# A Negative Answer to Melnikov's Valency-Variety Problem

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**Abstract**

The valency-variety $w(G)$ of a graph $G$ is the number of distinct vertex degrees of $G$. In a problem recorded by Vizing in 1968 and later listed by Jensen and Toft, Melnikov conjectured that every graph $G$ with $n \geq 2$ vertices satisfies $\chi(G) > \lceil \lfloor w(G)/2 \rfloor / (n - w(G)) \rceil$. We show that the conjecture is false. The inequality is equivalent to $(2\chi(G) - 1)(n - w(G)) \geq n - 1$. For every $k \geq 3$ we construct explicit connected $k$-chromatic graphs that attain equality in this form, a $k$-chromatic counterexample on $14k - 5$ vertices (it has one isolated vertex), and a connected $k$-chromatic counterexample on $22k - 9$ vertices. A blow-up construction gives connected $k$-chromatic graphs with $n - w = 16n/(32k - 13) < n/(2k - 1)$, so the inequality fails by an amount that grows linearly in $n$. On the positive side, the inequality holds for all bipartite graphs, and a counting argument based on Turán's theorem shows $n - w \geq \mu_k n - O(1)$ for every $K_{k+1}$-free graph, where $\mu_3 = (3 - \sqrt{5})/4 \approx 0.191$ (Melnikov's bound asks for $1/5$, and our constructions give $16/83 \approx 0.193$). The same argument, evaluated exactly by computer and combined with the bipartite case, shows that every graph with at most 36 vertices satisfies Melnikov's inequality. So the smallest counterexample has 37 vertices. For $3 \leq k \leq 60$, the smallest $k$-chromatic counterexample has $14k-5$ vertices, and the smallest one without isolated vertices (in particular, the smallest connected one) has $22k - 9$ vertices. This is an unrefereed note.

## 1 Introduction

All graphs are finite and simple. For a graph $G$ with $n$ vertices let $w(G)$ be the number of distinct values among its vertex degrees, the *valency-variety* of $G$, and let $m(G) = n - w(G)$. If $n \geq 2$, the degrees 0 and $n - 1$ cannot both occur, so $m(G) \geq 1$. Nettleton [Net60] and Dirac [Dir64] proved the upper bound $\chi(G) \leq n - \lfloor w(G)/2 \rfloor$, which is best possible. In his 1968 survey of unsolved problems in graph theory, Vizing reports that L. S. Melnikov conjectured a matching lower bound [Viz68, p. 128 of the Russian original]. In our notation it reads

$$\chi(G) \ > \ \left\lceil \frac{\lfloor w(G)/2 \rfloor}{n - w(G)} \right\rceil \qquad \text{for every graph } G \text{ with } n \geq 2 \text{ vertices.} \tag{1}$$

Vizing writes the bound as $\gamma(L) \geq ]\lfloor w/2 \rfloor/(n-w)[+1$, where $]x[$ denotes the least integer $\geq x$; this is exactly (1). The problem appears in the Open Problem Garden [Ope13], and as record OPG-46575 of the UnsolvedMath dataset [ula26], as the question whether (1) holds. According to the Open Problem Garden, Jensen and Toft list it as *Melnikov's valency-variety problem* [JT95, p. 90], and it was also mentioned by Zykov [Zyk68], who reports that Melnikov showed the suggested lower bound would be best possible. We have not seen [JT95] or [Zyk68] ourselves; these statements are quoted from [Ope13]. We found no later work on the problem.

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We show that the answer is no. The inequality fails for every value $\chi(G) \geq 3$ of the chromatic number, and it holds when $\chi(G) \leq 2$. The following equivalent form is more convenient (Lemma 2.1):

$$(2\chi(G) - 1)\, m(G) \;\geq\; n - 1. \tag{2}$$

**Theorem 1.1.** *For all integers $k \geq 3$ and $c \geq 1$, let $a = c(c+1)/2$. The graph $F_c^{(k)}$ of Section 3 is connected, has chromatic number $k$ and minimum degree 1, and satisfies*

$$|V(F_c^{(k)})| = (2k-1)(a+1) + c - 2, \qquad m(F_c^{(k)}) = a + 1, \qquad (n-1) - (2k-1)\, m = c - 3.$$

*Consequently:*

*(a) $F_3^{(k)}$ attains equality in (2), for every $k \geq 3$;*

*(b) the disjoint union of $F_3^{(k)}$ and an isolated vertex has $14k - 5$ vertices, chromatic number $k$ and violates (1);*

*(c) for every $c \geq 4$ the connected graph $F_c^{(k)}$ violates (1); in particular $F_4^{(k)}$ has $22k - 9$ vertices.*

For $k = 3$ the counterexample in (b) has 37 vertices and 30 distinct degrees. Here (1) asks for $\chi > \lceil 15/7 \rceil = 3$, but the graph is 3-chromatic. It is described completely in Remark 3.3.

The violation in Theorem 1.1 grows only like $\sqrt{n}$. A blow-up makes it linear.

**Theorem 1.2.** *For all integers $k \geq 3$ and $t \geq 1$ there is a connected $k$-chromatic graph with $n = t(32k - 13)$ vertices and $m = 16t$, so that $(n-1) - (2k-1)\, m = 3t - 1$. In particular, the weakened inequality $(2\chi - 1)\, m \geq n - C$ fails for every constant $C$.*

On the positive side we prove the following.

**Theorem 1.3.** *(a) Inequality (1) holds for every graph with $\chi(G) \leq 2$.*

*(b) For every $k \geq 2$ there is a constant $C_k$ such that every $K_{k+1}$-free graph on $n$ vertices satisfies $m(G) \geq \mu_k n - C_k$, where*

$$\mu_k = \frac{2k - \sqrt{2(k-1)(2k-1)}}{3k-1}, \qquad \mu_3 = \frac{3 - \sqrt{5}}{4} = 0.19098\ldots$$

*(c) Every graph with at most 36 vertices satisfies (1), and so does every graph without isolated vertices with at most 56 vertices. For $3 \leq k \leq 60$, every $k$-chromatic graph violating (1) has at least $14k - 5$ vertices, and at least $22k - 9$ vertices if it has no isolated vertex.*

*Part (c) rests on an exact computer evaluation of the bound in Theorem 5.3.*

By Theorems 1.1 and 1.3(c), the smallest counterexample to (1) has 37 vertices, the smallest one without isolated vertices has 57 vertices, and both are attained by the graphs of Theorem 1.1. For $k = 3$, Theorems 1.2 and 1.3(b) give

$$0.19098\ldots = \frac{3 - \sqrt{5}}{4} \;\leq\; \liminf_{n\to\infty} \min_{|V(G)|=n,\ \chi(G)\leq 3} \frac{m(G)}{n} \;\leq\; \frac{16}{83} = 0.19277\ldots,$$

whereas (2) would require $1/5$.

The graphs $F_c^{(k)}$ have a simple shape: a complete $k$-partite "core" and an independent "periphery" whose neighbourhoods in the core are nested. We found the first counterexample by a computer search. The explicit family was then read off from the structure of the solutions.

## 2 Preliminaries

**Lemma 2.1.** *Let $n \geq 2$, $1 \leq w \leq n-1$ and $\chi \geq 1$ be integers. Then $\chi > \lceil \lfloor w/2 \rfloor/(n-w) \rceil$ if and only if $(2\chi-1)(n-w) \geq n-1$.*

*Proof.* Since $\chi$ is an integer, $\chi > \lceil x \rceil$ is equivalent to $\chi - 1 \geq x$. So the first inequality says $2(\chi-1)(n-w) \geq 2\lfloor w/2 \rfloor$. The left side is even. If $w$ is even, $2\lfloor w/2 \rfloor = w$, and an even number is $\geq w$ if and only if it is $\geq w-1$. If $w$ is odd, $2\lfloor w/2 \rfloor = w-1$. In both cases the condition is $2(\chi-1)(n-w) \geq w-1 = (n-1)-(n-w)$, that is, $(2\chi-1)(n-w) \geq n-1$. □

We write $d(v)$ for the degree of $v$, $\delta(G)$ for the minimum degree, and $[r] = \{1, \ldots, r\}$. A graph violating (1) is called a *counterexample*. By Lemma 2.1, a graph with $\chi(G) = k$ is a counterexample if and only if $(2k-1)\, m(G) \leq n-2$.

## 3 The graphs $F_c^{(k)}$

Fix integers $k \geq 3$ and $c \geq 1$, and let $a = T_c$, where $T_j = j(j+1)/2$ is the $j$-th triangular number.

**Definition 3.1.** The graph $F_c^{(k)}$ has the following vertices.

- *Core.* Sets $X_1 = \{x_1^1, \ldots, x_a^1\}$ and $X_i = \{x_1^i, \ldots, x_{a+1}^i\}$ for $2 \leq i \leq k-1$, and $C = \{\gamma_1, \ldots, \gamma_c\}$. Any two core vertices in different sets among $X_1, \ldots, X_{k-1}, C$ are adjacent.
- *Periphery.* An independent set consisting of the following vertices.
  - $Y_2 = \{y_0, \ldots, y_{a-1}\}$. The vertex $y_i$ is adjacent to $x_1^1, \ldots, x_i^1$. If $i = a - T_j$ with $1 \leq j \leq c$, then $y_i$ is also adjacent to $\gamma_1, \ldots, \gamma_j$.
  - For $3 \leq j \leq k-1$, $Y_j = \{y_1^j, \ldots, y_{a+1}^j\}$. The vertex $y_t^j$ is adjacent to all of $X_1 \cup \cdots \cup X_{j-2}$ and to $x_1^{j-1}, \ldots, x_t^{j-1}$.
  - $Z = \{z_1, \ldots, z_a\} \cup \{p_1, \ldots, p_{a+2}\}$. The vertex $z_t$ is adjacent to all of $X_1 \cup \cdots \cup X_{k-2}$ and to $x_1^{k-1}, \ldots, x_t^{k-1}$. Each $p_r$ is adjacent to all of $X_1 \cup \cdots \cup X_{k-1}$.

There are no other edges.

The numbers $a - T_j$ $(1 \leq j \leq c)$ are distinct, lie in $\{0, \ldots, a-1\}$, and $a - T_c = 0$. For $k = 3$ there are no sets $Y_j$ with $j \geq 3$.

Colour $X_1$ with colour 1, $X_i \cup Y_i$ with colour $i$ $(2 \leq i \leq k-1)$, and $C \cup Z$ with colour $k$. This colouring is proper: core edges join different parts; $Y_2$ is joined only to $X_1$ and $C$; $Y_j$ only to $X_1, \ldots, X_{j-1}$; $Z$ only to $X_1, \ldots, X_{k-1}$; and the periphery is independent. The vertices $x_1^1, x_1^2, \ldots, x_1^{k-1}, \gamma_1$ form a $K_k$. Hence $\chi(F_c^{(k)}) = k$. Every periphery vertex has a core neighbour (for $y_0$ this is $\gamma_1$), and the core is connected, so $F_c^{(k)}$ is connected.

The number of vertices is

$$n = a + (k-2)(a+1) + c + a + (k-3)(a+1) + (2a+2) = (2k-1)(a+1) + c - 2.$$

**Proposition 3.2.** *Put $N = a + (k-2)(a+1)$. The degrees in $F_c^{(k)}$ are as follows.*

*(i) The degrees of $y_0, \ldots, y_{a-1}$ are $1, 2, \ldots, a$ in some order.*

*(ii) $d(y_t^j) = a + (j-3)(a+1) + t$ for $3 \leq j \leq k-1$ and $t \in [a+1]$.*

*(iii) $d(z_t) = N - (a+1) + t$ for $t \in [a]$, and $d(p_r) = N$ for $r \in [a+2]$.*

*(iv)* $d(\gamma_l) = N + c + 1 - l$ *for* $l \in [c]$.

*(v)* $d(x^i_q) = N + c + (k-i)(a+1) + 1 - q$ *for* $2 \le i \le k-1$ *and* $q \in [a+1]$.

*(vi)* $d(x^1_q) = N + c + (k-1)(a+1) - q$ *for* $q \in [a]$.

*Consequently every integer from* $1$ *to* $W = N + c + (k-1)(a+1) - 1$ *is the degree of exactly one vertex, except* $N$*, which is the degree of the* $a+2$ *vertices* $p_r$*. So* $w(F^{(k)}_c) = W$ *and* $m(F^{(k)}_c) = a+1$.

*Proof.* (i) If $i$ is not of the form $a - T_j$, then $d(y_i) = i$. If $i = a - T_j$, then $d(y_i) = a - T_j + j = a - T_{j-1}$. Let $P = \{a - T_j : 1 \le j \le c\}$. Then $0 \in P$ and $\{a - T_{j-1} : 1 \le j \le c\} = (P \setminus \{0\}) \cup \{a\}$. So the degrees of $Y_2$ are the elements of $(\{0, \ldots, a-1\} \setminus P) \cup (P \setminus \{0\}) \cup \{a\} = \{1, \ldots, a\}$, each once.

(ii) The vertex $y^j_t$ has $a + (j-3)(a+1)$ neighbours in $X_1 \cup \cdots \cup X_{j-2}$ and $t$ in $X_{j-1}$.

(iii) $|X_1 \cup \cdots \cup X_{k-2}| = a + (k-3)(a+1) = N - (a+1)$, and $|X_1 \cup \cdots \cup X_{k-1}| = N$.

(iv) The vertex $\gamma_l$ is adjacent to all of $X_1 \cup \cdots \cup X_{k-1}$, which has $N$ vertices, and to the $y_{a-T_j}$ with $j \ge l$, of which there are $c + 1 - l$.

(v) Let $2 \le i \le k-1$. The vertex $x^i_q$ has $N - (a+1) + c$ core neighbours. Its periphery neighbours are as follows.

- If $i \le k-2$: the $a+2-q$ vertices $y^{i+1}_t$ with $t \ge q$; all $a+1$ vertices of each $Y_j$ with $i+2 \le j \le k-1$; and all $2a+2$ vertices of $Z$. The total is $(a+2-q) + (k-2-i)(a+1) + 2(a+1)$.
- If $i = k-1$: the $z_t$ with $t \ge q$ and all $p_r$, which is $(a+1-q) + (a+2)$ vertices.

It has no neighbours in $Y_2$ or in the $Y_j$ with $j \le i$. In both cases $d(x^i_q) = N + c + (k-i)(a+1) + 1 - q$.

(vi) The vertex $x^1_q$ has $(k-2)(a+1) + c$ core neighbours. In the periphery it is adjacent to the $a - q$ vertices $y_i$ with $i \ge q$, to all of $Y_3, \ldots, Y_{k-1}$, which have $(k-3)(a+1)$ vertices, and to all $2a+2$ vertices of $Z$. Summing gives $(2k-2)(a+1) + c - 1 - q = N + c + (k-1)(a+1) - q$.

Now read the values from the bottom. $Y_2$ gives $[1, a]$. $Y_j$ gives $[a + (j-3)(a+1) + 1,\ a + (j-2)(a+1)]$, for $j = 3, \ldots, k-1$ in turn. The $z_t$ give $[N-a, N-1]$, and the $p_r$ give $N$. $C$ gives $[N+1, N+c]$. Next comes $X_i$ with $[N + c + (k-i-1)(a+1) + 1,\ N + c + (k-i)(a+1)]$, for $i = k-1, k-2, \ldots, 2$. Last, $X_1$ gives $[N + c + (k-2)(a+1) + 1,\ N + c + (k-1)(a+1) - 1]$. These intervals are consecutive and disjoint, and within each interval every value occurs once. The only exception is $N$, which occurs $a+2$ times. $\square$

*Proof of Theorem 1.1.* By Proposition 3.2, $n - w = a + 1$. Hence $(n-1) - (2k-1)m = (2k-1)(a+1) + c - 3 - (2k-1)(a+1) = c - 3$. With Lemma 2.1 this gives (a) and (c), since $\chi(F^{(k)}_c) = k$. For $c = 4$, $n = (2k-1) \cdot 11 + 2 = 22k - 9$. For (b), adding an isolated vertex to $F^{(k)}_3$ increases $n$ and $w$ by one, because 0 was not a degree. So $m$ is unchanged and $(n-1) - (2k-1)m$ becomes $1 > 0$. The new graph has $(2k-1) \cdot 7 + 1 + 1 = 14k - 5$ vertices. $\square$

*Remark* 3.3. For $k = c = 3$ we have $a = 6$, $N = 13$, and $F^{(3)}_3$ has 36 vertices. Write $A = X_1 = \{x_1, \ldots, x_6\}$ and $B = X_2 = \{x'_1, \ldots, x'_7\}$. The core is $K_{6,7,3}$ on $A \cup B \cup C$. The periphery consists of the following vertices.

- $y_1, y_2, y_4$, adjacent to $x_1, \ldots, x_i$ (degree $i$).
- $y_0$, adjacent to $\gamma_1, \gamma_2, \gamma_3$ (degree 3).
- $y_3$, adjacent to $x_1, x_2, x_3, \gamma_1, \gamma_2$ (degree 5).
- $y_5$, adjacent to $x_1, \ldots, x_5, \gamma_1$ (degree 6).

- $z_1, \ldots, z_6$, where $z_t$ is adjacent to $A$ and to $x'_1, \ldots, x'_t$ (degrees $7, \ldots, 12$).
- $p_1, \ldots, p_8$, adjacent to $A \cup B$ (degree 13).

The core degrees are $d(\gamma_l) = 17 - l$, $d(x'_q) = 24 - q$ and $d(x_q) = 30 - q$. So the degrees are $1, \ldots, 29$, with 13 repeated eight times. The colour classes $A$, $B \cup Y_2$ and $C \cup Z$ have sizes 6, 13 and 17. Adding an isolated vertex gives a 3-chromatic graph with 37 vertices and 30 distinct degrees.

# 4 Blow-ups

**Lemma 4.1.** *Let $G$ be a graph without isolated vertices, and let $M$ be a matching of $G$ such that every degree value of $G$ is the degree of some vertex covered by $M$. For $t \geq 1$ let $G^{(t)}$ be the graph with vertex set $V(G) \times \{0, \ldots, t-1\}$ in which $(u, i)$ and $(v, j)$ are adjacent if and only if $uv \in E(G)$ and either $uv \notin M$ or $i + j \leq t - 1$. Then $|V(G^{(t)})| = t|V(G)|$, $w(G^{(t)}) = t\, w(G)$, $m(G^{(t)}) = t\, m(G)$ and $\chi(G^{(t)}) = \chi(G)$. If $G$ is connected, so is $G^{(t)}$.*

*Proof.* Let $v$ be a vertex. Each neighbour $u$ of $v$ with $uv \notin M$ contributes $t$ neighbours $(u, j)$ to $(v, i)$. If $uv \in M$, then $u$ contributes those $(u, j)$ with $j \leq t - 1 - i$, which are $t - i$ of them. Hence $d(v, i) = t\, d(v) - i$ if $v$ is covered by $M$, and $d(v, i) = t\, d(v)$ otherwise. All degrees of copies of degree-$d$ vertices lie in $\{td - t + 1, \ldots, td\}$. For different $d \geq 1$ these sets are disjoint. Each of them is filled completely by the copies of an $M$-covered vertex of degree $d$. So $w(G^{(t)}) = t\, w(G)$.

The map $(v, i) \mapsto v$ is a homomorphism, so $\chi(G^{(t)}) \leq \chi(G)$. The layer $V(G) \times \{0\}$ induces a copy of $G$, so equality holds. Finally, $(v, i)$ is adjacent to $(u, 0)$ for every neighbour $u$ of $v$. So every vertex has a neighbour in the connected layer $V(G) \times \{0\}$. $\square$

**Lemma 4.2.** *$F_c^{(k)}$ has a matching covering a vertex of every degree value.*

*Proof.* Take the following edges:

- $y_{a-T_j}\gamma_j$ for $j \in [c]$;
- $y_i x_i^1$ for $1 \leq i \leq a - 1$ with $i \notin \{a - T_j\}$;
- $y_t^j x_t^{j-1}$ for $3 \leq j \leq k - 1$ and $t \in [a + 1]$;
- $z_t x_t^{k-1}$ for $t \in [a]$;
- $p_1 x_{a+1}^{k-1}$.

These are edges of $F_c^{(k)}$ with distinct ends. They cover all of $C$, all of $Y_2, \ldots, Y_{k-1}$, all $z_t$, $p_1$, all of $X_2, \ldots, X_{k-1}$, and the vertices $x_i^1$ with $i \leq a - 1$ and $i \neq a - T_j$. The uncovered vertices of $X_1$ are the $x_q^1$ with $q \in \{a - T_j : 1 \leq j \leq c - 1\} \cup \{a\}$, which are $c$ vertices. Match them to $p_2, \ldots, p_{c+1}$. This is possible because $a + 2 \geq c + 1$, and each $p_r$ is adjacent to all of $X_1$. Every vertex other than the $p_r$ has a degree of its own, and the value $N$ is covered by $p_1$. $\square$

*Proof of Theorem 1.2.* Take $G = F_5^{(k)}$, where $a = 15$, $n = 16(2k - 1) + 3 = 32k - 13$ and $m = 16$. Apply Lemma 4.1 with the matching of Lemma 4.2. Then $G^{(t)}$ is connected and $k$-chromatic, with $n = t(32k - 13)$ and $m = 16t$, so that $(n - 1) - (2k - 1)m = t(32k - 13) - 1 - 16t(2k - 1) = 3t - 1$. $\square$

For a given $k$, the value $c = 5$ minimizes the ratio $m/n = (a + 1)/((2k - 1)(a + 1) + c - 2)$ over the graphs $F_c^{(k)}$.

## 5 Lower bounds

**Proposition 5.1.** *Every bipartite graph with $n \geq 2$ vertices has at most $\lfloor (2n+1)/3 \rfloor$ distinct degrees. Equivalently,* (1) *holds whenever $\chi(G) \leq 2$. For $n = 3p$ the bound is attained.*

*Proof.* If $\chi(G) = 1$, then $w = 1$ and (1) reads $1 > 0$. Let $G$ be bipartite with parts $A, B$, where $|A| = a' \geq |B| = b \geq 1$, and let $W = \lfloor (2n+1)/3 \rfloor$. By Lemma 2.1 with $\chi = 2$, the two forms are equivalent.

Degrees of vertices in $A$ lie in $[0, b]$, and $B$ has only $b$ vertices, so $w \leq 2b + 1$. All degrees are at most $a'$, so $w \leq a' + 1$. Suppose $w \geq W + 1$. Then $a' \geq W$ and $2b \geq W$. Since $W \geq (2a' + 2b - 1)/3$, this gives $2b - 1 \leq a' \leq 2b$, and in both cases $W = a'$. Hence $w = a' + 1$, and every value $0, 1, \ldots, a'$ is a degree. If $b = 1$, then $n \leq 3$, and no graph on at most 3 vertices has more than $W$ distinct degrees. So let $b \geq 2$; then $a' \geq 2b - 1 > b$.

The value $a' > b$ is the degree of some $y \in B$, which is then adjacent to all of $A$. The value 0 is therefore the degree of some $z \in B$ with $z \neq y$. The $a' - b$ values $b+1, \ldots, a'$ occur only in $B \setminus \{z\}$, which has $b - 1$ vertices. Hence $a' \leq 2b - 1$, so $a' = 2b - 1$, and the vertices of $B \setminus \{z\}$ have exactly the degrees $b+1, \ldots, 2b-1$. Now the value $b$ must occur. It does not occur in $B$, so some $x \in A$ has degree $b = |B|$ and is adjacent to $z$. This contradicts $d(z) = 0$.

For sharpness, let $n = 3p$ and take parts $\{u_1, \ldots, u_p\}$ and $\{v_1, \ldots, v_{2p}\}$, with $u_i v_j$ an edge if and only if $i+j \leq 2p+1$. Then $d(u_i) = 2p+1-i$ takes the values $p+1, \ldots, 2p$, and $d(v_j) = \min\{p,\, 2p+1-j\}$ takes the values $1, \ldots, p$. So there are $2p = \lfloor (2n+1)/3 \rfloor$ distinct degrees. □

For $K_{k+1}$-free graphs, a Turán-type counting argument gives a lower bound. Let $t_k(s)$ be the number of edges of the Turán graph $T_k(s)$, the complete balanced $k$-partite graph on $s$ vertices. By Turán's theorem [Tur41], every $K_{k+1}$-free graph on $s$ vertices has at most $t_k(s) \leq (1 - 1/k)s^2/2$ edges.

**Lemma 5.2.** *Let $G$ be $K_{k+1}$-free with degrees $d_1 \geq \cdots \geq d_n$. Then $\sum_{i \leq s} d_i - \sum_{i > s} d_i \leq 2t_k(s)$ for every $s \in [n]$.*

*Proof.* Let $S$ be a set of $s$ vertices with the largest degrees, and $\bar{S} = V(G) \setminus S$. Then $\sum_{v \in S} d(v) - \sum_{v \in \bar{S}} d(v) = 2e(S) - 2e(\bar{S}) \leq 2e(S) \leq 2t_k(s)$. □

**Theorem 5.3.** *Let $G$ be $K_{k+1}$-free with $n \geq 2$ vertices. For all integers $s$ and $l$ with $\lceil n/2 \rceil \leq s \leq n$ and $l \geq 1$,*

$$l \cdot m(G) \;\geq\; \sum_{j=1}^{n-1} \min\{j,\, 2s-j,\, l\} \;+\; (2s-n)\,\delta(G) \;-\; 2t_k(s).$$

*Proof.* Let $d_1 \geq \cdots \geq d_n$ be the degrees, let $\Delta_j = d_j - d_{j+1} \geq 0$ for $j \in [n-1]$, and let $F = \{j : \Delta_j = 0\}$. The number of distinct degrees is $1 + (n - 1 - |F|)$, so $|F| = m(G)$. Writing $d_i = d_n + \sum_{j \geq i} \Delta_j$, we get

$$\sum_{i \leq s} d_i - \sum_{i > s} d_i = (2s-n)d_n + \sum_{j=1}^{n-1} \kappa_j \Delta_j, \qquad \kappa_j = \min\{j, s\} - \max\{0, j-s\} = \min\{j, 2s-j\}.$$

Since $2s \geq n$, every $\kappa_j \geq 1$. Using $\Delta_j \geq 1$ for $j \notin F$ and $d_n = \delta(G)$,

$$\sum_{i \leq s} d_i - \sum_{i > s} d_i \;\geq\; (2s-n)\delta(G) + \sum_{j \notin F} \min\{\kappa_j, l\} \;\geq\; (2s-n)\delta(G) + \sum_{j=1}^{n-1} \min\{\kappa_j, l\} - l\,|F|.$$

Combining this with Lemma 5.2 proves the claim. □

*Proof of Theorem 1.3.* Part (a) is Proposition 5.1.

(b) Let $\sigma \in (\frac{1}{2}, 1)$ and $2\sigma - 1 \le \lambda \le \sigma$. Take $s = \lceil \sigma n \rceil$ and $l = \lceil \lambda n \rceil$ in Theorem 5.3 (for $n$ large, so that $s \le n$). The function $x \mapsto \min\{x, 2s - x, l\}$ is 1-Lipschitz. So the sum in Theorem 5.3 differs by $O(n)$ from

$$\int_0^n \min\{x, 2s-x, l\}\, dx = n^2\Big(2\sigma\lambda - \lambda^2 - \tfrac{1}{2}(2\sigma-1)^2\Big) + O(n).$$

The integral has three pieces: $[0, l]$, $[l, 2s - l]$ and $[2s - l, n]$. With $2t_k(s) \le (1 - 1/k)s^2$ we obtain $m(G) \ge n\,\varphi(\sigma, \lambda) - O(1)$, where

$$\varphi(\sigma,\lambda) = 2\sigma - \lambda - \frac{(2\sigma-1)^2 + 2(1-1/k)\sigma^2}{2\lambda}.$$

If $2\lambda^2 = (2\sigma - 1)^2 + 2(1 - 1/k)\sigma^2$, then $\varphi(\sigma, \lambda) = 2\sigma - 2\lambda$. Put $r = \sqrt{2(k-1)(2k-1)}$,

$$\sigma = \frac{k}{4k-2-r}, \qquad \lambda = (2\sigma - 1) + \Big(1 - \frac{1}{k}\Big)\sigma.$$

From $2k-2 < r < 2k-1$ we get $\frac{1}{2} < \sigma < k/(2k-1)$, and hence $2\sigma - 1 \le \lambda \le \sigma$. A direct computation shows that $u = 2\sigma - 1$ satisfies $u^2 + 4c\sigma u + 2c(c-1)\sigma^2 = 0$ with $c = 1 - 1/k$, which is equivalent to $2\lambda^2 = u^2 + 2c\sigma^2$. Therefore

$$\varphi = 2\sigma - 2\lambda = 2 - 2\sigma\,\frac{2k-1}{k} = \frac{2(2k-1-r)}{4k-2-r} = \frac{2k-r}{3k-1} = \mu_k.$$

(This is the maximum of $\varphi$ over the admissible region.) For $k = 3$ we have $\sigma = (15 + 3\sqrt{5})/40$, $\lambda = 1/\sqrt{5}$ and $(2\sigma - 1)^2 + \frac{4}{3}\sigma^2 = \frac{2}{5}$, so $\mu_3 = (3 - \sqrt{5})/4$.

(c) Let $G$ be a counterexample with $\chi(G) = k$ and $n$ vertices. By (a), $k \ge 3$. By Lemma 2.1, $m(G) \le \lceil (n-1)/(2k-1) \rceil - 1$. Since $m(G) \ge 1$, this forces $\lceil (n-1)/(2k-1) \rceil \ge 2$. $G$ is $K_{k+1}$-free, so Theorem 5.3 applies with this $k$, with $\delta(G) \ge 0$, or $\delta(G) \ge 1$ if $G$ has no isolated vertex. For every pair $(n, k)$ in the stated ranges, the program described below finds integers $s, l$ for which the right-hand side of Theorem 5.3, divided by $l$, exceeds $\lceil (n-1)/(2k-1) \rceil - 1$. This is a contradiction. For $n \le 36$, and for $n \le 56$ when $\delta \ge 1$, it succeeds for every $k \ge 3$ with $\lceil (n-1)/(2k-1) \rceil \ge 2$. For $3 \le k \le 60$ it succeeds for all $n < 14k - 5$, and for all $n < 22k - 9$ when $\delta \ge 1$. □

*Remark* 5.4. For $n \le 60$ the only pairs $(n, k)$ with $k \ge 3$ that Theorem 5.3 does not exclude are

$$(37,3),\ (42,3),\ (47,3),\ (51,4),\ (52,3),\ (57,3),\ (58,3),\ (58,4),$$

and with $\delta \ge 1$ only $(57, 3)$. For $k = 2$, Theorem 5.3 gives only $\mu_2 = (4 - \sqrt{6})/5 \approx 0.310$. This is weaker than the sharp value $1/3$ given by Proposition 5.1, so bipartiteness is used there in an essential way.

# 6 Concluding remarks

Melnikov's inequality holds for $\chi \le 2$ and fails for every $\chi \ge 3$. For every $k \ge 3$ it is attained with equality by the connected graphs $F_3^{(k)}$. This agrees with Zykov's report that the bound would be best possible.

The conjecture fails by the smallest possible margin. The weaker inequality $\chi(G) \ge \lceil \lfloor w/2 \rfloor / (n - w) \rceil$ is equivalent, by the argument of Lemma 2.1, to $(2\chi + 1)\, m \ge n - 1$. None of our graphs violates it. By Theorem 1.3(b) it holds for each fixed $\chi$ and all large $n$, since $\mu_k > 1/(2k+1)$. An exact evaluation of Theorem 5.3, together with Proposition 5.1, shows that it holds for all graphs with at most 400 vertices. Two natural questions remain.

**Problem 6.1.** *Determine* $\rho_k = \liminf_{n\to\infty} \min\{m(G)/n :\ |V(G)| = n,\ \chi(G) \le k\}$. *For* $k \ge 3$ *we know* $\mu_k \le \rho_k \le 16/(32k - 13) < 1/(2k - 1)$. *For* $k = 2$, *Proposition 5.1 gives* $\rho_2 = 1/3$.

**Problem 6.2.** *Is the least order of a* $k$*-chromatic counterexample equal to* $14k - 5$ *for every* $k \ge 3$, *and* $22k - 9$ *without isolated vertices? This holds for* $3 \le k \le 60$.

**Verification.** The claims were checked as follows.

1. A self-contained program by the author, `verify_melnikov.py` (standard library only, about twenty seconds), does the following.
   - It builds $F_c^{(k)}$ for $3 \le k \le 7$ and $1 \le c \le 6$. It checks Proposition 3.2, the colouring, the $K_k$, connectivity and Lemma 4.2.
   - It builds the blow-ups for $t = 2, 3$ and checks Lemma 4.1.
   - It checks Lemma 2.1 for $n \le 200$. It checks Proposition 5.1 on all 108,622 spanning subgraphs of the complete bipartite graphs $K_{a',b}$ with $a' \ge b \ge 1$ and $a' + b \le 8$.
   - It checks the inequality of Theorem 5.3 on all 33,866 labelled graphs with $2 \le n \le 6$, taking $k$ to be the clique number.
   - It evaluates Theorem 5.3 in exact arithmetic for Theorem 1.3(c). The checks for $n \le 60$ use rational arithmetic over all $(s, l)$. The checks for $k \le 60$ use integer arithmetic. It also writes a certificate, a pair $(s, l)$, for every excluded pair $(n, k)$ with $n \le 36$, and for every excluded pair with $n \le 56$ when $\delta \ge 1$. For example, for $(n, k) = (36, 3)$ the pair $(s, l) = (19, 16)$ gives $\sum_{j\le 35} \min\{j, 38 - j, 16\} - 2t_3(19) = 349 - 240 = 109 > 16 \cdot 6$, so $m \ge 7$.
2. The counterexamples of Theorem 1.1(b), for $k = 3, 4, 5$ (37, 51 and 65 vertices), and $F_4^{(3)}$ (57 vertices) were also exported as edge lists. A separate program computes their chromatic numbers exactly, by backtracking and a clique search.
3. The first counterexamples were found by a different route, with separate code. A mixed-integer program searched for degree sequences and colour classes satisfying the necessary conditions of Lemma 5.2. An edge-level integer program then realized the sequences as graphs, which were checked independently. An earlier SAT computation of the same kind found no counterexample with at most 20 vertices.
4. An independent referee checked all proofs and wrote separate code from the text of the paper.
   - It rebuilt $F_c^{(k)}$ for $3 \le k \le 7$ and $1 \le c \le 7$, and it rebuilt the blow-ups.
   - It confirmed with a SAT solver that $\chi = k$ for the graphs on 37, 51, 57, 65 and 79 vertices.
   - It evaluated Theorem 5.3 exactly with its own code, which confirms Theorem 1.3(c) and Remark 5.4.
   - It checked (1), Proposition 5.1 and Theorem 5.3 on all labelled graphs with at most 8 vertices.

**Scope and priority.** Theorem 1.1 answers the problem as stated by Vizing [Viz68] and in the Open Problem Garden [Ope13]. The answer stays negative if only connected graphs are allowed, and it is negative for every chromatic number $\ge 3$. We read Vizing's survey in a scan of the original. We did not see the book of Jensen and Toft [JT95], Zykov's problem collection [Zyk68] or Dirac's paper [Dir64]. What we say about them is taken from [Viz68, Ope13]. We searched arXiv, Crossref, OpenAlex and zbMATH and found no solution, and no partial result beyond the Nettleton–Dirac bound. The Jensen–Toft update pages do not mention the problem. A web search found nothing beyond the Open Problem Garden page. Russian-language follow-up literature was not checked systematically. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the counterexamples and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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