 1} P^n = \text{id}_U \cup \{(i, i') : i < i' \text{ in } \mathbb{N}\}$. Also $K_2 \supseteq \text{id}_U \cup F_j$ for some $j \geq 1$. No pair of F_j has first entry in $\{c_1, c_2, \dots\}$. So a path in F_j runs along P and may end with one pair (n, c_n) , $n \geq j$, and

$$S(F_j) = Q \cup \{(i, c_n) : i \in \mathbb{N}, n \geq \max(i, j)\}.$$

For such a pair (i, c_n) we have $(i, n) \in Q$ and $(n, c_n) \in F_j$, and $Q = \text{id}_U \circ Q$. Hence $S(F_j) \subseteq (\text{id}_U \cup F_j) \circ Q \subseteq K_2 \circ K_1 \subseteq K'$, and $K' \in \text{up } S^*(f)$. This shows $\text{up}(S(f) \circ S(f)) \subseteq \text{up } S^*(f)$, that is, $S^*(f) \sqsubseteq S(f) \circ S(f)$, and Proposition 5.2(c) gives the equalities.

The same example works for S_1 . Along a path in F_m the natural numbers increase and no pair starts at a point c_n . So no set $(F_m)^m$ meets id_U , and $K \setminus \text{id}_U = \bigcup_{m \geq 1} (F_m)^m$ belongs to $\text{up } S_1(f) = \bigcap_{n \geq 1} \text{up } f^n$. Step 3 uses only $K_2 \in \text{up } f$ and $K_1 \in \text{up } f^j$, so it applies to members of $\text{up } S_1(f)$. Hence $K \setminus \text{id}_U \notin \text{up}(S_1(f) \circ S_1(f))$ and $S_1(f) \circ S_1(f) \not\sqsubseteq S_1(f)$.

Remark 5.4 (Two further examples). (a) Let $U = \mathbb{N} \times \mathbb{R}$, let $F_\varepsilon = \{((i, x), (i + 1, y)) : |x - y| < \varepsilon\}$, and let f have the base $\{F_\varepsilon : \varepsilon > 0\}$. Then $(F_\varepsilon)^n = \{((i, x), (i + n, y)) : |x - y| < n\varepsilon\}$. The relation $K = \text{id}_U \cup \bigcup_{n \geq 1} \{((i, x), (i + n, y)) : |x - y| < 1/n\}$ contains $(F_{1/n^2})^n$ for every n , so $K \in \text{up } S(f)$. Suppose $K_2 \circ K_1 \subseteq K$ with $K_1, K_2 \in \text{up } S(f)$. Then $K_1 \supseteq F_\delta$ for some $\delta > 0$, and for each $m \geq 1$ there is $\eta > 0$ with $K_2 \supseteq (F_\eta)^m$. For $|x - y| < \delta$ the pairs $((i, x), (i + 1, y)) \in K_1$ and $((i + 1, y), (i + 1 + m, y)) \in K_2$ give $((i, x), (i + m + 1, y)) \in K$, so $|x - y| < 1/(m + 1)$. Hence $\delta \leq 1/(m + 1)$ for all m , a contradiction. This was the first example we found.

(b) Let $U = \mathbb{R}$ and let f have the base $\{T_\varepsilon : \varepsilon > 0\}$ with $T_\varepsilon = \{(x, y) : |y - x - 1| < \varepsilon\}$; this is the translation by 1 composed with the usual uniformity of $\mathbb{R}$. Then $(T_\varepsilon)^n = \{(x, y) : |y - x - n| < n\varepsilon\}$, and the argument of (a) works with $K = \text{id}_U \cup \bigcup_{n \geq 1} \{(x, y) : |y - x - n| < 1/(4n)\}$ and $\delta \leq 1/4$: for $|s| < \delta$ the pair $(x, x + m + 1 + s) \in K_2 \circ K_1$ can lie in K only through the term $n = m + 1$.

6 Funcoids

For a set A let $\mathfrak{F}(A)$ be the set of filters on A , ordered by reverse inclusion (in [P, Section 5.38, p. 131] a poset isomorphic to it). Its least element is $\mathcal{P}(A)$, and $\uparrow X$ denotes the principal filter of $X \subseteq A$. Two filters $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}(A)$ *intersect*, written $\mathcal{X} \not\neq \mathcal{Y}$, if they have a common lower bound other than the least element [P, Def. 29, p. 19], that is, if some filter without the member $\emptyset$ contains $\mathcal{X} \cup \mathcal{Y}$. The supersets of the sets $X \cap Y$ with $X \in \mathcal{X}, Y \in \mathcal{Y}$ form the smallest filter containing $\mathcal{X} \cup \mathcal{Y}$, so

$$\mathcal{X} \not\neq \mathcal{Y} \iff X \cap Y \neq \emptyset \text{ for all } X \in \mathcal{X}, Y \in \mathcal{Y}; \quad \uparrow Y \not\neq \mathcal{X} \iff A \setminus Y \notin \mathcal{X}. \quad (4)$$

The second statement follows from the first: Y meets every member of $\mathcal{X}$ if and only if no member of $\mathcal{X}$ lies in $A \setminus Y$.

A *functoid* from A to B is a quadruple $f = (A, B, \alpha, \beta)$ with maps $\alpha: \mathfrak{F}(A) \rightarrow \mathfrak{F}(B)$ and $\beta: \mathfrak{F}(B) \rightarrow \mathfrak{F}(A)$ such that $\mathcal{Y} \not\neq \alpha\mathcal{X}$ if and only if $\mathcal{X} \not\neq \beta\mathcal{Y}$, for all $\mathcal{X} \in \mathfrak{F}(A)$ and $\mathcal{Y} \in \mathfrak{F}(B)$ [P, Def. 831, p. 172]. One writes $\langle f \rangle = \alpha$ and $f^{-1} = (B, A, \beta, \alpha)$; for filters $\mathcal{X}, \mathcal{Y}$ and sets X, Y one writes $\mathcal{X} [f] \mathcal{Y}$ if $\mathcal{Y} \not\neq \alpha\mathcal{X}$, and $X [f]^* Y$ if $\uparrow X [f] \uparrow Y$ [P, Defs. 834, 835, 839, p. 172; Def. 853, p. 174]. The composition is $(B, C, \alpha_2, \beta_2) \circ (A, B, \alpha_1, \beta_1) = (A, C, \alpha_2 \circ \alpha_1, \beta_1 \circ \beta_2)$, which is again a functoid (apply the defining condition twice), the identity functoid of A is $(A, A, \text{id}, \text{id})$, and functoids from A to B are ordered by $f \sqsubseteq g$ if and only if $[f] \subseteq [g]$ [P, Def. 847, p. 174; Def. 885, p. 184; Def. 877, p. 180; Section 25.8, p. 186]. An *endofunctoid* on U is a functoid from U to U . Def. 1220 of [P] (p. 256) defines S for the endomorphisms of any category with countable joins of morphisms. For functoids, with the composition, the identities and the order just recalled, it reads $S(\varphi) = \bigsqcup_{n \geq 0} \varphi^n$, where φ^0 is the identity functoid of U , $\varphi^{n+1} = \varphi \circ \varphi^n$, and the join is the least upper bound among the endofunctoids on U . It exists by Lemma 6.2.

Lemma 6.1. *Let $f = (A, B, \alpha, \beta)$ be a functoid.*

- (a) $\mathcal{X} [f] \mathcal{Y}$ if and only if $X [f]^* Y$ for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.
- (b) α and β are monotone.
- (c) f is determined by the relation $[f]^*$. For functoids f, g from A to B , $f \sqsubseteq g$ if and only if $[f]^* \subseteq [g]^*$. In particular $\sqsubseteq$ is a partial order.

Proof. (a) By (4), a filter $\mathcal{Y}$ intersects a filter $\mathcal{A}$ if and only if $\uparrow Y \not\neq \mathcal{A}$ for all $Y \in \mathcal{Y}$. We use this twice, and the definition of a functoid twice: $\mathcal{Y} \not\neq \alpha\mathcal{X}$ iff $\uparrow Y \not\neq \alpha\mathcal{X}$ for all $Y \in \mathcal{Y}$, iff $\mathcal{X} \not\neq \beta\uparrow Y$ for all Y , iff $\uparrow X \not\neq \beta\uparrow Y$ for all $X \in \mathcal{X}$ and Y , iff $\uparrow Y \not\neq \alpha\uparrow X$ for all X and Y . (This is [P, Thm. 856, 2°, p. 174].)

(b) Let $\mathcal{X} \subseteq \mathcal{X}'$, that is, $\mathcal{X}' \subseteq \mathcal{X}$, and let $W \in \alpha\mathcal{X}'$. If $W \notin \alpha\mathcal{X}$, then by (4) $\uparrow(B \setminus W) \neq \alpha\mathcal{X}$, hence $\mathcal{X} \neq \beta\uparrow(B \setminus W)$. Then also $\mathcal{X}' \neq \beta\uparrow(B \setminus W)$ by (4), so $\uparrow(B \setminus W) \neq \alpha\mathcal{X}'$ and $W \notin \alpha\mathcal{X}'$, a contradiction. So $\alpha\mathcal{X}' \subseteq \alpha\mathcal{X}$. The same holds for β , since f^{-1} is a functor. $\square$

(c) By (a), $[f]$ is determined by $[f]^*$, and $[f] \subseteq [g]$ if and only if $[f]^* \subseteq [g]^*$. By (4), $W \in \alpha\mathcal{X}$ iff not $\mathcal{X} [f] \uparrow(B \setminus W)$, and $V \in \beta\mathcal{Y}$ iff not $\uparrow(A \setminus V) [f] \mathcal{Y}$. So $[f]$ determines α and β . $\square$

Lemma 6.2 ([P, Thm. 880, p. 181]). *Let R be a nonempty set of functors from A to B . Then R has a least upper bound $\sqcup R$. For all $X \subseteq A$ and $Y \subseteq B$, $X [\sqcup R]^* Y$ if and only if $X [f]^* Y$ for some $f \in R$, and $\langle \sqcup R \rangle \uparrow X = \bigcap_{f \in R} \langle f \rangle \uparrow X$.*

Proof. For $X \subseteq A$ let $\mathcal{A}_X = \bigcap_{f \in R} \langle f \rangle \uparrow X$, a filter on B . If $X' \subseteq X$, then $\mathcal{A}_X \subseteq \mathcal{A}_{X'}$ by Lemma 6.1(b). For $\mathcal{X} \in \mathfrak{F}(A)$ put $\alpha\mathcal{X} = \bigcup_{X \in \mathcal{X}} \mathcal{A}_X$. This is a filter: it contains B and is closed under supersets because every $\mathcal{A}_X$ is a filter, and if $W \in \mathcal{A}_X$ and $W' \in \mathcal{A}_{X'}$ with $X, X' \in \mathcal{X}$, then $W \cap W' \in \mathcal{A}_{X \cap X'}$. By (4), for $\mathcal{Y} \in \mathfrak{F}(B)$,

$$\begin{aligned} \mathcal{Y} \neq \alpha\mathcal{X} &\iff B \setminus Y \notin \mathcal{A}_X \text{ for all } X \in \mathcal{X}, Y \in \mathcal{Y} \\ &\iff \text{for all } X \in \mathcal{X}, Y \in \mathcal{Y} \text{ there is } f \in R \text{ with } \uparrow Y \neq \langle f \rangle \uparrow X. \end{aligned}$$

Define β in the same way from the functors f^{-1} . Then $\mathcal{X} \neq \beta\mathcal{Y}$ if and only if for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ there is $f \in R$ with $\uparrow X \neq \langle f^{-1} \rangle \uparrow Y$. Since each f is a functor, the two conditions agree. So $h = (A, B, \alpha, \beta)$ is a functor, and $\mathcal{X} [h] \mathcal{Y}$ if and only if for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ there is $f \in R$ with $X [f]^* Y$. For principal filters this gives $X [h]^* Y$ if and only if $X [f]^* Y$ for some $f \in R$: by Lemma 6.1(a), $X [f]^* Y$ implies $X' [f]^* Y'$ for all $X' \supseteq X$ and $Y' \supseteq Y$. So $[h]^* = \bigcup_{f \in R} [f]^*$, and by Lemma 6.1(c) h is the least upper bound of R . Finally $\langle h \rangle \uparrow X = \bigcup_{X' \supseteq X} \mathcal{A}_{X'} = \mathcal{A}_X$. $\square$

Proof of Theorem 1.3. Let U , P and F_j be as in the proof of Theorem 1.2, and let $C = \{c_n : n \geq 1\}$. For $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}(U)$ let $\alpha\mathcal{X}$ be the set of all $W \subseteq U$ with $W \supseteq F_j[X]$ for some $j \geq 1$ and $X \in \mathcal{X}$, and let $\beta\mathcal{Y}$ be the set of all $V \subseteq U$ with $V \supseteq F_j^{-1}[Y]$ for some $j \geq 1$ and $Y \in \mathcal{Y}$. These are filters, because $F_j[X] \cap F_{j'}[X'] \supseteq F_{\max(j,j')}[X \cap X']$. By (4), $\mathcal{Y} \neq \alpha\mathcal{X}$ if and only if $F_j \cap (X \times Y) \neq \emptyset$ for all j , $X \in \mathcal{X}$, $Y \in \mathcal{Y}$, and this is also equivalent to $\mathcal{X} \neq \beta\mathcal{Y}$. So $\varphi = (U, U, \alpha, \beta)$ is an endofunctor on U . (It is the functor that [P, Section 27.1, p. 224] assigns to the reloid f of Theorem 1.2; we do not use this.)

Step 1: $\alpha^n \uparrow \{0\} = \uparrow \{n\}$ for $n \geq 0$. Let $k \in \mathbb{N}$. Every set $F_j[X]$ with $k \in X$ contains $k+1$, and $F_{k+1}[\{k\}] = \{k+1\}$. So $\alpha \uparrow \{k\} = \uparrow \{k+1\}$, and the claim follows by induction.

Step 2. By Lemma 6.2 with $R = \{\varphi^n : n \geq 0\}$, and since $\langle \varphi^n \rangle = \alpha^n$, we get $\langle S(\varphi) \rangle \uparrow \{0\} = \bigcap_{n \geq 0} \uparrow \{n\} = \uparrow \mathbb{N}$. As $C \cap \mathbb{N} = \emptyset$, the filters $\uparrow C$ and $\uparrow \mathbb{N}$ do not intersect. So $\{0\} [S(\varphi)]^* C$ does not hold.

Step 3. By the definition of composition and by Lemma 6.2, $\langle S(\varphi) \circ S(\varphi) \rangle \uparrow \{0\} = \langle S(\varphi) \rangle \uparrow \mathbb{N} = \bigcap_{n \geq 0} \alpha^n \uparrow \mathbb{N} \subseteq \alpha \uparrow \mathbb{N}$. Every member of $\alpha \uparrow \mathbb{N}$ contains a set $F_j[\mathbb{N}]$, and $c_j \in F_j[\mathbb{N}]$. So every member of $\langle S(\varphi) \circ S(\varphi) \rangle \uparrow \{0\}$ meets C , and by (4) $\{0\} [S(\varphi) \circ S(\varphi)]^* C$ holds.

Conclusion. By Steps 2 and 3, $S(\varphi) \circ S(\varphi) \neq S(\varphi)$. The join $S(S(\varphi))$ is an upper bound of $S(\varphi)^2 = S(\varphi) \circ S(\varphi)$. By Lemma 6.1(c), $\{0\} [S(S(\varphi))]^* C$ holds, so $S(S(\varphi)) \neq S(\varphi)$. $\square$

Corollary 6.3. *Part 2° of Conjecture 1253 and Conjecture 1256 of [P] (online: Conjectures 1259 and 1262) are false, and the answer to the question of [Por08b] is negative.*

Verification. Every item of [P] cited above was read on the pages of the copy built on 16 August 2026, the conjectures and the main definitions also in the copy built on 1 October 2026, and a program compared the text of the cited items in the two copies. On 8 October 2026 the online copy was still the latter, with (I)–(III) stated as conjectures; the Open Problem Garden pages, read on the same day, carry no solution. The two answers were first obtained with the pair $\uparrow G, \uparrow H$ of Remark 4.5 and

the example of Remark 5.4(a). Two independent verification runs, both AI-assisted, followed. One checked these first proofs line by line against the definitions of [P] and added Remark 5.4(b). The other derived both answers from the definitions of [P] before it saw the first proofs; it found the test pair of Theorem 1.1 and the example of Theorem 1.2, proved Proposition 4.2 and Remark 4.4, and then confirmed the first proofs. All proofs printed here were checked again when this note was written. Both runs also concluded that the funcoïd conjectures fail, by arguments that rest on Theorems 856, 859 and 880 of [P], whose proofs they did not re-check. The proof of Theorem 1.3 in Section 6 was written for this note. It uses only definitions of [P]; Lemmas 6.1 and 6.2 reprove what it needs from those theorems. A third independent verification run, also AI-assisted, then checked the finished note. It read every proof again line by line, compared Section 6 with the definitions of [P] on the pages of the book (the order of funcoïds, their composition, their joins, and the series S for funcoïds), and confirmed Theorem 1.3. It also completed the argument of Remark 5.3 and made the statement of Remark 4.5 on test pairs more precise. Finally, programs performed sanity checks on finite instances; no proof depends on them. They confirm Corollary 4.3 and Remark 4.4 for all relations F between sets with at most 3 elements (with all pairs of relations and, in small cases, all nonempty families to be composed with F), Proposition 5.2(a) for all relations on at most 4 points, and the path arguments of Theorem 1.2 and Remark 5.3 on a finite window of the example. The third run repeated these checks with separate programs. It also confirmed by enumeration that between sets with at most 2 elements the funcoïds, as defined in Section 6, are exactly those given by binary relations, and that their joins are given by unions. On a finite set every filter is principal, so these computations test the combinatorial steps only.

Scope and priority. Theorem 1.1 proves the conjecture as stated in [P] and in [Por13], for reloïds between arbitrary sets and under both readings of the definition. Theorem 1.2 answers the question of [Por08c] for the series S with the identity term; this is the meaning of S in [P] and in Porton’s draft of 2008 [Por08a], to which [Por08c] refers. Theorem 1.3 does the same for [Por08b]. The notions, the conjectures and the theorems quoted from [P] are due to V. Porton. The results of this note are elementary, and they belong to a theory that has so far been developed essentially by its author, in a self-published book. We searched for earlier solutions in the current copy of [P], on the author’s site (list of posts; searches for “metamonovalued”, “endoreloïd”, “connectedness”), on the Open Problem Garden pages, on SciLag (groups G-181201.1 and G-181201.3 of 2018, whose member problems, among them (I) and the four statements on S , are listed as open), in arXiv and zbMATH Open (no record for reloïd, endoreloïd, funcoïd or metamonovalued), in Crossref (no record), in OpenAlex (only texts of Porton himself and unrelated records), and with web searches. We found no earlier proof of (I) and no earlier counterexample to (II) or (III). This negative search is not a proof of priority; unindexed or private communications cannot be excluded.

AI-assisted tools were used to search the problem corpus and the literature, to find the counterexamples and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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