

Three-Colourability of Random Lifts of K_5

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Abstract

Let L_n be the random n -lift of K_5 , obtained by choosing independent uniform perfect matchings over its ten edges. We prove that $\mathbb{P}(\chi(L_n) = 3) \rightarrow 1$. The main ingredient is an entropy inequality for the exact edge couplings of two fibrewise balanced three-colourings. For doubly stochastic overlap matrices normalized to have total mass one, it bounds the edge entropy by three quarters of the sum of the endpoint entropies plus $\log(4/3)$, with equality only at the uniform coupling. This proves the missing global maximum in the balanced second-moment calculation at degree four. We derive the Gaussian constants, treat all integer lift sizes by bounded translates of the counting lattice, and apply small subgraph conditioning. The proof uses the random-lift moment framework of Nir and Pérez-Giménez; it does not transfer a theorem about uniform random regular graphs to a different probability model.

1 The random-lift question

Replace each vertex of K_5 by a labelled fibre of size n . Independently for each base edge, put a uniformly random perfect matching between its two fibres. The resulting simple four-regular graph is denoted by L_n . All limits below are as $n \rightarrow \infty$, through all integers.

Theorem 1.1. *The chromatic number of L_n is asymptotically almost surely three:*

$$\mathbb{P}(\chi(L_n) = 3) \rightarrow 1.$$

More specifically, asymptotically almost surely there is a proper three-colouring having prescribed colour counts $\lfloor n/3 \rfloor + 1$ for the first $n \bmod 3$ colours and $\lfloor n/3 \rfloor$ for the remaining colours in every fibre.

Record OPG-37656 in `ulamai/UnsolvedMath` v1.6.0 asks precisely this K_5 concentration question. Our result does not settle the corresponding question for every complete base. Farzad and Theis [1] proved three-colourability for lifts of K_5 with one base edge deleted. Nir and Pérez-Giménez [2] developed balanced-colouring moment calculations and small subgraph conditioning for regular graph lifts. Their degree condition for three-colourability is $d < (16/3) \log 2 < 4$; together with their other bounds it leaves the K_5 case in the two-value window $\{3, 4\}$. Shi and Wormald's theorem for uniform random four-regular graphs [3] does not by itself apply to random lifts.

Our new ingredient is the exact-marginal entropy estimate in Section 3. It replaces the global optimization step in [2, Proposition 5.1]. For transparency, we derive the specialized moment constants instead of invoking their second-moment proposition outside its stated degree range. All logarithms are natural, and $H(p) = -\sum p_i \log p_i$, with $0 \log 0 = 0$.

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2 A sharp three-coordinate inequality

Lemma 2.1. *If $x_i \geq 0$ and $x_1 + x_2 + x_3 = 1$, then*

$$\sum_{i=1}^3 x_i^{3/2} + \left(\sum_{i=1}^3 x_i^{3/4} \right)^2 \leq \frac{4}{\sqrt{3}}. \quad (1)$$

Equality holds only at $x_1 = x_2 = x_3 = 1/3$.

Proof. The left side has a maximum on the compact simplex. A boundary point cannot maximize it: moving ε mass from a positive coordinate to a zero coordinate produces the positive term $2S\varepsilon^{3/4}$, where $S = \sum x_i^{3/4} > 0$, and all other changes are $O(\varepsilon)$ or smaller. At an interior stationary point the multiplier equations give

$$x_i^{1/2} + Sx_i^{-1/4} = \text{constant}.$$

The function $u^2 + S/u$ decreases and then increases on $(0, \infty)$. Consequently the coordinates take at most two values. Up to a permutation they have the form $(t^4, 1, 1)/(t^4 + 2)$, for some $t > 0$. Substitution into (1), followed by squaring positive sides, reduces the assertion to

$$\begin{aligned} 4(t^4 + 2)^3 - 3(t^6 + 2t^3 + 3)^2 &= (t - 1)^2 Q(t) \geq 0, \\ Q(t) &= t^{10} + 2t^9 + 3t^8 - 8t^7 + 5t^6 + 18t^5 \\ &\quad + t^4 - 16t^3 + 15t^2 + 10t + 5. \end{aligned}$$

For $u \geq 0$, both $Q(1 + u)$ and $(1 + u)^{10}Q(1/(1 + u))$ have strictly positive coefficients. Their coefficient lists, in ascending order, are respectively

$$\begin{aligned} (36, 112, 261, 444, 558, 552, 411, 208, 66, 12, 1), \\ (36, 248, 873, 1932, 2910, 3048, 2199, 1064, 330, 60, 5). \end{aligned}$$

These identities cover $t \geq 1$ and $0 < t \leq 1$, respectively. Thus $Q(t) > 0$; equality forces $t = 1$. The only maximizing point is the uniform vector, where the claimed value is attained. $\square$

3 Exact edge entropy

Let $\mathcal{S} = \{1, 2, 3\}^2$. Two states $(i, j), (i', j')$ are *allowed* if $i \neq i'$ and $j \neq j'$. Their adjacency matrix is

$$K = (J_3 - I_3) \otimes (J_3 - I_3).$$

There are 36 allowed ordered pairs. A normalized overlap matrix is a nonnegative 3×3 matrix with every row and column sum $1/3$.

Lemma 3.1. *For normalized overlap matrices A, C ,*

$$\sum_{i \neq i', j \neq j'} A_{ij}^{3/4} C_{i'j'}^{3/4} \leq \frac{4}{3}. \quad (2)$$

Equality is possible only when $A = C = J_3/9$.

Proof. Write $F = A^{3/4}$ entrywise and regard it as a vector. The spectral absolute value of K is

$$|K| = (I_3 + J_3/3) \otimes (I_3 + J_3/3),$$

because $J_3 - I_3$ has eigenvalues 2, -1, -1. Let r_i, c_j be the row and column sums of F , and let $T = \sum F_{ij}$. Using $T^2 \leq 3 \sum r_i^2$ and $T^2 \leq 3 \sum c_j^2$, we obtain

$$\begin{aligned} F^T |K| F &= \sum F_{ij}^2 + \frac{1}{3} \left(\sum r_i^2 + \sum c_j^2 \right) + \frac{T^2}{9} \\ &\leq \sum F_{ij}^2 + \frac{1}{2} \left(\sum r_i^2 + \sum c_j^2 \right) \leq \frac{4}{3}. \end{aligned}$$

For the last inequality, apply Lemma 2.1 to each probability vector $(3A_{ij})_j$ and $(3A_{ij})_i$. Every row or column bracket $\sum F_{ij}^2 + (\sum F_{ij})^2$ is at most 4/9, and we sum half of each of the six brackets. Equality forces every entry of A to be 1/9. Putting $G = C^{3/4}$, spectral Cauchy–Schwarz gives

$$F^T K G \leq \sqrt{F^T |K| F} \sqrt{G^T |K| G} \leq 4/3.$$

Indeed apply ordinary Cauchy–Schwarz to $|K|^{1/2} F$ and $\text{sign}(K)|K|^{1/2} G$; the sign matrix is orthogonal. The equality assertion follows from the preceding bounds. $\square$

Proposition 3.2. *If B is a probability distribution on allowed state pairs, with normalized overlap marginals A, C , then*

$$H(B) \leq \frac{3}{4}(H(A) + H(C)) + \log(4/3). \quad (3)$$

Equality holds exactly when the marginals and the allowed-pair distribution are uniform.

Proof. Let $Z = \sum_{s \sim t} A_s^{3/4} C_t^{3/4}$, and put $Q_{st} = A_s^{3/4} C_t^{3/4} / Z$ on the allowed pairs. Feasibility of B ensures $Z > 0$, and every positive entry of B has positive marginals. Hence relative entropy is defined and

$$0 \leq D(B \| Q) = -H(B) + \frac{3}{4}(H(A) + H(C)) + \log Z.$$

Lemma 3.1 proves the bound, including zero entries in the marginals. Its equality case and $D(B \| Q) = 0$ prove the assertion. $\square$

4 Balanced-colouring moments

Write Y_n for the number of proper colourings with the prescribed near-balanced counts in Theorem 1.1. Define

$$a = \frac{1024}{243}, \quad C_1 = \frac{2^{20}}{3^{5/2}}, \quad C_2 = \frac{2^{120}}{3^{175} 10^3 23^8}.$$

Proposition 4.1. *For all integer lift sizes tending to infinity,*

$$\mathbb{E} Y_n \sim C_1 (2\pi n)^{-5} a^n, \quad \mathbb{E} Y_n^2 \sim C_2 (2\pi n)^{-10} a^{2n}. \quad (4)$$

In particular,

$$\frac{\mathbb{E} Y_n^2}{(\mathbb{E} Y_n)^2} \longrightarrow R := \frac{2^{80}}{3^{125} 10^3 23^8}. \quad (5)$$

We first prove the formulas for $36 \mid n$, then justify the removal of the divisibility condition. This is the moment framework of [2, Sections 4–5], with the global optimization supplied by Proposition 3.2.

4.1 The factorial sums and their unique maxima

For a pair of colourings let $a_{v,s}$ count vertices with colour pair $s \in \mathcal{S}$ in fibre v , and let $b_{e,st}$ count matched pairs of states over $e = vw$. The 3×3 array a_v has all row and column totals $n/3$, while b_e has marginals a_v, a_w and support on the allowed pairs. Choosing the classes, their matching subclasses, and the matchings gives exactly

$$\mathbb{E}Y_n^2 = (n!)^{-5} \sum_{a,b} \frac{\prod_{v,s} a_{v,s}!^3}{\prod_{e,s \sim t} b_{e,st}!}. \quad (6)$$

For the first moment the same expression holds with three states, the six unequal-colour cells, and fixed $a_{v,i} = n/3$.

Normalize counts by n , writing A_v, B_e . The second-moment exponential rate in (6) is

$$f(A, B) = \sum_e H(B_e) - 3 \sum_v H(A_v). \quad (7)$$

Every base vertex has degree four, so summing (3) cancels all the vertex terms and proves

$$f(A, B) \leq 10 \log(4/3) = 2 \log a.$$

Equality forces $A_v = J_3/9$, $B_e = 1/36$ everywhere. For the first moment, strict concavity of each edge entropy gives the unique maximum at its six equal cells, with rate $10 \log 6 - 15 \log 3 = \log a$. Uniform Stirling bounds, valid also at zero counts, add only polynomial factors. Compactness and uniqueness make the contribution outside any fixed neighbourhood of these maxima exponentially negligible.

4.2 Integer coordinates and the second-moment Hessian

Let W be the four-dimensional space of 3×3 matrices with zero row and column sums. Its integer lattice has basis

$$E_{ij} - E_{i3} - E_{3j} + E_{33}, \quad i, j \in \{1, 2\}.$$

The Gram matrix is $D = (I_2 + J_2) \otimes (I_2 + J_2)$, with $\det D = 81$.

The allowed-cell bipartite graph $\mathcal{H}$ has nine states on each side and 36 edges. It is connected: any two left states have a common right neighbour. Fix a spanning tree. Its 19 fundamental cycles form an integer kernel basis C for the marginal equations. The tree incidence matrix is unimodular, so integral marginals admit an integral tree solution, and every other integral solution is that solution plus Cz , $z \in \mathbb{Z}^{19}$. Cauchy–Binet gives $\det(C^T C) = \tau(\mathcal{H})$: a square cycle minor is zero unless its complementary edges are a spanning tree, in which case its determinant is ± 1 .

The matrix K has eigenvalues 4, -2 , 1, of multiplicities 1, 4, 4. Thus the nonzero Laplacian eigenvalues of $\mathcal{H}$ are 8, and 2, 6, 3, 5 each four times. The matrix-tree theorem yields

$$\tau(\mathcal{H}) = \frac{8 \cdot 2^4 6^4 3^4 5^4}{18} = 2^{10} 3^6 5^4. \quad (8)$$

For infinitesimal endpoint marginals $x, y \in W$, the minimum squared norm of an edge perturbation with those marginals is

$$\min_b \|b\|^2 = \frac{4\|x\|^2 + 4\|y\|^2 - 2\langle x, y \rangle}{15}. \quad (9)$$

Indeed the unsigned incidence matrix times its transpose is $\begin{pmatrix} 4I & K \\ K & 4I \end{pmatrix}$. On W , $K = I$; invert $\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ on each of the four coordinates to obtain (9).

The negative Hessian of (7) at its uniform maximum is

$$36 \sum_e \|b_e\|^2 - 27 \sum_v \|x_v\|^2.$$

Minimizing over the edge-cycle coordinates and using (9) gives the Schur complement

$$M \otimes I_4, \quad M = \frac{3}{5}(19I_5 - 4 \operatorname{Adj}(K_5))$$

in orthonormal overlap coordinates. The eigenvalues of M are $9/5$ once and $69/5$ four times. In integer coordinates the Schur-complement determinant is therefore

$$\det S = (9/5)^4 (69/5)^{16} 81^5.$$

Each edge-cycle block has determinant $36^{19} \tau(\mathcal{H})$. There are 20 overlap coordinates and 190 edge-cycle coordinates, and the full negative Hessian $\mathcal{Q}$ satisfies

$$\det \mathcal{Q} = (36^{19} \tau(\mathcal{H}))^{10} \det S. \quad (10)$$

It is positive definite, since both its cycle blocks and Schur complement are positive definite. The chosen coordinates parametrize the whole integer lattice, not a proper sublattice.

4.3 Laplace evaluation

The negative Hessian gives quadratic decay in a sufficiently small fixed neighbourhood. Together with the compactness bound above, this localizes the sum to distance at most $\log n / \sqrt{n}$ from the saddle. The cubic Taylor remainder multiplied by n is $O((\log n)^3 / \sqrt{n}) = o(1)$ there; the Stirling prefactor is uniform to relative error $o(1)$. Hence the usual Gaussian Riemann sum applies in the integer coordinates just described.

For the second moment, Stirling's prefactor at the saddle in (6) is

$$(2\pi n)^{-115} 2^{360} 3^{225}.$$

There are 210 integer coordinates, so their Gaussian sum contributes $(2\pi n)^{105} / \sqrt{\det \mathcal{Q}}$. Equations (8)–(10) give

$$\frac{2^{360} 3^{225}}{\sqrt{\det \mathcal{Q}}} = \frac{2^{120}}{3^{175} 5^{10} 23^8} = C_2.$$

For the first moment, the six allowed colour cells form a bipartite six-cycle. Its marginal kernel has the single integral generator with alternating entries $+1, -1$. Its negative Hessian is $6 \cdot 6 = 36$. There are ten independent edge coordinates. The central Stirling prefactor is $(2\pi n)^{-10} 2^{30} 3^{15/2}$, and the Gaussian factor is $(2\pi n)^5 / 6^{10}$. Their constant is $2^{20} / 3^{5/2} = C_1$. This proves (4) for $36 \mid n$.

4.4 All integer sizes

We supply the lattice argument to avoid any assumption of monotonicity in lift size. The homogeneous integer constraint lattice is the same for every residue class of n . It remains to exhibit a feasible count vector within bounded distance of the uniform real vector nu .

A 36-lift can have nine overlap classes of size four in every fibre and one matching edge in every allowed cell. For each base edge, assign the four vertices of each class to its four allowed partners. Each side uses every vertex once, giving the required matching.

For every $t \geq 2$, a lift with the prescribed near-balanced colour counts also exists. For $t = 3q$, take q three-lifts with one vertex of each colour per fibre and cyclic-shift matchings. For $t = 3q + 2$, add a bipartite two-lift on colours $1, 2$. For $t = 3q + 1$, take $q - 1$ three-lifts and two bipartite two-lifts on colours $\{1, 2\}$ and $\{1, 3\}$. The two-lift matchings swap the colours. Use each such colouring twice to obtain overlap and edge count arrays.

Write $n = 36m + r$, $0 \leq r < 36$. For $m \geq 1$, combine $m - 1$ uniform 36-blocks with the fixed $t = 36 + r$ block. The resulting count vector is $nu + t_r$, with t_r bounded independently of n . It is interior for large n , and all integer solutions form this vector plus the same homogeneous lattice as above, restricted to nonnegative coordinates. Projection onto the first colours gives the analogous reference vectors for the first-moment lattice.

The feasible normalized polytopes converge to the balanced one. Compactness and continuity give the same exponential gap off a fixed neighbourhood of u . In this neighbourhood write $x = u + t_r/n + z$, with z a homogeneous tangent vector. The gradient of the entropy rate at u is constant in each vertex and edge block, whereas both t_r and z have zero total in every block. The linear term vanishes. Negative definiteness on the tangent space gives, locally, an upper bound

$$f(x) \leq f(u) + O(n^{-2}) + O(\|z\|/n) - c\|z\|^2$$

for some $c > 0$. This again localizes the sum to the central window. There, the bounded translate changes the exponent by $o(1)$ and the prefactor by $1 + o(1)$. A bounded translate in integer coordinates does not change the leading Gaussian Riemann sum. The same argument applies to the first moment. Thus (4) and (5) hold through all integers, completing the proof of Proposition 4.1.

5 Small subgraph conditioning

Let $Z_{j,n}$ count unoriented simple j -cycles, $j \geq 3$. Let B be the nonbacktracking matrix on the 20 oriented edges of K_5 : $B_{(u,v),(v,w)} = 1$ if $w \neq u$, and zero otherwise. Put

$$\lambda_j = \frac{\text{tr}(B^j)}{2j}, \quad \delta_j = \frac{(-1)^j}{2^{j-1}}. \quad (11)$$

Lemma 5.1. *For every fixed finite set of indices, the cycle counts converge jointly to independent Poisson variables with means λ_j . Moreover, for every finite sequence of nonnegative integers $p_3, \dots, p_\ell$,*

$$\frac{\mathbb{E} \left(Y_n \prod_{j=3}^{\ell} [Z_{j,n}]_{p_j} \right)}{\mathbb{E} Y_n} \longrightarrow \prod_{j=3}^{\ell} (\lambda_j (1 + \delta_j))^{p_j}, \quad (12)$$

where $[x]_p = x(x-1) \cdots (x-p+1)$.

Proof. We give the standard counting argument, as in [2, Section 6], to specify its applicability to our prescribed near-balanced counts. A rooted oriented lifted j -cycle projects to a closed nonbacktracking base walk. For each of the $\text{tr}(B^j)$ such walks, there are $n^j(1 + O(n^{-1}))$ injective lifted vertex choices and probability $n^{-j}(1 + O(n^{-1}))$ of the required matches. Division by $2j$ gives λ_j . Disjoint cycles factor to leading order. A collection with intersecting distinct cycles has more edges than vertices in some component and is $O(n^{-1})$. The joint factorial moments therefore converge to those of independent Poisson variables.

Under the probability measure biased by $Y_n/\mathbb{E}Y_n$, first choose uniformly a colouring with the prescribed fibre counts, then choose each matching uniformly among those that avoid equal colours. This description is exact by fibre-permutation symmetry. The colours on any fixed finite set of distinct vertices are asymptotically independent and uniform.

The first-moment Laplace estimate implies that each of the six colour-transition counts is $n/6 + o(n)$, with exponentially small probability outside any fixed relative neighbourhood. Conditional on these counts, specifying h disjoint matching edges of allowed colours has probability given by a product of falling transition counts divided by the corresponding falling endpoint class counts. It is therefore $(3/(2n))^h(1+o(1))$. The same formula bounds every fixed consistent pinning by $O(n^{-h})$; atypical transition arrays contribute only an exponentially small amount. These statements are unchanged by bounded shifts of the class sizes.

A rooted j -cycle has $2^j + 2(-1)^j$ proper three-colourings. Consequently its factor relative to the unconditioned expectation is

$$3^{-j}(3/2)^j(2^j + 2(-1)^j) = 1 + \delta_j.$$

For disjoint cycles these factors multiply. Intersecting distinct cycles are again $O(n^{-1})$ by the pinning bound and the excess of edges over vertices. This proves every mixed factorial moment in (12). $\square$

The nonbacktracking characteristic factorization, equivalently the regular-graph formula in [2, Section 2], gives

$$\det(I - zB) = (1 - z^2)^5(1 - 4z + 3z^2)(1 + z + 3z^2)^4. \quad (13)$$

Thus the spectral radius is 3, and

$$\det(I - B/4) = \frac{15^5 \cdot 3 \cdot 23^4}{4^{20}}.$$

There are no nonbacktracking closed walks of lengths one and two. The convergent logarithmic determinant series at $z = 1/4$ yields

$$\sum_{j \geq 3} \lambda_j \delta_j^2 = 2 \sum_{j \geq 3} \frac{\text{tr}(B^j)}{j 4^j} = -2 \log \det(I - B/4) = \log R < \infty. \quad (14)$$

Also $\lambda_j > 0$ for every $j \geq 3$: K_5 has 3-, 4-, and 5-cycles, and nonbacktracking concatenations of a triangle and a four-cycle give all lengths at least six. Finally $\delta_j > -1$.

The small subgraph conditioning theorem in [4, Theorem 4.1] now applies. Its hypotheses are the Poisson limits, the mixed factorial moments (12), convergence of the sum in (14), and the second-moment bound $\mathbb{E}Y_n^2/(\mathbb{E}Y_n)^2 \leq \exp(\sum \lambda_j \delta_j^2) + o(1)$. The latter is an equality asymptotically by Proposition 4.1. Since no δ_j equals -1 , the theorem gives

$$\mathbb{P}(Y_n > 0) \longrightarrow 1.$$

6 The lower bound and verification

Fix a triangle in the base. After relabelling two of its matchings to identities, the remaining matching is a uniform permutation of $[n]$. Its cycles of length j lift to graph cycles of length $3j$. The probability that a uniform permutation has only even cycles is zero when n is odd. For $n = 2m$, the exponential generating function

$$\exp\left(\sum_{r \geq 1} \frac{z^{2r}}{2r}\right) = (1 - z^2)^{-1/2}$$

gives that probability as $\binom{2m}{m}/4^m = O(m^{-1/2})$. Hence L_n contains an odd cycle asymptotically almost surely, so $\chi(L_n) \geq 3$. Together with $Y_n > 0$ this proves Theorem 1.1.

The accompanying standard-library Python program checks the polynomial certificates, integer cycle lattice, minimum-norm edge calculation, Gaussian determinants, nonbacktracking determinant, and bounded reference arrays for all 36 residue classes using exact arithmetic. It also enumerates all $6^6 = 46,656$ gauge-fixed three-lifts and independently checks the overlap counting formula. For that finite test, $\mathbb{E}Y_3 = 32/243$ and $\mathbb{E}Y_3^2 = 283/162$. These finite calculations are checks on the proof, not substitutes for its inequalities or asymptotic arguments.

This is an unrefereed preprint. The mathematical argument and its reproducible checks were developed with AI assistance; the author takes responsibility for the manuscript. No independent review, proof-assistant verification, or absolute priority certification is claimed. The random-lift moment framework and small subgraph conditioning are credited to the sources above; the assertion of this paper is the stated K_5 theorem, not a solution of all random-lift colouring problems.

References

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