

A Three-Round Peeling Bound for Internal Partitions of 5- and 8-Regular Graphs

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Abstract

Lin and Zeng proved that every r -regular graph with at least 10^7 vertices has an internal partition when $r \in \{5, 8\}$. We refine their argument to the explicit threshold 500 000. The change is a third round of degree- p deletion in their weighted local-density argument. It replaces the bound on the number of degree- p vertices by $N/p^3 + (2 + 2/p + 2/p^2)|X|$, under a local inequality on sets of at most 53 vertices. Explicit integer choices in the final small-surplus-set argument then yield the improved threshold. More precisely, each part of every minimum bisection contains a $\lceil r/2 \rceil$ -cohesive set. The existence theorem for these two degrees is prior work; our result is a quantitative refinement and does not resolve the internal-partition conjecture for arbitrary regular degree.

1 The bound and its scope

All graphs are finite, simple and undirected. For $S \subseteq V(G)$, write $e_G(S) = |E(G[S])|$ and $d_S(v) = |N_G(v) \cap S|$. A nonempty set S is k -cohesive if $\delta(G[S]) \geq k$. An *internal partition* is a partition into two nonempty parts in which each vertex has at least as many neighbors in its own part as in the other. In an r -regular graph, both parts must therefore be $\lceil r/2 \rceil$ -cohesive.

DeVos's question [4], also recorded as OPG-37163 in *UnsolvedMath* [5], asks whether for each fixed positive integer r only finitely many r -regular graphs fail to have an internal partition. Ban and Linial [2] and Bärnkopf, Nagy and Paulovics [3] develop the cohesive-set approach. Lin and Zeng [1, Theorems 1.2 and 1.3] establish the cases $r = 5, 8$ with threshold 10^7 . We use their minimum-bisection reduction and small-surplus-set lemma, and extend the two-round deletion estimate in their Lemma 3.1.

A *bisection* (A, B) has $||A| - |B|| \leq 1$. It is minimum if its cut has the fewest edges among all bisections.

Theorem 1. *Let $r \in \{5, 8\}$ and let G be an r -regular graph on at least 500 000 vertices. Each part of every minimum bisection of G contains a $\lceil r/2 \rceil$ -cohesive set. Consequently, G has an internal partition.*

The threshold is not claimed optimal. Finding a minimum bisection is not asserted to take polynomial time. The general question for arbitrary r remains open in this work.

2 A local inequality from a minimum bisection

We give the reduction explicitly, keeping the two parameters $k = \lceil r/2 \rceil$ and $p = r - k + 1$ distinct. For $r = 8$, they are 4 and 5.

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Lemma 2 (Minimum-bisection reduction, after Lin-Zeng). *Let $r \in \{5, 8\}$, let G be an r -regular graph on at least 40 vertices, and let (A, B) be a minimum bisection. Suppose B has no k -cohesive set, where $k = \lceil r/2 \rceil$. Put $p = r - k + 1$, $b = |B|$. There are an ordering $v_1, \dots, v_b$ of B and a graph $H = G[A \cup \{v_1\}]$, of order $N = |A| + 1$, such that*

$$p \leq d_H(v) \leq r \leq p + 3, \quad 2e(H) \leq (p + 1)N. \quad (1)$$

For every integer $2 \leq L \leq b$, set

$$X_L = \bigcup_{i=2}^L (N_G(v_i) \cap V(H)).$$

Then $|X_L| \leq r(L - 1)$ and every nonempty $S \subseteq V(H) \setminus X_L$ with $|S| \leq L$ satisfies

$$e_H(S) \leq \sum_{v \in S} (d_H(v) - p). \quad (2)$$

Proof. Write $a = |A|$, $c = e_G(A, B)$ and $q = r - 2k + 2$. Since $G[B]$ is $(k - 1)$ -degenerate,

$$c = rb - 2e_G(B) \geq qb + k(k - 1).$$

The change in the cut on exchanging $u \in A$ and $v \in B$ is $2(d_A(u) + d_B(v) - r + \mathbf{1}_{uv \in E(G)})$. Minimality implies

$$d_A(u) + d_B(v) + \mathbf{1}_{uv \in E(G)} \geq r. \quad (3)$$

Let $\delta_B = \delta(G[B])$. Summing (3) over $u \in A$ with v of degree δ_B gives $c \leq a\delta_B + r - \delta_B$. If $\delta_B \leq k - 2$, this is incompatible with the lower bound: for $r = 5$, the order of G is even, $a = b$, and $c \leq b + 4 < b + 6$; for $r = 8$, $c \leq 2b + 8 < 2b + 12$. Hence $\delta_B = k - 1$.

Choose v_1 of degree $k - 1$ in B . Equation (3) gives $d_H(u) \geq p$ for $u \in A$, and $d_H(v_1) = p$. Furthermore $2e(H) = ra - c + 2p$. For $r = 5$, this is at most $4a$. For $r = 8$, it is at most $6a + 2(a - b) - 2 \leq 6a$. This proves (1).

Extend v_1 to a $(k - 1)$ -degeneracy ordering of B . After moving $v_1, \dots, v_t$ to A , write $A_t = A \cup \{v_1, \dots, v_t\}$. Each move decreases the cut by at least q , so the new cut is at most $c - qt$. Moving any t -element set $S \subseteq A_t$ back to the other part restores the original part sizes and changes this cut by

$$2 \sum_{v \in S} d_{A_t}(v) - rt - 2e_G(S).$$

Minimality of the original bisection therefore implies

$$e_G(S) \leq \sum_{v \in S} d_{A_t}(v) - \frac{r + q}{2}t = \sum_{v \in S} (d_{A_t}(v) - p).$$

If $S \subseteq V(H) \setminus X_L$ and $t = |S| \leq L$, its vertices have no neighbors among $v_2, \dots, v_t$. Thus $d_{A_t}(v) = d_H(v)$ on S , proving (2). $\square$

3 Three rounds of deletion

The next estimate is the quantitative change to the argument.

Lemma 3. *Let $p \geq 3$, and let H be a graph on N vertices with $p \leq d_H(v) \leq p + 3$ and $2e(H) \leq (p + 1)N$. Let $X \subseteq V(H)$, $x = |X|$, and suppose*

$$e_H(S) \leq \sum_{v \in S} (d_H(v) - p) \quad \text{whenever } S \subseteq V(H) \setminus X, \quad |S| \leq 53.$$

If n_0 counts the vertices of degree p , then

$$n_0 \leq \frac{N}{p^3} + \left(2 + \frac{2}{p} + \frac{2}{p^2}\right)x. \quad (4)$$

For $D = \{v : d_H(v) = p + 1\}$, it follows that

$$e_H(D) - |D| \geq \left(\frac{p-1}{2} - \frac{2}{p^2}\right)N - \left(4p + 4 + \frac{4}{p}\right)x. \quad (5)$$

Proof. Put $w(v) = d_H(v) - p \in \{0, 1, 2, 3\}$. Starting with $H_0 = H$, let P_j be the vertices outside X of degree exactly p in H_j , and set $H_{j+1} = H_j - P_j$, for $j = 0, 1, 2$. We first justify that each P_j is independent and that every surviving vertex outside X has degree at least p .

The vertices of P_0 have weight zero, so an edge between two of them would contradict the local inequality. A vertex of P_j has exactly $w(v)$ neighbors in earlier deleted layers. Include all these neighbors, and recursively their earlier-layer neighbors, in a set C_v together with v . By induction, C_v has at most

$$W_j = 1 + 3 + \dots + 3^j \quad (W_0 = 1, W_1 = 4, W_2 = 13)$$

vertices. It is closed under taking earlier-layer neighbors.

For any such closed set, each vertex has exactly $w(v)$ neighbors in earlier layers. Since earlier layers are independent, its internal edge count equals the sum of its weights. If two vertices of the current layer were adjacent, the union of their sets C_v would have an additional edge. Its order is at most $2W_j \leq 26$, contradicting the local inequality. This proves independence of P_j .

After deleting P_j , suppose a surviving vertex $v \notin X$ had degree less than p . It would have at least $w(v) + 1 \leq 4$ neighbors in the deleted layers. Take the union of the predecessor sets of that many neighbors, and then add v . The union before adding v is closed under earlier neighbors and has edge count equal to its total weight. Adding v creates more than $w(v)$ edges. The resulting set is outside X and has at most $1 + 4W_j \leq 53$ vertices, again impossible. This completes the induction through all three rounds.

Write $Q_j = |P_0 \cup \dots \cup P_{j-1}|$. Independence and the exact degree at deletion give $e(H_j) = e(H) - pQ_j$. Thus

$$(p+1)|V(H_j)| - 2e(H_j) = (p+1)N - 2e(H) + (p-1)Q_j \geq (p-1)Q_j.$$

Outside X , a vertex contributes positively to this deficit only if it is in P_j , in which case its contribution is one. Each vertex of X contributes at most $p+1$. Consequently

$$Q_{j+1} \geq pQ_j - (p+1)x.$$

Since $Q_1 \geq n_0 - x$, two further applications give

$$Q_3 \geq p^2n_0 - (2p^2 + 2p + 1)x. \quad (6)$$

The degree lower bound in H_3 also gives

$$p(N - Q_3 - x) \leq 2e(H_3) = 2e(H) - 2pQ_3 \leq (p+1)N - 2pQ_3,$$

so $Q_3 \leq N/p + x$. Combining this with (6) proves (4).

Let n_j count the vertices of original degree $p+j$. The average-degree hypothesis gives $n_0 \geq n_2 + 2n_3$. Bounding the edges from D to its complement by the total degree of the complement yields

$$\begin{aligned} e_H(D) - |D| &\geq \frac{p-1}{2}N - \frac{2p-1}{2}n_0 - \frac{2p+1}{2}n_2 - (p+1)n_3 \\ &\geq \frac{p-1}{2}N - 2pn_0. \end{aligned}$$

Now apply (4). □

4 Explicit constants

We use the following elementary lemma from [1, Lemma 3.2]. Its brief proof is included to make the numerical step self-contained.

Lemma 4 (Small surplus set). *Let $d, \ell \geq 2$ be integers and let F have at most N vertices and maximum degree at most d . If $e(F) - |V(F)| > N^2/\ell^2$, there is a set S of at most $2d\ell$ vertices with $e_F(S) > |S|$.*

Proof. Suppose no such set exists. Discard components of order below ℓ ; their edge surplus is nonpositive. Each remaining component can be partitioned into connected sets of orders between ℓ and $d\ell$. Indeed, root a spanning tree at a leaf and repeatedly remove a deepest descendant subtree of order at least ℓ . Its order is at most $1 + (d-1)(\ell-1)$. Merge a final remainder of order below ℓ , if any, into an adjacent removed set; its combined order is at most $d\ell$.

There are at most N/ℓ resulting sets. Each has at most as many internal edges as vertices. Any two have at most two edges between them: three such edges, together with their connectedness, would give a union with positive surplus and order at most $2d\ell$. The total surplus is therefore at most $2\binom{\lfloor N/\ell \rfloor}{2} \leq N^2/\ell^2$, a contradiction. $\square$

Proof of Theorem 1. Fix a minimum bisection and suppose one part B has no k -cohesive set, $k = \lceil r/2 \rceil$. Apply Lemma 2. Then $N \geq 250\,000$ and $b \geq N - 2$. Use Lemma 3 with $X = X_{53}$ and $|X| \leq 52r$. With $D = \{v : d_H(v) = p + 1\}$, this gives

$$e_H(D) - |D| \geq a_p N - C_p, \quad \begin{array}{c|c|c|c} r & p & a_p & C_p \\ \hline 5 & 3 & 7/9 & 13520/3 \\ 8 & 5 & 48/25 & 51584/5 \end{array}.$$

Set $s = \lceil \sqrt{N} \rceil$. Choose

$$\ell = \begin{cases} 2s, & r = 5, \\ \lceil 5s/4 \rceil, & r = 8, \end{cases} \quad T = 2(p+1)\ell.$$

We have $T \geq 53$ and $T \leq 16s + 9 \leq b$: indeed $s \leq N/500 + 1$ and $16s + 9 \leq 4N/125 + 25 \leq N - 2$.

Let $F = H[D \setminus X_T]$. Its maximum degree is at most $p + 1$. Deleting a vertex of D reduces edge surplus by at most p , and $|X_T| \leq r(T - 1)$. Therefore

$$e(F) - |V(F)| \geq a_p N - C_p - pr(T - 1). \quad (7)$$

We show this is strictly greater than N^2/ℓ^2 .

For $r = 5$, $T - 1 \leq 16s$ and $N^2/\ell^2 \leq N/4$. The difference between the right side of (7) and N^2/ℓ^2 is at least

$$\frac{19}{36}N - 240s - \frac{13520}{3} \geq \frac{43}{900}N - \frac{14240}{3} \geq \frac{64780}{9} > 0.$$

For $r = 8$, $T - 1 \leq 15s + 8$ and $N^2/\ell^2 \leq 16N/25$. The corresponding difference is at least

$$\frac{32}{25}N - 600s - 320 - \frac{51584}{5} \geq \frac{2}{25}N - \frac{56184}{5} \geq \frac{43816}{5} > 0.$$

Here we again used $s \leq N/500 + 1$ and $N \geq 250\,000$.

Lemma 4, with $d = p + 1$, produces $S \subseteq D \setminus X_T$ such that $|S| \leq T$ and $e_H(S) > |S|$. But (2) gives $e_H(S) \leq \sum_{v \in S} (d_H(v) - p) = |S|$, a contradiction. Hence both sides of the bisection contain disjoint k -cohesive sets.

For completeness, the standard extension argument [1, Lemma 2.1] turns these into an internal partition. Keep the two cores U, W disjoint, and repeatedly add to U a vertex outside $U \cup W$ with at least k neighbors in U . At termination, the enlarged U has minimum degree at least k . Put all other vertices into the second part. Vertices in W keep their k neighbors there; each other vertex has at most $k - 1$ neighbors in U , so has at least $r - k + 1 \geq k$ neighbors in the second part. Both parts are nonempty. $\square$

5 Limitations and reproducibility

The existence of internal partitions for all sufficiently large 5- and 8-regular graphs is due to Lin and Zeng [1]. The present change is the three-round estimate (4) and the resulting factor-twenty improvement of their stated threshold. The small-surplus-set lemma and the minimum-bisection strategy are theirs. We make no claim of an optimal threshold or of an exhaustive priority search.

The argument does not extend by substitution to $r = 7$. Then the corresponding degeneracy bound only gives an asymptotic average-degree bound of 6 on H , whereas $p + 1 = 5$; the minimum-degree argument for the discarded side also fails. These missing hypotheses cannot be dropped.

The accompanying checker uses integer arithmetic and rational constants. It tests the three-round lemma on eligible atlas graphs through order seven and the bisection reduction on seeded regular graphs through order fourteen. It also verifies the rounding inequalities on an explicit finite domain. Normal and optimized Python runs agree on 796 323 checked conditions, including 1 102 accepted configurations in which all three deletion layers are nonempty. These checks supplement the written proof; they neither test all large graphs nor constitute formal verification.

AI systems assisted with the literature search, proof development, computation and exposition. The argument and computations were checked within that workflow. This is an unrefereed preprint; no independent review or proof-assistant verification is claimed.

References

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