Logarithmic Equivalence Covers of Powers of Cycles

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Abstract

An equivalence graph is a vertex-disjoint union of cliques. We give an explicit cover of every noncomplete cycle power by a logarithmic number of equivalence subgraphs. More precisely, for integers $k \geq 1$ and $n \geq 2k+2$,

$$\lceil \log_2(2k+2) \rceil \leq \operatorname{eq}(C_n^k) \leq 4 \lceil \log_2(k+1) \rceil + 1.$$

The upper bound partitions the cycle into short clique blocks and uses binary encodings for threshold adjacency between blocks. The lower bound is an application of Alon's multilinear rank method. Consequently the equivalence covering number has order $\log(k+1)$ uniformly in n, giving a negative answer to the linear-growth conjecture recorded in Open Problem Garden, including its intended large-n regime. We credit earlier logarithmic co-chain encodings and record a related 2010 conference abstract whose numerical bounds were not available in the located text. No absolute priority claim is made.

1 The question and the result

All graphs here are finite, simple and undirected. An *equivalence subgraph* of a graph G is a subgraph whose components are cliques. Isolated vertices may be added to make it spanning. The equivalence covering number $\operatorname{eq}(G)$ is the smallest number of such subgraphs whose edge union is $E(G)$. The subgraphs need not be induced, and their edge sets need not be disjoint.

For $n \geq 3$, identify the vertices of C_n with $\mathbb{Z}/n\mathbb{Z}$. Distinct vertices are adjacent in its kth power C_n^k precisely when their cyclic distance is at most k. If $n \leq 2k+1$, this graph is complete and $\operatorname{eq}(C_n^k) = 1$. We will work in the noncomplete range $n \geq 2k+2$.

King's problem posting [3] asks whether $\operatorname{eq}(C_n^k) = \Omega(k)$, with the discussion expressly including n very large compared with k. The related REGS page [5] asks for the asymptotic order and a sublinear upper bound. Our result addresses this growing-k regime, not only the trivial complete-graph exception.

Theorem 1. *For every pair of integers $k \geq 1$ and $n \geq 2k+2$,*

$$\lceil \log_2(2k+2) \rceil \leq \operatorname{eq}(C_n^k) \leq 4 \lceil \log_2(k+1) \rceil + 1. \tag{1}$$

If $n = mk$, the upper bound improves to

$$\operatorname{eq}(C_n^k) \leq \begin{cases} 2\lceil \log_2(k+1) \rceil + 1, & m \text{ even}, \\ 3\lceil \log_2(k+1) \rceil + 1, & m \text{ odd}. \end{cases} \tag{2}$$

All these upper bounds are given by explicit partitions of the vertex set.

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The lower-bound tool is established: it is Alon's Theorem 1.1 [1], presented below in elementary matrix form. Logarithmic encodings of co-bipartite chain graphs are also known; see Golumbic, Morgenstern and Rajendraprasad [2, Theorem 4.2]. The upper construction here makes the application to cycle powers explicit, including the remainder block when k does not divide n. The resulting bound depends on k, not on the number of blocks.

2 The rank lower bound

Suppose C_n^k has a cover by s equivalence subgraphs. Since this graph has edges, $s \geq 1$. In each layer ℓ, label its components by distinct real numbers, and denote the label of the component containing v by $c_\ell(v)$. Define

$$F(v, w) = \prod_{\ell=1}^{s} (c_\ell(w) - c_\ell(v)).$$

Then $F(v, w) = 0$ if $v = w$ or vw is an edge. It is nonzero for distinct nonadjacent vertices, since such vertices cannot share a component in any layer. The expansion

$$F(v, w) = \sum_{S \subseteq \{1,\ldots,s\}} (-1)^{|S|} \prod_{\ell \in S} c_\ell(v) \prod_{\ell \notin S} c_\ell(w) \tag{3}$$

shows that any matrix with entries $F(u_i, w_j)$ has rank at most 2^s: each summand is a matrix of rank at most one. This is the separated-product version of the multilinear argument in [1].

Set $h = 2k + 2$ and take

$$u_i = i, \qquad w_i = i + k + 1 \pmod n, \qquad 0 \leq i < h.$$

These are legitimate vertex sequences because $h \leq n$. The two cyclic distances from u_i to w_i are $k+1$ and $n - k - 1$, both at least $k + 1$. Thus the diagonal entries $F(u_i, w_i)$ are nonzero. If $i > j$, then

$$w_j - u_i \equiv k + 1 - (i - j) \pmod n, \qquad -k \leq k + 1 - (i - j) \leq k.$$

Hence $u_i = w_j$ or $u_i w_j$ is an edge, so $F(u_i, w_j) = 0$. The h-by-h matrix is upper triangular with nonzero diagonal. Its rank is h, and (3) gives $2^s \geq 2k + 2$. This proves the lower bound in (1).

The sequences may overlap; this is allowed and is handled by $F(v, v) = 0$. At the boundary $n = 2k + 2$ the graph is the complement of a perfect matching. Alon's Corollary 1.2 [1] gives equality in the lower bound there. That exact boundary case is not a new result of this note.

3 A binary cover of threshold adjacency

Lemma 2. *Let $A = \{a_0, \ldots, a_{a-1}\}$ and $B = \{b_0, \ldots, b_{b-1}\}$ be disjoint cliques, with $b \geq 1$. Suppose the permitted cross edges are exactly $a_i b_j$ with $j < t_i$, where $0 \leq t_i \leq b$ are integers. The cross edges can be covered by $q = \lceil \log_2(b + 1) \rceil$ equivalence subgraphs, each contained in this two-clique graph.*

Proof. Represent all t_i and j with q binary digits. For a digit $0 \leq d < q$ and higher-bit prefix $p \geq 0$, put

$$\begin{aligned} K_{d,p} = \{a_i : \lfloor t_i/2^{d+1} \rfloor = p,\ \mathrm{bit}_d(t_i) = 1\} \\ \cup \{b_j : \lfloor j/2^{d+1} \rfloor = p,\ \mathrm{bit}_d(j) = 0\}. \end{aligned}$$

Same-side pairs in $K_{d,p}$ are edges because A and B are cliques. For a cross pair, the higher bits agree and the dth bit is larger for t_i than for j. Therefore $t_i > j$, whatever the lower bits are, and the cross pair is also an edge. Thus every $K_{d,p}$ is a clique.

For fixed d the sets $K_{d,p}$ are pairwise disjoint. Make them the nontrivial components of a layer and add all unused vertices as singletons. If $j < t_i$, choose the most significant digit where t_i and j differ. At this digit their higher prefixes agree, t_i has bit one and j has bit zero. Thus some $K_{d,p}$ covers $a_i b_j$. No forbidden cross edge is included, by the preceding clique check. □

No ordering or distinctness assumption on the thresholds is needed. The values $t_i = 0$ and $t_i = b$ are included. One extra layer with components A and B covers all same-side edges if needed. The lemma is an explicit binary-prefix realization of the known logarithmic co-chain encoding principle [2]; we do not claim that principle as new.

4 The cyclic block construction

Let $q = \lceil \log_2(k+1) \rceil$. Partition $0, \ldots, n-1$ into consecutive blocks $B_0, \ldots, B_{m-1}$ of size k, except that the last block may have size $r = n \bmod k \in \{1, \ldots, k-1\}$. Thus $m = \lceil n/k \rceil \geq 3$. Each block is a clique. Start the cover with the equivalence subgraph having these blocks as its components.

Since $2k < n$, an edge between different blocks has a unique cyclic direction whose forward arc has length at most k. Such an arc cannot traverse an entire full block between its endpoints: crossing a k-vertex intermediate block requires at least $k+1$ edges. Therefore the arc either crosses one block boundary, or skips the one short last block. These are all possibilities.

To organize the remaining edges, introduce the following oriented *activities*: the m pairs

$$B_a \longrightarrow B_{a+1 \bmod m} \quad (0 \leq a < m)$$

with gap $g = 0$, and, if the last block is short, the extra pair $B_{m-2} \longrightarrow B_0$ with gap $g = r$. For an activity with left-block size a and right-block size b, write i, j for the local forward indices. The forward distance is $a + g + j - i$. Consequently the activity's edges are precisely $j < t_i$, where

$$t_i = \max\{0, \min\{b, k - a - g + i + 1\}\}. \tag{4}$$

Both endpoint blocks are cliques. Lemma 2 therefore supplies a cover of the activity's cross edges. Use q digits for every activity, padding with leading zeros when necessary since $b \leq k$.

If m is even, color the cyclic activities alternately with colors zero and one. If m is odd, color activities $0, \ldots, m-2$ by the parity of their indices and give the final activity color two. In either case, activities of the same color have disjoint endpoint blocks. Assign the optional extra activity color three.

For each color and each digit, combine the binary cliques from all activities of that color into one equivalence subgraph. An activity's cliques are disjoint by Lemma 2, and different activities of that color use disjoint blocks. Component names retain the activity identity, so equal prefixes in different activities do not merge cliques. Add any unused vertices as singleton components.

Every between-block edge belongs to an activity, and the lemma covers it at its highest differing digit. Every within-block edge is in the initial layer. All components used are cliques of C_n^k. We have constructed a cover with at most $1 + 4q$ layers, proving the upper bound in (1). If $k \mid n$, there is no extra activity: two colors suffice when m is even and three when m is odd. This gives (2) and completes the proof of Theorem 1.

Remark 3. When $m = 3$ and the last block is short, the extra activity can join the same unordered pair of full blocks as a cyclic activity, but in the other direction. We keep these activities separate and assign the extra one color three. Their separate edge sets are both valid subgraphs, so this case does not require a simple-graph edge-coloring assertion or an induced-subgraph assumption.

5 Consequences, certificates and scope

Corollary 4. *Uniformly for $n \geq 2k+2$, $\mathrm{eq}(C_n^k) = \Theta(\log(k+1))$. For every integer-valued choice $n(k) \geq 2k+2$,*

$$\frac{\mathrm{eq}(C_{n(k)}^k)}{k} \longrightarrow 0 \qquad (k \longrightarrow \infty).$$

Proof. Both conclusions follow directly from (1), whose constants are independent of n. □

In particular, requiring n to be arbitrarily large compared with k cannot restore the proposed $\Omega(k)$ lower bound. The explicit family $n(k) = 2k(k+1)$ already has $n(k)/k \to \infty$ and admits at most $2\lceil \log_2(k+1) \rceil + 1$ layers. The theorem is stronger because it treats every n in the noncomplete range. It does not give the exact covering number for each pair or the optimal asymptotic constant.

For a small concrete certificate, C_{112}^7 is covered by seven layers; these cover all $112 \cdot 7 = 784$ edges. Since $8 \mid 112$, this also shows that the assertion $\mathrm{eq}(C_n^k) = k+1$ when $(k+1) \mid n$ on the REGS page [5] cannot hold as an equality as written. The usual $k+1$-layer construction is not an optimality proof. This finite observation is not needed for our asymptotic conclusion.

The accompanying standard-library Python implementation outputs every partition and checks every component pair against cyclic distance. It verifies 3,114 parameter pairs, including complete powers, cyclic seams, short remainders and binary thresholds. Four deliberately invalid cases test rejection of a nonedge, a repeated vertex, a removed necessary layer and an out-of-range lower-bound witness. Identical deterministic evidence was obtained with Python 3.9.6 and 3.13.5. Full certificates for $(n, k) = (112, 7)$ and $(1984, 31)$ contain seven and eleven layers, respectively. These checks test the implementation; the general theorem is proved above and is not inferred by extrapolating finite computations.

An earlier MathFest abstract by Blankenship and coauthors [4] announces improved bounds for cycle powers but does not state numerical bounds in the located abstract. We therefore make no absolute priority claim. In particular, both Alon's rank method and the logarithmic co-chain principle are established prior work, expressly credited above.

This unrefereed manuscript was developed with AI assistance in proof construction, literature search, drafting and exact computational checks. The argument was self-audited within the originating research workflow; no independent human review or proof-assistant certification is claimed.

References

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