

The Schrijver–Seymour Bound for Four Weight Classes

Alper Ferudun*

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Abstract

Let a finite matroid be labeled by an arbitrary abelian group, and let S be the set of its basis-weight sums. We prove the Schrijver–Seymour stabilizer bound whenever the nonloop labels occupy at most four cosets modulo $\text{stab}(S)$. There is no restriction on the rank, the size of the matroid, or the ambient group. The proof uses the classical four-variable submodular cone. Its primitive generators have zero–one elemental slacks, which allows integral subtraction inside the minimal face and yields an integer decomposition of every tight four-class rank function. Rado’s theorem transfers that decomposition to an exact sumset identity. Nine atomic types have immediate bounds; the remaining Vámos type reduces to a four-element difference set. Kneser’s theorem completes the proof. The unrestricted Schrijver–Seymour conjecture is not settled.

1 The four-class theorem

Let M be a finite matroid on E , with rank function r_M , and let $w : E \rightarrow G$ label its elements in an additive abelian group. Write $r = r_M(E)$ and

$$S(M, w) = \left\{ \sum_{e \in B} w(e) : B \text{ is a basis of } M \right\}, \quad \text{stab}(S) = \{h \in G : S + h = S\}.$$

The set S is finite and nonempty, including when $r = 0$. Its stabilizer H is therefore finite. A finite set is *aperiodic* if its stabilizer is trivial.

Schrijver and Seymour proved a prime-order bound and proposed the corresponding aperiodic assertion for arbitrary abelian groups [1, Theorem 2.1 and Conjecture 4.2]. The stabilizer formulation is

$$|S(M, w)| \geq |H| \left(1 - r + \sum_{Q \in G/H} r_M(w^{-1}(Q)) \right). \quad (1)$$

Only finitely many summands are nonzero. DeVos, Goddyn and Mohar established the bound for matroids obtained from uniform matroids by adding parallel elements [2]. Our restriction is instead on the number of distinct weight classes, after quotienting by the stabilizer.

Theorem 1.1. *If the nonloop labels of M occupy at most four cosets of $H = \text{stab}(S(M, w))$, then (1) holds.*

The proof uses the classical classification of the four-variable submodular cone due to Promislow and Young [3, Theorem 5.2]; a convenient explicit form appears in Živný and Jeavons [4, Theorem 3.7 and Figure 2], and the classification is also discussed in [5, Section 3.3]. Neither

*Mercury Software GmbH. Email: alper@mercurycodelab.com.

that classification nor Rado's or Kneser's theorem is claimed as new. We give the integral reduction and the atomic weight-set argument explicitly. The result addresses a restricted case of the source problem *Bases of many weights* (OPG-369 in the frozen `ulamai/UnsolvedMath` v1.6.0 corpus); it makes no assertion for general five-or-more-class labelings and no absolute priority claim.

2 Profiles and tight rank functions

For a partition $E = C_1 \sqcup \cdots \sqcup C_k$, put

$$f(I) = r_M \left(\bigcup_{i \in I} C_i \right) \quad (I \subseteq [k]).$$

The function f is normalized, nondecreasing, integral, and submodular. For a vector x , write $x(I) = \sum_{i \in I} x_i$.

Lemma 2.1 (Rado's profile criterion). *The vectors $(|B \cap C_i|)_{i=1}^k$, as B ranges over the bases of M , are exactly*

$$\mathcal{P}(f) = \{x \in \mathbb{Z}_{\geq 0}^k : x([k]) = f([k]), x(I) \leq f(I) \text{ for every } I \subseteq [k]\}.$$

The necessity follows from independence of $B \cap \bigcup_{i \in I} C_i$. For sufficiency apply Rado's independent-transversal theorem to x_i copies of C_i . This is precisely the formulation in Schrijver and Seymour [1, Theorem 3.2]. Two partitioned matroids with the same function f consequently have the same integer profiles and the same basis-weight set when their classes receive the same labels. This exact statement, rather than an equality of real polytopes alone, will be important.

Let $N = [4]$. Empty classes may be added to obtain four classes. For $i \in N$, define

$$\ell_i = f(N) - f(N \setminus \{i\}), \quad g(I) = f(I) - \sum_{i \in I} \ell_i.$$

Diminishing returns gives

$$f(I \cup \{i\}) - f(I) \geq \ell_i \quad (i \notin I).$$

Thus g is again normalized, nondecreasing, integral, and submodular, and

$$g(N \setminus \{i\}) = g(N) \quad (i \in N). \tag{2}$$

We call (2) *tightness*. The defect

$$d(f) = \sum_{i=1}^4 f(\{i\}) - f(N)$$

is unchanged by subtraction of a modular function, so $d(f) = d(g)$.

3 An integral consequence of the classical cone

Let $\mathcal{C}$ be the cone of real submodular functions on 2^N that vanish at the empty set and satisfy (2). All triple values are the full-set value R , leaving eleven coordinates: four singleton values, six pair values, and R . The cone is defined by the 24 elemental inequalities

$$\Delta_{ij|A}(g) = g(A \cup \{i\}) + g(A \cup \{j\}) - g(A) - g(A \cup \{i, j\}) \geq 0, \tag{3}$$

where $i < j$ and $A \subseteq N \setminus \{i, j\}$. These inequalities are equivalent to submodularity. Tightness makes the full-set marginals zero, and diminishing returns then implies monotonicity. Hence every member is nonnegative, $\mathcal{C}$ is pointed, and a nonzero member has $R > 0$.

Type	(g_1, g_2, g_3, g_4)	$(g_{12}, g_{13}, g_{14}, g_{23}, g_{24}, g_{34})$	R	Orbit size
1	(0, 0, 1, 1)	(0, 1, 1, 1, 1, 1)	1	6
2	(0, 1, 1, 1)	(1, 1, 1, 1, 1, 1)	1	4
3	(0, 1, 1, 1)	(1, 1, 1, 2, 2, 2)	2	4
4	(1, 1, 1, 1)	(1, 1, 1, 1, 1, 1)	1	1
5	(1, 1, 1, 1)	(1, 2, 2, 2, 2, 2)	2	6
6	(1, 1, 1, 1)	(2, 2, 2, 2, 2, 2)	2	1
7	(1, 1, 1, 1)	(2, 2, 2, 2, 2, 2)	3	1
8	(1, 1, 1, 2)	(2, 2, 2, 2, 2, 2)	2	4
9	(1, 1, 1, 2)	(2, 2, 3, 2, 3, 3)	3	4
10	(2, 2, 2, 2)	(3, 3, 3, 3, 3, 4)	4	6

Table 1: Primitive generators of the tight four-variable cone. Here $g_i = g(\{i\})$, and similarly for pairs; every triple has value R . The orbit sizes sum to 37.

The classical classification quoted above gives 37 primitive integral extreme rays in ten permutation orbits. Table 1 lists representatives in the tight normalization. Adding or subtracting a modular function preserves all the slacks (3); it converts the published pseudo-Boolean generators to this table.

Lemma 3.1. *Every integral member of $\mathcal{C}$ is a sum of the primitive generators in Table 1, with nonnegative integer multiplicities.*

Proof. Substitution in (3) shows that every primitive generator v has

$$\Delta_{ij|A}(v) \in \{0, 1\}.$$

Let $g \neq 0$ be integral, and choose an extreme ray v in the minimal face of $\mathcal{C}$ containing g , scaled to its primitive representative. Every defining inequality tight at g is tight at v . If $\Delta_{ij|A}(v) = 1$, the corresponding slack of g is a positive integer, hence at least one. Therefore all elemental slacks of $g - v$ are nonnegative. The function $g - v$ is integral and belongs to $\mathcal{C}$.

Its full-set value is a nonnegative integer strictly smaller than $g(N)$, since $v(N) \geq 1$. Induction terminates at zero and proves the assertion. $\square$

This argument is not an inference of integer decomposability from rational generation. The zero-one slack property is the additional ingredient that makes integral subtraction possible.

4 Ten atomic matroids

Each row v of Table 1 is the color-union rank function of a small partitioned matroid M_v . For types 1–9, take respectively:

- 1 $U_{1,2}$ with colors 3, 4
- 2 $U_{1,3}$ with colors 2, 3, 4
- 3 $U_{2,3}$ with colors 2, 3, 4
- 4 $U_{1,4}$ with colors 1, 2, 3, 4
- 5 rank two on colors 1, 2, 3, 4, with 1, 2 parallel
- 6 $U_{2,4}$ with colors 1, 2, 3, 4
- 7 $U_{3,4}$ with colors 1, 2, 3, 4
- 8 $U_{2,5}$ with colors 1, 2, 3, 4, 4
- 9 $U_{3,5}$ with colors 1, 2, 3, 4, 4.

In type 5 all pairs except $\{1, 2\}$ are bases. The rank functions of these models give the first nine rows directly.

For type 10, take four disjoint two-element classes C_1, C_2, C_3, C_4 . Its bases are all four-element subsets of their union except the five sets

$$C_i \cup C_j, \quad \{i, j\} \neq \{3, 4\}.$$

This is the Vámos matroid in its usual pair partition. For completeness, its basis exchange axiom follows because distinct forbidden four-sets intersect in at most two elements. Given bases B, D and $e \in B \setminus D$, if $|D \setminus B| \geq 2$, at most one of the sets $B - e + f$, $f \in D \setminus B$, can be forbidden: any two intersect in the three-element set $B - e$. If $|D \setminus B| = 1$, the exchange gives D . Its singleton class ranks are two; the first five pair-union ranks are three, and $r(C_3 \cup C_4) = 4$. Every triple union has rank four, as required.

Lemma 4.1. *Give distinct labels to the nonempty colors of any atomic model M_v . If its basis-weight set T_v is aperiodic, then*

$$|T_v| \geq d(v) + 1.$$

Proof. Write the color labels as a, b, c, d . For types 1–9 the assertion holds even without aperiodicity. The following subsets of T_v exhibit sufficiently many distinct elements:

Type	$d(v)$	Subset of T_v
1	1	$\{c, d\}$
2	2	$\{b, c, d\}$
3	1	$(b + c + d) - \{b, c, d\}$
4	3	$\{a, b, c, d\}$
5	2	$c + \{a, b, d\}$
6	2	$a + \{b, c, d\}$
7	1	$(a + b + c + d) - \{a, b, c, d\}$
8	3	$d + \{a, b, c, d\}$
9	2	$2d + \{a, b, c\}$.

Translation and negation preserve distinctness in every abelian group.

For type 10, the basis-count profiles are exactly

$$(1, 1, 1, 1), \quad \text{all permutations of } (2, 1, 1, 0), \quad (0, 0, 2, 2).$$

Indeed these are all possible four-element profiles except the five forbidden pair unions. Put $A = \{a, b, c, d\}$ and $t = a + b + c + d$. The profile list gives the exact identity

$$T_v - t = (A - A) \cup \{c + d - a - b\}. \quad (4)$$

Here $d(v) = 4$. If $|A - A| \geq 5$, the bound follows. Otherwise $|A - A| = 4$, since $A - a \subseteq A - A$. For a fixed $a_0 \in A$, the set $B = A - a_0$ contains zero and satisfies $B \subseteq B - B$, with both sets of size four. Thus $B - B = B$, so $B = K$ is a subgroup of order four. Consequently $A = a_0 + K$, $A - A = K$, and $c + d - a - b \in K$. Equation (4) now gives $T_v = t + K$, contradicting aperiodicity. This proves the remaining case. $\square$

5 Exact transfer and proof of the theorem

Proof of Theorem 1.1. Quotient G by $H = \text{stab}(S(M, w))$. The image $\bar{S}$ is aperiodic. Indeed S is H -saturated, so any period of its image lifts to a period of S , necessarily in H . Also $|S| = |H| |\bar{S}|$. It suffices to prove the aperiodic bound for the quotient labeling.

Delete loops, which affect neither basis sums nor the ranks in (1). Partition the remaining elements by their at most four distinct quotient labels. Pad with empty classes if necessary, assigning arbitrary labels to those empty classes. Form f, ℓ_i, g as in Section 2. By Lemma 3.1,

$$g = v_1 + \cdots + v_m$$

for primitive generators, with repetitions allowed.

Construct M' as the direct sum of the corresponding atomic matroids M_{v_j} , with their colors aligned, and ℓ_i coloops of color i . The ranks of color unions add under direct sums, so the color-union rank function of M' is

$$\sum_j v_j(I) + \sum_{i \in I} \ell_i = f(I).$$

Lemma 2.1 gives equality of the integer basis profiles, and hence the exact identity

$$\bar{S} = \sum_i \ell_i a_i + T_{v_1} + \cdots + T_{v_m}. \quad (5)$$

If an original class is empty, $f(\{i\}) = 0$. Nonnegativity then forces $\ell_i = 0$ and $v_j(\{i\}) = 0$ for every j . Thus no atomic component uses an artificial empty color. The labels on all its nonempty classes are distinct, as required by Lemma 4.1.

Any period of a summand in (5) is a period of $\bar{S}$; the same holds for any partial sum. All these sets are therefore aperiodic. Kneser's theorem in its aperiodic form [1, Theorem 4.1] yields, when $m \geq 1$,

$$\begin{aligned} |\bar{S}| &\geq \sum_{j=1}^m |T_{v_j}| - (m-1) \\ &\geq 1 + \sum_{j=1}^m d(v_j) = 1 + d(g) = 1 + d(f). \end{aligned}$$

If $m = 0$, the profile is fixed and both sides are one. Multiplying by $|H|$ gives (1). $\square$

Corollary 5.1. *Suppose $M = M_1 \oplus \cdots \oplus M_q$. If each component has at most four nonloop label classes modulo the stabilizer of the entire basis-weight set, then (1) holds, even if their union uses more than four classes.*

Proof. Work modulo the full stabilizer. Each component weight set and each partial sum is aperiodic. Apply Theorem 1.1 to the components and combine their bounds by Kneser. The rank of each weight class and the total rank are additive under direct sums. $\square$

6 Scope and reproducibility

The four-class hypothesis is essential to this proof method as presented. Three-class profiles can already be described by coordinate minima and maxima; with four classes, genuine pair-union rank inequalities remain. Replacing the profile set by its bounding box may enlarge the weight set, which cannot establish the desired lower bound for the original set. Likewise, aperiodicity need not survive contraction. For $U_{2,3}$ with labels 0, 1, 2 in $\mathbb{Z}/4\mathbb{Z}$, the basis sums are $\{1, 2, 3\}$, but contraction of the element labeled 1 leaves the periodic weight set $\{0, 2\}$. Our direct-sum transfer avoids that inference.

The source package includes exact-arithmetic checks of the 37-ray cone, with two different enumeration implementations and different inequality orders; the zero-one slacks; all ten matroid basis-exchange axioms and color rank tables; and their integer profiles. A separate normalization check matches the ten published pseudo-Boolean generator orbits to Table 1. Finite regression tests cover 275,280 ordered four-label assignments over the groups

$$\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/8\mathbb{Z}, \mathbb{Z}/12\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2, (\mathbb{Z}/2\mathbb{Z})^3, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}.$$

They include the periodic Vámos example in $\mathbb{Z}/4$, which has only four sums and shows why aperiodicity cannot be omitted from its atomic bound. A bounded enumeration through full-set

rank six gives the same 37 additive atoms, but is not used as a proof of the unbounded integer decomposition. Normal and optimized Python runs agree.

These finite checks supplement the written proof; they are not proof-assistant verification. This is an unrefereed preprint. AI systems assisted with literature retrieval, mathematical development, exact checking, and manuscript preparation. No independent-review certification is asserted.

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