

Sending Loxodromic Elements of Right-Angled Artin Groups to Pseudo-Anosov Mapping Classes: A Partial Answer to Problem 2.28 of the K3 List

Alper Ferudun*

October 9, 2026

Abstract

Problem 2.28 of the list *K3: A New Problem List in Low-Dimensional Topology* (Baykur, Kirby and Ruberman, eds., 2026), recorded by T. Koberda, asks: if Γ is a finite graph that is not a join, is there an injective homomorphism from the right-angled Artin group $A(\Gamma)$ to the mapping class group of a surface which sends every loxodromic element to a pseudo-Anosov mapping class? We give a partial answer; the problem remains open in general. We prove that such a homomorphism exists, with a closed surface as target, when Γ is chordal, when the complement graph of Γ is chordal, when Γ is the complement of a cycle of length at least five (this includes the pentagon), and more generally when the complement of Γ is obtained from complete graphs and cycles by gluing along cliques. The answer is also positive for disjoint unions of graphs of these kinds and of joins, with isolated vertices added, and for connected graphs whose right-angled Artin groups have finite index in the group of a connected graph with a positive answer. Further constructions treat the hexagon, graphs obtained by duplicating vertices, and complements of cycles with one more vertex. Together with a computer enumeration of small graphs, the constructions give a positive answer for every graph with at most six vertices that is not a join and for 758 of the 853 such graphs with seven vertices; four more of these are settled by the finite-index property, and for the remaining 91 we have no answer. The homomorphisms are those of Clay, Leininger and Mangahas: the generators are sent to pseudo-Anosov maps of subsurfaces, and by a theorem of these authors the property asked for is equivalent to a filling condition, which we verify. The proofs use subsurface projections (Masur–Minsky, Behrstock) and, for complements of cycles and for the hexagon, planar configurations of discs lifted to totally ramified cyclic covers of the sphere. This is an unrefereed note.

1 Introduction

1.1 The problem

All graphs in this note are finite and simplicial, and subgraphs are induced. For a graph Γ with vertex set $V = V(\Gamma)$, the *right-angled Artin group* is

$$A(\Gamma) = \langle V \mid [v, w] = 1 \text{ whenever } v \text{ and } w \text{ are adjacent in } \Gamma \rangle.$$

The *complement* of Γ (also called the opposite graph) has the same vertices and the reversed adjacency; we always denote it by Λ . The graph Γ is a *join* if V is the disjoint union of two non-empty sets J_1, J_2 such that every vertex of J_1 is adjacent to every vertex of J_2 ; we then write $\Gamma = J_1 * J_2$. Equivalently,

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
Unrefereed preprint. CC BY 4.0.

Λ is disconnected, and equivalently $A(\Gamma)$ is a non-trivial direct product. A *join* of Γ is an induced subgraph J that is a join, and $A(J) \leq A(\Gamma)$ is then called a *join subgroup*. Following Koberda, Mangahas and Taylor [KMT17] and Kim and Koberda [KK14a], an element of $A(\Gamma)$ is *elliptic* if it is conjugate into a join subgroup, and *loxodromic* if it is not the identity and not elliptic; see Section 2.2 for a discussion of this definition.

The following problem was recorded by T. Koberda in the problem list [BKR26]. It is record KP-2.28 of the UnsolvedMath dataset [ula26].

Problem ([BKR26, Problem 2.28]). *Let Γ be a graph that is not a join. Is there an injective homomorphism from $A(\Gamma)$ to the mapping class group of a surface such that every loxodromic element of $A(\Gamma)$ is sent to a pseudo-Anosov mapping class?*

In [BKR26] the problem is worded for a graph that is “not a nontrivial join” and asks for an “injective map”; we read this as above, in accordance with the remarks that accompany the problem. These remarks explain its background. Injective homomorphisms from right-angled Artin groups to mapping class groups exist in abundance [CP01, CLM12, Kob12]. An injective homomorphism never sends an elliptic element to a pseudo-Anosov class [KK14a, Theorem 38], so the problem asks for an embedding under which an element has a pseudo-Anosov image exactly when this is not excluded for algebraic reasons. Not every embedding has this property: Kim and Koberda [KK14a, Proposition 40] describe an embedding of $A(P_4)$, where P_4 is the path on four vertices, into the mapping class group of the closed surface of genus two under which a loxodromic element becomes reducible. (Theorem and proposition numbers of [KK14a] refer to the arXiv version.) By [MT16, KMT17], the restriction of an embedding as in the problem to a subgroup all of whose non-trivial elements are loxodromic is convex cocompact; such subgroups are free. Finally, Remark (4) of the problem reformulates it, in light of [KMT17], as a question about subsurfaces: it suffices to find a surface S and pairwise non-nested, non-annular subsurfaces S_v ($v \in V$), each supporting a pseudo-Anosov map, such that S_v and S_w intersect essentially exactly when v and w are not adjacent in Γ , and such that for every connected subgraph Y of Λ with the property that every vertex of Λ is adjacent to a vertex of Y , every curve of S essentially intersects S_v for some $v \in Y$. This reformulation rests on the work of Clay, Leininger and Mangahas [CLM12], and it is the form in which we treat the problem.

We use the following terminology. An *LPA embedding* of $A(\Gamma)$ is an injective homomorphism $A(\Gamma) \rightarrow \text{Mod}(S)$, for some connected orientable surface S of finite type, that sends every loxodromic element to a pseudo-Anosov mapping class. An *LPA realization* of Γ on S is a family $\{X_v\}_{v \in V}$ of proper, connected, non-annular essential subsurfaces of S (the conventions are fixed in Section 2.3) such that

- (R1) X_v and X_w are disjoint, up to isotopy, if v and w are adjacent in Γ ;
- (R2) X_v and X_w overlap if $v \neq w$ are not adjacent in Γ : they cannot be isotoped to be disjoint, and neither can be isotoped into the other;
- (F) for every non-empty set $Y \subset V$ that is not contained in a join of Γ , the family $\{X_y\}_{y \in Y}$ fills S : no curve of S can be isotoped to be disjoint from all X_y , $y \in Y$.

Conditions (R1) and (R2) say that $\{X_v\}$ realizes Γ nicely in the sense of [CLM12]. By two theorems of Clay, Leininger and Mangahas, an LPA realization gives LPA embeddings: one sends v to a mapping class supported on X_v and pseudo-Anosov there, with large translation length (Proposition 3.2). For homomorphisms of this kind, (F) is also necessary. A non-empty vertex set is not contained in a join of Γ if and only if it induces a connected subgraph of Λ and every vertex outside it has a neighbour in it with respect to Λ (Lemma 3.1); so for graphs without isolated vertices an LPA realization is exactly what Remark (4) of the problem asks for.

1.2 What was known

We found the following positive answers in the literature. For a graph without edges, $A(\Gamma)$ is free and all its non-trivial elements are loxodromic, and LPA embeddings are the free purely pseudo-Anosov subgroups of mapping class groups, which are classical; see [FM02]. Loa [Loa21, Theorem 1.1] proved that two groups generated by multitwists about multicurves that are far apart in the curve graph generate their free product, and that every element of it which is not conjugate into a factor is pseudo-Anosov. This is an LPA embedding of $A(\Gamma)$ for the disjoint union Γ of two complete graphs with at least two vertices each. Aougab, Bray, Dowdall, Hoganson, Maloni and Whitfield [ABD⁺26, Theorem A and Corollary 1.1] treat homomorphisms of the kind of [CLM12] for a graph that is a disjoint union of at least two subgraphs: if the subgroup of each subgraph is reducible and the boundaries of supports belonging to different subgraphs are at distance at least 3 in the curve graph, then for large powers the homomorphism is injective and every element that is not conjugate into a factor is pseudo-Anosov. When every subgraph is a join, these are exactly the loxodromic elements, so their theorem gives a positive answer for disjoint unions of at least two joins. For connected graphs Γ we found no positive answer in the literature. The idea of obtaining subsurfaces for the pentagon from a planar picture and a branched cover is in [CLM12, Section 2.4], where five discs with three punctures each on a sphere with eight punctures are lifted to a two-fold branched cover of genus three; this gives a nice realization, and the filling property (F) is not discussed there. Mangahas and Taylor [MT16] (Section 8 of the arXiv version) realize the complement of the cycle C_{2n} , $n \geq 2$, nicely on a surface of genus $n + 1$, by rotated copies of a one-holed torus and of a sphere with four holes, and find convex cocompact free subgroups of the resulting group; property (F) is not considered there either.

1.3 Results

The problem remains open in general. We answer it positively for the following classes of graphs. In all cases the surface is closed, and when Γ has no isolated vertex the embedding comes from an LPA realization. A graph is *chordal* if it has no induced cycle of length at least four.

Theorem 1.1 (chordal graphs). *Let Γ be a chordal graph with $n \geq 2$ vertices and no isolated vertex, and let S be a closed orientable surface of genus at least $n + 2$. Then Γ has an LPA realization on S by one-holed tori.*

Theorem 1.2 (graphs with chordal complement). *Let Γ be a graph without isolated vertices whose complement Λ is connected and chordal, and let r be the number of maximal cliques of Λ . Then Γ has an LPA realization by one-holed tori on every closed orientable surface of genus at least $2r$.*

Chordal graphs include forests, interval graphs and block graphs. Theorem 1.2 applies, for instance, to the complements of trees that are not stars.

Theorem 1.3 (complements of cycles). *Let $m \geq 5$, let Γ be the complement of the cycle C_m , and let $d \geq 5$ be an odd integer. Then Γ has an LPA realization on the closed orientable surface of genus $(d - 1)(m - 2)/2$ by subsurfaces of complexity at least two. For $m = 5$, where Γ is the pentagon, and $d = 5$, this is a realization on the surface of genus 6 by five subsurfaces of genus 2 with one boundary component.*

A graph is a *clique-sum* of two induced subgraphs Λ_1 and Λ_2 if their vertex sets cover it, their intersection K is a non-empty clique, both contain vertices outside K , and there is no edge between $V(\Lambda_1) \setminus K$ and $V(\Lambda_2) \setminus K$. Let $\mathcal{K}$ be the smallest class of graphs that contains all complete graphs and all cycles of length at least four and is closed under clique-sums. A connected graph is chordal if and only if it can be built from complete graphs by clique-sums, so $\mathcal{K}$ contains the connected chordal graphs and all cycles of length at least four.

vertices	not joins				of these, connected			
	all	(i)	(ii)	(iii)	all	(i)	(ii)	(iii)
4	6	6	6	6	1	1	1	1
5	21	21	21	21	8	8	8	8
6	112	93	105	112	68	49	61	68
7	853	477	578	758	662	305	394	567

Table 1: Numbers of graphs, up to isomorphism, for which a positive answer follows from (i) Theorems 1.1 to 1.3 and the closure properties of Theorem 1.5; (ii) Theorem 1.5; (iii) Proposition 9.3, which combines Theorems 1.5 and 1.6. The numbers are the result of a computer enumeration (Section 11). The four graphs with seven vertices that are settled by Remark 3.9 are not counted. For graphs with one, two or three vertices that are not joins (1, 1 and 2 graphs) the answer is positive by Theorem 1.5.

Theorem 1.4 (gluing along cliques). *Let Γ be a graph without isolated vertices whose complement lies in $\mathcal{K}$. Then Γ has an LPA realization by one-holed tori on a closed orientable surface.*

Theorem 1.4 contains the existence statements of Theorems 1.2 and 1.3; the subsurfaces of Theorem 1.3 are used as building blocks for it. The genera that occur are described in Remark 7.3.

Theorem 1.5 (closure properties; small graphs). *Let Γ be a graph that is not a join, and let Γ^- be the graph obtained by removing its isolated vertices. Suppose that every connected component K of Γ^- is of one of the following kinds:*

- (a) *K is chordal;*
- (b) *the complement of K lies in $\mathcal{K}$;*
- (c) *K is a join.*

Then $A(\Gamma)$ has an LPA embedding into the mapping class group of a closed surface. Every graph with at most five vertices that is not a join satisfies this hypothesis.

In addition, the class of connected graphs Γ that are not joins and whose groups $A(\Gamma)$ have LPA embeddings is closed under passing to right-angled Artin subgroups of finite index (Proposition 3.8).

For graphs with isolated vertices, and more generally for graphs that are not connected, Theorem 1.5 depends on the reading of the word “loxodromic” explained in Section 2.2: the definition above is applied verbatim, which makes the powers of an isolated vertex loxodromic. For connected graphs no such choice is involved.

The next theorem collects constructions that go beyond clique-sums of complete graphs and cycles. A *false twin* of a vertex v is a new vertex with the same neighbours as v and not adjacent to v ; a *true twin* of v is a new vertex adjacent to v and to all neighbours of v .

Theorem 1.6 (further graphs).

- (1) *The hexagon C_6 has an LPA realization on the closed orientable surface of genus 6 by six subsurfaces of genus 2 with one boundary component.*
- (2) *Let $m \geq 5$, and let I be a non-empty set of pairwise non-consecutive vertices of the cycle C_m with $2|I| < m$. The graph obtained from the complement of C_m by adding one vertex adjacent exactly to the vertices of I has an LPA realization by one-holed tori on a closed surface. For $m = 5$ these graphs are the pentagon with a pendant vertex and the pentagon with a triangle attached along an edge.*

- (3) *Let Γ be a graph as in Theorem 1.4, or the hexagon, or a graph as in (2). Every graph obtained from Γ by adding false twins repeatedly, and then adding one true twin to each of some pairwise distinct vertices of the graph obtained, has an LPA realization by one-holed tori on a closed surface.*
- (4) *If Γ is a graph with at most six vertices that is not a join, then $A(\Gamma)$ has an LPA embedding into the mapping class group of a closed surface.*

Part (4) follows from the other statements by an enumeration of the 112 graphs with six vertices that are not joins: Theorem 1.5 covers 105 of them, and the remaining seven are the hexagon and six graphs covered by parts (2) and (3); see Table 2. The enumeration was done by computer (Section 11). A more general statement, which combines Theorems 1.5 and 1.6 and covers more graphs than the two theorems as they are stated here, is Proposition 9.3. Table 1 shows how many graphs with at most seven vertices are reached.

1.4 What stays open

We do not know whether the answer to the problem is positive for all graphs. Among the 853 graphs with seven vertices that are not joins, 758 satisfy the hypothesis of Proposition 9.3, and 95 do not; all of these 95 graphs are connected. Four of them are settled by the finite-index property (Remark 3.9). For the other 91, among them the heptagon C_7 , the hexagon with a pendant vertex and the pentagon with a pendant path of length two, we have no answer. We found no obstruction. A negative answer for some graph would require control of all injective homomorphisms into all mapping class groups, not only of those considered here. The surfaces produced in this note have large genus, and we make no attempt to minimize it; the powers of the pseudo-Anosov maps that are needed come from [CLM12] and are not explicit; see Section 10.

1.5 About the proofs

Everything rests on the criterion of Section 3: one has to construct subsurfaces that are disjoint or overlapping as prescribed by Γ and such that the right subfamilies fill. For chordal graphs (Section 4) the tori are chosen one at a time along a perfect elimination ordering. Each new torus is placed in the complement of the tori of its earlier neighbours, which are pairwise disjoint because the graph is chordal, and it is moved by a high power of a pseudo-Anosov map of that complement. The filling property then follows from the Lipschitz property of subsurface projections [MM00].

For the other theorems the subsurfaces are obtained in two steps. First we construct *territories*: subsurfaces U_v of genus at least two that are disjoint or not as prescribed, and fill as prescribed, but may be nested or equal. Then the *territory lemma* (Lemma 5.2) places a generic one-holed torus in each territory; this repairs nesting and keeps the filling property. Its proof uses the Behrstock inequality [Beh06, Man10]. For graphs with chordal complement the territories are unions of pieces of a surface that is glued along a clique tree (Section 5). For complements of cycles they are preimages of discs under a totally ramified cyclic branched cover of the sphere (Section 6); the planar picture is the cyclic chain of arcs joining m points on a circle. Territory systems for two graphs can be glued along a clique when the territories of every clique have a common interior point (Section 7); this gives Theorem 1.4. Section 8 contains the constructions of Theorem 1.6, and Section 9 the proof of Theorem 1.5 and the enumeration of small graphs.

2 Preliminaries

2.1 Graphs and right-angled Artin groups

Let Γ be a graph with vertex set V and complement Λ . For $Y \subset V$ we write $\Gamma[Y]$ and $\Lambda[Y]$ for the induced subgraphs. The *link* $\text{lk}(v)$ of a vertex v is the set of its neighbours in Γ , and its *star* is $\text{st}(v) = \{v\} \cup \text{lk}(v)$. A vertex is *isolated* if its link is empty. Unless Λ is mentioned, “adjacent” refers to Γ . A set $Y \subset V$ is *dominating in Λ* if every vertex outside Y has a neighbour in Y with respect to Λ . We write $\Gamma_1 \sqcup \Gamma_2$ for the disjoint union of two graphs, K_r for the complete graph, C_m for the cycle and P_m for the path with m vertices.

For $Y \subset V$ the subgroup of $A(\Gamma)$ generated by Y is the right-angled Artin group $A(\Gamma[Y])$: sending the generators outside Y to the identity defines a retraction $A(\Gamma) \rightarrow A(\Gamma[Y])$. For $y \in V$, the *exponent sum* in y is the homomorphism $A(\Gamma) \rightarrow \mathbb{Z}$ that sends y to 1 and the other generators to 0. It is constant on conjugacy classes and vanishes on $A(\Gamma[Y])$ if $y \notin Y$. A word in the generators is a product of *syllables* v^k with $v \in V$ and $k \neq 0$.

2.2 Loxodromic elements

Definition 2.1. An element $g \in A(\Gamma)$ is *elliptic* if it is conjugate into a join subgroup $A(J)$, where J is a join of Γ . It is *loxodromic* if $g \neq 1$ and g is not elliptic.

This is the definition of Koberda, Mangahas and Taylor [KMT17, Sections 2.2 and 2.4], which is the reference given for the problem in [BKR26], with the identity excluded explicitly; Kim and Koberda [KK14a] and [KMT17, Theorem 2.1(5)] classify the non-identity elements into elliptic and loxodromic ones. (If Γ has no edge, the identity is not conjugate into a join subgroup; it has to be excluded.) Three remarks on this definition.

(1) Let Γ be connected, not a join, and with at least two vertices. Then g is loxodromic if and only if its centralizer in $A(\Gamma)$ is cyclic, and if and only if g acts as a loxodromic isometry on the extension graph of Γ [KMT17, Theorem 2.2]. The name comes from the second characterization. We use the first one in Proposition 3.8; the extension graph is not used in our proofs.

(2) The problem only assumes that Γ is not a join. For a graph that is not connected, [KMT17] do not discuss loxodromic elements, and the extension graph is not connected. In this note Definition 2.1 is applied verbatim to all graphs. This is an extension of the terminology of [KMT17], and it is the demanding one: if $g = g_1 g_2$ with $g_i \in A(J_i)$ lies in a join subgroup $A(J_1 * J_2)$, then, for the action by conjugation on the extension graph, g^n maps a vertex $v_2 \in J_2$ to $g_2^n v_2 g_2^{-n}$, which is adjacent to $g_2^n v_1 g_2^{-n} = v_1$ for $v_1 \in J_1$; so an elliptic element has a bounded orbit, and an element all of whose orbits are unbounded is loxodromic in our sense. With our definition, $A(\Gamma)$ contains loxodromic elements if and only if Γ is not a join, as in Remark (2) of the problem: if Γ is not a join, the product of all generators is loxodromic (see the proof of Proposition 3.2). If Γ has an isolated vertex y , then y^k is loxodromic for $k \neq 0$, because y lies in no join. For connected graphs none of this matters.

(3) The converse of the property asked for in the problem holds for every injective homomorphism $\phi: A(\Gamma) \rightarrow \text{Mod}(S)$: if $g \neq 1$ is elliptic, then $\phi(g)$ is not pseudo-Anosov [KK14a, Theorem 38]. Indeed, an element $g = g_1 g_2 \neq 1$ of a join subgroup $A(J_1) \times A(J_2)$ lies in a subgroup isomorphic to $\mathbb{Z}^2$ (generated by g_1 and g_2 if both are non-trivial, and by g_1 and a vertex of J_2 if $g_2 = 1$), whereas the centralizer of a pseudo-Anosov class is virtually cyclic.

2.3 Surfaces and subsurfaces

We follow the conventions of [CLM12, Section 2.3]. A surface S is connected, orientable and of finite type, with genus g and n punctures, and its complexity $\xi(S) = 3g - 3 + n$ is positive. The mapping

class group $\text{Mod}(S)$ is the group of isotopy classes of orientation preserving homeomorphisms of S . A simple closed curve is *essential* if it does not bound a disc or a once-punctured disc. A *curve* of S is an essential simple closed curve, usually considered up to isotopy. A mapping class is *pseudo-Anosov* if none of its non-zero powers fixes the isotopy class of a curve. The geometric intersection number of two curves is $i(\cdot, \cdot)$. Two transverse simple closed curves are in *minimal position* if they realize the intersection number of their classes. By the bigon criterion [FM12, Section 1.2.4], this is the case if and only if they do not form a bigon, that is, there is no embedded disc whose boundary is the union of an arc of one curve and an arc of the other. If there is a bigon, there is one whose interior is disjoint from both curves. Distinct simple closed geodesics of a complete hyperbolic metric of finite area are in minimal position (loc. cit.).

All surfaces constructed in this note are closed of genus at least two. Let S be such a surface. A *subsurface* of S is a compact connected subsurface W whose boundary components are essential curves of S and which is neither an annulus nor a pair of pants; $W = S$ is allowed. These are the non-annular essential subsurfaces of [CLM12]. A *curve* of W is a simple closed curve in W that is essential in S and not parallel to a boundary component of W . A subsurface is π_1 -injective, and two curves of W that are isotopic in S are isotopic in W . Two curves of W that are in minimal position in W are in minimal position in S : a bigon in S formed by them lies in W , because the boundary curves of W do not bound discs. A *one-holed torus* is a subsurface homeomorphic to a torus with an open disc removed.

Let W, W' be subsurfaces. They *can be isotoped apart* if W' is isotopic to a subsurface disjoint from W ; they are *nested* if one of them is isotopic to a subsurface contained in the other (this includes the case that they are isotopic); and they *overlap* if they cannot be isotoped apart and are not nested. A curve c *misses* W if it is isotopic to a curve disjoint from W , and *cuts* W otherwise. A curve that misses W has intersection number zero with every simple closed curve contained in W . A family of subsurfaces *fills* S if every curve of S cuts at least one of them.

2.4 Curve graphs and subsurface projections

Let W be a subsurface of S with $\xi(W) \geq 2$ (boundary components count as punctures); all subsurfaces to which we project have genus at least two. The *curve graph* $\mathcal{C}(W)$ has the isotopy classes of curves of W as vertices, and two vertices are adjacent if the curves can be made disjoint. It is connected; d_W denotes its path metric. For subsets $A, B \subset \mathcal{C}(W)$ we write $d_W(A, B) = \text{diam}(A \cup B)$, as in [CLM12].

The *subsurface projection* of a curve c of S to W is the finite set $\pi_W(c) \subset \mathcal{C}(W)$ defined as follows [MM00, Section 2.3], [CLM12, Section 3.1]. Put c in minimal position with ∂W . If c is contained in W and is a curve of W , then $\pi_W(c) = \{c\}$. If c meets ∂W , then $\pi_W(c)$ consists of the curves of W that are boundary components of regular neighbourhoods of $a \cup \partial W$, where a runs through the arcs of $c \cap W$. Otherwise $\pi_W(c) = \emptyset$. Thus $\pi_W(c) = \emptyset$ if and only if c misses W ; this includes the curves parallel to a boundary component of W . For $W = S$ we have $\pi_S(c) = \{c\}$. For a multicurve, the projection is the union of the projections of its components. We use the following published results.

- (K1) *Lipschitz projection* [MM00, Lemma 2.3]. If c and c' are disjoint or equal curves of S and $\pi_W(c), \pi_W(c')$ are non-empty, then $d_W(\pi_W(c), \pi_W(c')) \leq 2$. We mostly use the contrapositive: if $d_W(x, x') \geq 3$ for some $x \in \pi_W(c)$ and $x' \in \pi_W(c')$, then $i(c, c') > 0$.
- (K2) *Pseudo-Anosov maps move curves linearly* [MM99, Proposition 4.6] (Proposition 3.6 of the arXiv version). There is a constant $\kappa > 0$, depending on W , such that $d_W(\rho^N(a), a) \geq \kappa|N|$ for every pseudo-Anosov mapping class ρ of W , every curve a of W and every integer N . Consequently, for finitely many curves $a_1, \dots, a_p, b_1, \dots, b_q$ of W and every D there is N with $d_W(\rho^N(a_i), b_j) \geq D$ for all i, j .

- (K3) *Existence of pseudo-Anosov maps.* There are homeomorphisms of S that are the identity outside W and whose restriction to W is pseudo-Anosov, for instance suitable products of Dehn twists about two multicurves that fill W [Thu88, Pen88].
- (K4) *Behrstock inequality* [Beh06, Theorem 4.3], in the form of Mangahas [Man10, Lemma 2.5] (numbering of the arXiv version). Let W, W' be overlapping subsurfaces and x a multicurve with non-empty projections to both. If $d_W(x, \partial W') \geq 10$, then $d_{W'}(x, \partial W) \leq 4$.

In (K2) the mapping class group of W is that of the interior of W , a surface with punctures, and (K2) is applied to it. The same constant is used in [CLM12]: a mapping class f of S supported on W has a *translation length* $\tau_W(f) = \lim_{k \rightarrow \infty} d_W(a, f^k(a))/k$, which does not depend on the curve a of W , is positive if and only if f is pseudo-Anosov on W , and satisfies $\tau_W(f^N) = N \tau_W(f)$. This is the stable translation length. It is also defined for subsurfaces of complexity one, such as one-holed tori, with the usual modification of the curve graph [MM00, CLM12]; we need this case only through Theorems 2.4 and 2.5.

2.5 Elementary facts about subsurfaces

The following facts are standard. We include the short proofs, because the arguments below use them constantly.

Lemma 2.2. *Let S be a closed orientable surface of genus at least two, and let W, W' be subsurfaces of S .*

- (a) *If a curve c contained in W misses W' , then c is parallel in W to a boundary component of W . In particular a curve of W cuts W .*
- (b) *If $W \subset W'$ and c is a curve of W , then c is a curve of W' .*
- (c) *If $W \subset W'$ and a curve misses W' , then it misses W .*
- (d) *If W is isotopic to a subsurface of W' and not isotopic to W' , then some boundary component of W is isotopic to a curve of W' .*
- (e) *If W and W' overlap, then some boundary component of W' cuts W and some boundary component of W cuts W' .*
- (f) *Let X be a one-holed torus. A simple closed curve in X that is essential in S is either non-separating in S or parallel to ∂X .*
- (g) *Suppose that the genus of S is at least three. Two one-holed tori with isotopic boundary curves are isotopic, and the boundary curves of two disjoint one-holed tori are not isotopic.*

Proof. (a) Let c' be a curve disjoint from W and isotopic to c . We may assume that c lies in the interior of W . Then c and c' are disjoint and isotopic, so they bound an annulus A . The boundary of A is disjoint from ∂W , so $A \cap \partial W$ is a union of boundary components of W . They are essential in S , hence parallel to the core of A , and there is at least one, because c lies in W and c' does not. The part of A between c and the nearest of them lies in W .

(b) The curve c is essential in S . Suppose that it is parallel in W' to a boundary component δ' of W' , through an annulus $A \subset W'$. If $A \subset W$, then δ' is a boundary component of W and c is parallel to it in W . Otherwise A contains boundary components of W , which are parallel to its core, and the part of A between c and the nearest of them lies in W . In both cases c is not a curve of W .

(c) is clear.

(d) We may assume that $W \subset W'$ and that every boundary component of W lies in the interior of W' or is a boundary component of W' . Suppose that no boundary component of W is a curve of W' . Then every boundary component δ of W is parallel in W' to a boundary component of W' , through an annulus $A_\delta \subset W'$ (which degenerates to a circle if $\delta \subset \partial W'$). The boundary components of W in the interior of A_δ are parallel to its core and cut A_δ into annuli that lie alternately inside and outside W . None of them lies in W , because W is connected and not an annulus; so A_δ meets W only in δ . Hence W' is the union of W and of collars A_δ of its boundary, and W' is isotopic to W .

(e) Suppose that every boundary component of W' misses W . Then every boundary component of W' has intersection number zero with every boundary component of W , and after an isotopy of W' the multicurves $\partial W'$ and ∂W are disjoint (by the argument of Lemma 2.3 below, applied to the multicurve $\partial W'$), where those components of $\partial W'$ that are isotopic to boundary components of W are placed outside W . A component of $\partial W'$ in the interior of W would now be a curve of W , which cuts W by (a). So $\partial W'$ is disjoint from W , and the connected set W lies in W' or is disjoint from W' . Both are excluded. The second assertion follows by symmetry.

(f) A simple closed curve in X that separates X cuts off a disc or an annulus neighbourhood of ∂X , because the genus is one. A curve that does not separate X does not separate S .

(g) After an isotopy the two tori have the same boundary curve γ , and each is the closure of a component of $S \setminus \gamma$. These have genus 1 and $g - 1 \geq 2$, so both tori are the closure of the first one. If two disjoint one-holed tori had isotopic boundaries, the boundaries would bound an annulus outside the tori, and S would be the union of the tori and the annulus, of genus two. $\square$

Lemma 2.3 (simultaneous disjointness). *Let $\mathcal{A}$ be a finite set of essential simple closed curves on a surface, any two of which are disjoint or transverse and in minimal position. Let c be a simple closed curve with $i(c, \delta) = 0$ for all $\delta \in \mathcal{A}$. Then c is isotopic to a curve that is disjoint from all curves of $\mathcal{A}$.*

Proof. Put c in general position with respect to the curves of $\mathcal{A}$: transverse to each of them, and disjoint from the finitely many points in which two of them meet. Suppose that c meets some of them. A curve of $\mathcal{A}$ that meets c forms a bigon with c , by the bigon criterion. Among all bigons formed by c and a curve of $\mathcal{A}$ choose one, B , that is minimal with respect to inclusion; let it be formed by c and δ . An arc of c in the interior of B would have both end points on the side of B that lies on δ , and would cut off a smaller bigon; so c , and in the same way δ , does not meet the interior of B . Let $\delta' \in \mathcal{A}$ be different from δ . No component of δ' lies in B , because δ' is essential. An arc of $\delta' \cap B$ with both end points on c would cut off a smaller bigon formed by c and δ' , and an arc with both end points on δ would cut off a bigon formed by δ' and δ , which do not form bigons. So every arc of $\delta' \cap B$ joins the two sides of B . Pushing c across B therefore removes two points of $c \cap \delta$ and does not change the number of points of $c \cap \delta'$ for $\delta' \neq \delta$. After finitely many such steps c is disjoint from all curves of $\mathcal{A}$. The argument applies without change to a multicurve c with essential components in place of a single curve: no component of c lies in B , and no arc of c meets the interior of B . $\square$

2.6 The theorems of Clay, Leininger and Mangahas

Let $\{X_v\}_{v \in V}$ be proper subsurfaces of S satisfying (R1) and (R2) of Section 1; in the terminology of [CLM12, Section 2.4] they *realize* Γ *nicely*. Let $f_v \in \text{Mod}(S)$ be *fully supported* on X_v : it is represented by a homeomorphism that is the identity outside X_v and pseudo-Anosov on X_v . Mapping classes with disjoint supports commute, so $v \mapsto f_v$ defines a homomorphism $\phi: A(\Gamma) \rightarrow \text{Mod}(S)$. We quote two results with the numbering of the published version of [CLM12]; the first one is Theorem 2.1 in the arXiv version, and Theorem 2.1 of the published version is the statement (K2) above.

Theorem 2.4 ([CLM12, Theorem 2.2]). *For a nice realization $\{X_v\}$ of Γ in S there is a constant $C > 0$ such that, if the f_v are fully supported on the X_v and $\tau_{X_v}(f_v) \geq C$ for all v , then ϕ is a quasi-isometric embedding.*

In particular ϕ is then injective, since a homomorphism that is a quasi-isometric embedding has finite kernel and $A(\Gamma)$ is torsion free. For the second result let $\sigma \in A(\Gamma)$, and let σ' be a conjugate of σ that has the least number of syllables among all words representing conjugates of σ . Let $Y \subset V$ be the set of generators that occur in σ' ; it does not depend on the choice of σ' [CLM12, Section 6].

Theorem 2.5 ([CLM12, Theorem 6.1]). *Under the hypotheses of Theorem 2.4, and with C the constant found in its proof, $\phi(\sigma)$ is pseudo-Anosov if and only if the family $\{X_y\}_{y \in Y}$ fills S .*

In [CLM12] the condition is written $\text{Fill}(X_y : y \in Y) = S$, where the left-hand side is the smallest union of subsurfaces containing the X_y ; it is characterized there by the property that every curve inside it has non-empty projection to some X_y , so it is equal to S exactly when the X_y fill S in our sense. The constant C depends on S and on the realization and is not explicit. Mapping classes as required exist: by (K3) every X_v carries a fully supported class h_v , and $\tau_{X_v}(h_v^N) = N\tau_{X_v}(h_v) \geq C$ for large N .

2.7 Chordal graphs and clique-sums

A graph is chordal if it has no induced cycle of length at least four. We use three classical facts.

- (C1) A graph is chordal if and only if it has a *perfect elimination ordering*: an ordering $u_1, \dots, u_n$ of its vertices such that, for every t , the neighbours of u_t among $u_{t+1}, \dots, u_n$ are pairwise adjacent [Dir61, FG65].
- (C2) A connected chordal graph has a *clique tree*: a tree T whose nodes are the maximal cliques K_t of the graph, such that for every vertex v the set T_v of nodes t with $v \in K_t$ spans a subtree of T [Gav74, Bun74].
- (C3) A connected chordal graph that is not complete has a clique whose removal disconnects it [Dir61]. Consequently the connected chordal graphs are exactly the graphs obtained from complete graphs by repeated clique-sums.

(For the last sentence note that a clique-sum of chordal graphs is chordal: an induced cycle cannot meet both parts outside the common clique.) Splitting a connected graph repeatedly along cliques whose removal disconnects it ends with graphs that have no such clique, the *atoms* of the decomposition [Tar85]. The class $\mathcal{K}$ of Section 1 can be recognized by atoms.

Lemma 2.6. *A connected graph lies in $\mathcal{K}$ if and only if the atoms of one, equivalently of every, decomposition along clique separators are complete graphs or cycles of length at least four.*

Proof. If K is a clique whose removal disconnects a connected graph L and C is a component of the rest, then L is the clique-sum of $L[K \cup C]$ and $L[V(L) \setminus C]$ along K . So a graph all of whose atoms, for some decomposition, are complete graphs or cycles lies in $\mathcal{K}$. Conversely, let $L \in \mathcal{K}$ and let A be an atom of some decomposition of L ; it is a connected induced subgraph of L without clique separator. If a graph is the clique-sum of L_1 and L_2 along K , then such a subgraph A lies in L_1 or in L_2 : otherwise $A \cap K$ would be a clique whose removal disconnects A . By induction on the construction of L , the graph A is an induced subgraph of a complete graph or of a cycle. It is therefore complete, or the whole cycle, since a path with at least three vertices has a cut vertex. $\square$

3 The criterion and general reductions

3.1 Joins and connected dominating sets

Lemma 3.1. *Let Γ be a graph with complement Λ , and let Y be a non-empty set of vertices. Then Y is contained in a join of Γ if and only if $\Lambda[Y]$ is disconnected or Y is not dominating in Λ .*

Proof. If $\Lambda[Y]$ is disconnected, then $Y = Y_1 \sqcup Y_2$ with non-empty parts and no edge of Λ between them, so $\Gamma[Y] = Y_1 * Y_2$ is a join. If some vertex $u \notin Y$ has no neighbour in Y with respect to Λ , then $\Gamma[Y \cup \{u\}] = Y * \{u\}$ is a join. Conversely, let Y be contained in a join $J_1 * J_2$ of Γ . If Y meets both J_1 and J_2 , there is no edge of Λ between $Y \cap J_1$ and $Y \cap J_2$, and $\Lambda[Y]$ is disconnected. If $Y \subset J_1$, a vertex of J_2 lies outside Y and has no neighbour in Y with respect to Λ . $\square$

We record three consequences. First, a non-empty set Y that is not contained in a join induces a connected subgraph of Λ ; if Γ has no isolated vertex, then Y has at least two elements, because a vertex v with a neighbour w lies in the join $\{v\} * \{w\}$. Second, subsets of sets contained in a join are contained in a join. Hence, for a family $\{X_v\}$ of subsurfaces of S , property (F) is equivalent to the following: *for every curve c of S , the set*

$$M(c) = \{v \in V : c \text{ misses } X_v\}$$

is empty or contained in a join of Γ . Third, the sets Y in condition (iii) of Remark (4) of the problem are the non-empty sets that are not contained in a join. Here we read “every vertex of Λ is adjacent to a vertex of Y ” as “ Y is dominating”; for a connected Y with at least two vertices this is the same as the literal reading, and the two readings differ only for $Y = \{v\}$ with v an isolated vertex of Γ .

3.2 The criterion

Proposition 3.2 (criterion). *Let S be a closed orientable surface of genus at least two, and let $\{X_v\}_{v \in V}$ be proper subsurfaces of S that realize Γ nicely: they satisfy (R1) and (R2). Let C be the constant of Theorem 2.4, let $f_v \in \text{Mod}(S)$ be fully supported on X_v with translation length $\tau_{X_v}(f_v) \geq C$, and let $\phi: A(\Gamma) \rightarrow \text{Mod}(S)$ be the homomorphism with $\phi(v) = f_v$. Then ϕ is injective, and the following are equivalent:*

- (1) ϕ sends every loxodromic element to a pseudo-Anosov mapping class;
- (2) the family $\{X_v\}$ has property (F).

In particular, if Γ has an LPA realization on S , then $A(\Gamma)$ has an LPA embedding into $\text{Mod}(S)$.

Proof. Injectivity was noted after Theorem 2.4. Assume (F), and let g be loxodromic. Let g' be a conjugate of g with the least number of syllables, and let Y be the set of generators that occur in g' . Then $Y \neq \emptyset$ and $g' \in A(\Gamma[Y])$. If Y were contained in a join J , then g would be conjugate into $A(J)$. So Y is not contained in a join, $\{X_y\}_{y \in Y}$ fills S by (F), and $\phi(g)$ is pseudo-Anosov by Theorem 2.5.

Conversely, suppose that (F) fails: there is a non-empty set Y , not contained in a join, such that $\{X_y\}_{y \in Y}$ does not fill S . Let w be the product of the elements of Y , each taken once, in some order. For $y \in Y$ the exponent sum in y is 1 on every conjugate of w . Hence every word representing a conjugate of w contains every generator of Y ; so w has the least number of syllables among its conjugates, with set of generators Y , and a conjugate of w lies in $A(\Gamma[Z])$ only if $Y \subset Z$. Therefore w is loxodromic, and $\phi(w)$ is not pseudo-Anosov by Theorem 2.5. $\square$

Remark 3.3. The hypotheses that Proposition 3.2 inherits from [CLM12] are: the X_v are proper, connected and non-annular; they are disjoint up to isotopy for adjacent vertices and overlapping for non-adjacent ones; and the f_v have large stable translation length. One-holed tori are admitted. No hypothesis on the connectedness of Γ or Λ is needed. For a graph that is a join, property (F) is empty.

3.3 Transfer, induced subgraphs, isolated vertices

The following obvious principle is used several times.

Lemma 3.4 (transfer). *Let $\iota: A(\Gamma) \rightarrow A(\widehat{\Gamma})$ be an injective homomorphism that maps loxodromic elements to loxodromic elements. If $A(\widehat{\Gamma})$ has an LPA embedding, then so has $A(\Gamma)$.*

Lemma 3.5 (induced subgraphs). *Let Γ be an induced subgraph of $\widehat{\Gamma}$ such that, for every vertex x of $\widehat{\Gamma}$ outside Γ , the set $\text{lk}(x) \cap V(\Gamma)$ is empty or contained in a join of Γ . Then a subset of $V(\Gamma)$ that is contained in a join of $\widehat{\Gamma}$ is contained in a join of Γ , and the inclusion $A(\Gamma) \rightarrow A(\widehat{\Gamma})$ maps loxodromic elements to loxodromic elements. In particular, the restriction of an LPA realization of $\widehat{\Gamma}$ to the vertices of Γ is an LPA realization of Γ .*

Proof. Let $\widehat{J} = J_1 * J_2$ be a join of $\widehat{\Gamma}$. If both J_1 and J_2 meet $V(\Gamma)$, then $\widehat{J} \cap V(\Gamma)$ is a join of Γ . If, say, J_2 does not meet $V(\Gamma)$, then $\widehat{J} \cap V(\Gamma) \subset \text{lk}(x) \cap V(\Gamma)$ for any $x \in J_2$, and this set is empty or contained in a join of Γ . This proves the first statement. Now let $g \in A(\Gamma)$, $g \neq 1$, and suppose that hgh^{-1} lies in $A(\widehat{J})$ for some $h \in A(\widehat{\Gamma})$. Applying the retraction $r: A(\widehat{\Gamma}) \rightarrow A(\Gamma)$, we find that $r(h)gr(h)^{-1}$ lies in the subgroup generated by $\widehat{J} \cap V(\Gamma)$. This set is not empty because $g \neq 1$, so it is contained in a join of Γ , and g is elliptic in $A(\Gamma)$. The last statement follows from the first one, because (R1) and (R2) are inherited by subfamilies. $\square$

Proposition 3.6 (isolated vertices). *Let Γ be a graph, let I be its set of isolated vertices and $\Gamma^- = \Gamma - I$. Let Γ' be the disjoint union of Γ^- and of one copy P^y of the path P_4 for every $y \in I$. Let $\iota: A(\Gamma) \rightarrow A(\Gamma')$ be the homomorphism that is the identity on $A(\Gamma^-)$ and sends $y \in I$ to the product $a_y b_y$ of the two end vertices of P^y . Then ι is injective and maps loxodromic elements to loxodromic elements. The graph Γ' has no isolated vertex, and it is chordal if and only if Γ is.*

Proof. The group $A(\Gamma)$ is the free product of $A(\Gamma^-)$ and the infinite cyclic groups $\langle y \rangle$, $y \in I$, and $A(\Gamma')$ is the free product of $A(\Gamma^-)$ and the groups $A(P^y)$. Write P^y as $b - a' - b' - a$. Every join of this path is contained in $\{b, a', b'\}$ or in $\{a', b', a\}$, so no join contains both end vertices. For $k \neq 0$ the exponent sums of $(ab)^k$ in a and in b are k ; hence every conjugate of $(ab)^k$ in $A(P^y)$ involves a and b , and $(ab)^k$ is loxodromic in $A(P^y)$. In particular ab has infinite order. So ι is a free product of injective homomorphisms between the free factors; it is injective, and it maps a cyclically reduced element of length $\ell \geq 2$, in the sense of normal forms for free products, to a cyclically reduced element of length ℓ .

We use the conjugacy theorem for free products [LS01, Chapter IV, Section 1]: a cyclically reduced element of length at least two is not conjugate into a free factor, and if a non-trivial element of a free factor F is conjugated by h into a free factor F' , then $F' = F$ and $h \in F$. A join is connected, so every join subgroup of $A(\Gamma)$ or of $A(\Gamma')$ lies in one of the free factors named above. Let $g \in A(\Gamma)$ be loxodromic. If g is not conjugate into a free factor, it is conjugate to a cyclically reduced element of length at least two; so is $\iota(g)$, which is therefore not conjugate into a free factor, and not into a join subgroup. If g is conjugate to $g_0 \in A(\Gamma^-)$, $g_0 \neq 1$, and $\iota(g_0) = g_0$ is conjugate into a join subgroup $A(J)$ of $A(\Gamma')$, then $J \subset \Gamma^-$ and the conjugating element lies in $A(\Gamma^-)$; so g is elliptic in $A(\Gamma)$, a contradiction. If g is conjugate to y^k with $y \in I$ and $k \neq 0$, then $\iota(y^k) = (a_y b_y)^k$, and a conjugate of it in a join subgroup $A(J)$ would force $J \subset P^y$ and would make $(a_y b_y)^k$ elliptic in $A(P^y)$. The statements about Γ' are clear. $\square$

3.4 Disjoint unions

Proposition 3.7 (disjoint unions). *Let S be a closed orientable surface of genus at least two. Let Γ_1, Γ_2 be graphs without isolated vertices, with vertex sets V_1, V_2 , and let $\{X_u\}_{u \in V_1}$ and $\{X_w\}_{w \in V_2}$*

be proper subsurfaces of S that satisfy (R1), (R2) and (F) for Γ_1 and for Γ_2 , respectively. Let φ be a pseudo-Anosov mapping class of S . Then, for all large N , the family $\{X_u\}_{u \in V_1} \cup \{\varphi^N(X_w)\}_{w \in V_2}$ satisfies (R1), (R2) and (F) for the disjoint union $\Gamma_1 \sqcup \Gamma_2$.

For a join, property (F) is empty; so the proposition applies when Γ_1 or Γ_2 is a join with any nice realization.

Proof. By (K2), applied to S , for N large every boundary component of every $\varphi^N(X_w)$ is at distance at least 3 in $\mathcal{C}(S)$ from every boundary component of every X_u . Fix such an N , and let $u \in V_1$, $w \in V_2$. Then every boundary curve of X_u intersects every boundary curve of $\varphi^N(X_w)$ essentially. If the two subsurfaces could be isotoped apart or were nested, their boundaries could be made disjoint; so they overlap, as required for the non-adjacent vertices u , w . A curve that misses a proper subsurface is disjoint from, or equal to, each of its boundary curves, up to isotopy. So a curve missing both X_u and $\varphi^N(X_w)$ would put a boundary curve of the first at distance at most 2 from a boundary curve of the second. Hence X_u and $\varphi^N(X_w)$ together fill S .

Conditions (R1) and (R2) within V_1 and within V_2 are inherited, because φ^N is a homeomorphism. Let $Y \subset V_1 \cup V_2$ be non-empty and not contained in a join of $\Gamma_1 \sqcup \Gamma_2$. If Y meets both V_1 and V_2 , the corresponding family fills by the previous paragraph. Otherwise $Y \subset V_i$ for some i , and Y is not contained in a join of Γ_i ; then the family fills by (F) for Γ_i , which is preserved by the homeomorphism φ^N . $\square$

3.5 Subgroups of finite index

Proposition 3.8 (finite index). *Let Γ and $\widehat{\Gamma}$ be connected graphs with at least two vertices that are not joins, and let $\iota: A(\Gamma) \rightarrow A(\widehat{\Gamma})$ be an injective homomorphism whose image has finite index. Then ι maps loxodromic elements to loxodromic elements. Hence, if $A(\widehat{\Gamma})$ has an LPA embedding, so has $A(\Gamma)$.*

Proof. By [KMT17, Theorem 2.2], a non-trivial element of the group of a connected graph with at least two vertices that is not a join is loxodromic if and only if its centralizer is cyclic. Let $g \in A(\Gamma)$ be loxodromic. The centralizer of $\iota(g)$ in $A(\widehat{\Gamma})$ contains its intersection with the image of ι as a subgroup of finite index, and this intersection is the image of the centralizer of g , which is infinite cyclic. A torsion-free group with an infinite cyclic subgroup of finite index is infinite cyclic. So $\iota(g)$ is loxodromic, and the last statement follows from Lemma 3.4. $\square$

Remark 3.9 (doubles along stars). Let t be a vertex of a graph $\widehat{\Gamma}$, and let Γ be the double of $\widehat{\Gamma}$ along $\text{st}(t)$: two copies of $\widehat{\Gamma}$ glued along the copies of $\text{st}(t)$. By Lemma 50 of the arXiv version of [KK13] (see the proof of Theorem 2 given there), the kernel of the homomorphism $A(\widehat{\Gamma}) \rightarrow \mathbb{Z}/2$ that sends t to 1 and the other generators to 0 is isomorphic to $A(\Gamma)$. So Proposition 3.8 applies whenever $\widehat{\Gamma}$ and its double are connected and not joins. For example, the double of the pentagon along the star of a vertex consists of two pentagons that share a path with three vertices; by Theorem 1.3 and Proposition 3.8 its group has an LPA embedding. This graph has seven vertices and does not satisfy the hypothesis of Proposition 9.3. Among the 95 graphs with seven vertices that do not satisfy this hypothesis, exactly four are doubles along stars of connected graphs with at most six vertices that are not joins (Section 11); by Theorem 1.6(4) and Proposition 3.8 their groups have LPA embeddings. These four graphs are not counted in Table 1.

4 Chordal graphs

In this section Γ is a chordal graph with $n \geq 2$ vertices and no isolated vertex, and S is a closed orientable surface of genus $g \geq n + 2$. We prove Theorem 1.1.

4.1 The construction

By (C1), reversing a perfect elimination ordering, we can number the vertices $v_1, \dots, v_n$ so that for every k the set

$$L_k = \{j < k : v_j \text{ is adjacent to } v_k\}$$

is a clique of Γ . We write $j \sim k$ if v_j and v_k are adjacent. One-holed tori $X_1, \dots, X_n \subset S$, with X_k belonging to v_k , are chosen inductively so that for all $i \neq j$:

- (a) if $i \sim j$, then $X_i \cap X_j = \emptyset$;
- (b) if $i \not\sim j$, then $i(\partial X_i, \partial X_j) > 0$.

Property (b) implies that X_i and X_j overlap: if they could be isotoped apart or were nested, their boundary curves could be made disjoint.

Step k . Suppose that $X_1, \dots, X_{k-1}$ have been chosen with (a) and (b). Since L_k is a clique, the tori X_j , $j \in L_k$, are pairwise disjoint by (a). Let

$$R_k = S \setminus \bigcup_{j \in L_k} \text{int } X_j.$$

This is a connected compact surface of genus $g - |L_k| \geq g - (n-1) \geq 3$ with $|L_k|$ boundary components; it is a subsurface of S , and $R_k = S$ if L_k is empty. Choose a one-holed torus X_k^0 in the interior of R_k . Its boundary is a curve of R_k , because the complement of X_k^0 in R_k has genus at least two. Choose a homeomorphism ρ_k of S that is the identity outside R_k and pseudo-Anosov on R_k (K3), and put $X_k = \rho_k^N(X_k^0)$, where N is chosen below. Then X_k lies in the interior of R_k , so (a) holds for the pairs (i, k) with $i \in L_k$.

Lemma 4.1. *For $j < k$ with $j \notin L_k$, the projection $\pi_{R_k}(\partial X_j)$ is not empty.*

Proof. Otherwise ∂X_j misses R_k , that is, ∂X_j is isotopic to a curve in X_i for some $i \in L_k$; here $i \neq j$. The curve ∂X_j separates S , so by Lemma 2.2(f) it is isotopic to ∂X_i . This is impossible: if $i \sim j$, the tori X_i and X_j are disjoint and Lemma 2.2(g) applies, and if $i \not\sim j$ then $i(\partial X_i, \partial X_j) > 0$ by (b). $\square$

By (K2) we can choose N so large that

$$d_{R_k}(\partial X_k, x) \geq 10 \quad \text{for all } j < k \text{ with } j \notin L_k \text{ and all } x \in \pi_{R_k}(\partial X_j). \quad (4.1)$$

Since $\pi_{R_k}(\partial X_k) = \{\partial X_k\}$, (K1) gives $i(\partial X_k, \partial X_j) > 0$ for these j , which is (b) for the pairs (j, k) . After n steps we have one-holed tori $X_1, \dots, X_n$ with (a) and (b). They satisfy (R1) and (R2), and they are proper subsurfaces.

4.2 The filling property

For a curve c of S let $M(c)$ be the set of indices m such that c misses X_m . Note that c misses X_m if and only if $i(c, \partial X_m) = 0$ and c is not isotopic to a curve of X_m ; and that c is *isotopic into* X_m , meaning isotopic to a simple closed curve in X_m , if and only if it is isotopic to a curve of X_m or to ∂X_m .

Lemma 4.2. *Let c be a curve of S and $k \in \{1, \dots, n\}$.*

- (1) *If $\pi_{R_k}(c) \neq \emptyset$ and $i(c, \partial X_k) = 0$, then $i(c, \partial X_j) > 0$ for every $j < k$ with $j \notin L_k$.*
- (2) *If $\pi_{R_k}(c) = \emptyset$, then c is isotopic into X_i for some $i \in L_k$.*

(3) If c is isotopic into X_k , then c is isotopic to a curve of R_k , and $d_{R_k}(c, \partial X_k) \leq 1$.

Proof. (1) By (K1), every $x \in \pi_{R_k}(c)$ satisfies $d_{R_k}(x, \partial X_k) \leq 2$. If $i(c, \partial X_j) = 0$, then $d_{R_k}(x, y) \leq 2$ for $y \in \pi_{R_k}(\partial X_j)$, again by (K1); this set is non-empty by Lemma 4.1. So $d_{R_k}(\partial X_k, y) \leq 4$, which contradicts (4.1).

(2) The curve c can be isotoped off R_k , that is, into the union of the tori X_i , $i \in L_k$.

(3) We may assume $c \subset X_k \subset \text{int } R_k$. If c were parallel in R_k to a boundary component ∂X_i , $i \in L_k$, then c would separate S , so by Lemma 2.2(f) it would be parallel to ∂X_k , and the disjoint tori X_k and X_i would have isotopic boundaries, contradicting Lemma 2.2(g). So c is a curve of R_k , and it is disjoint from or equal to ∂X_k . $\square$

Lemma 4.3. *If $p \neq q$ are not adjacent, then no curve is isotopic into both X_p and X_q .*

Proof. Let $q < p$, so that $q \notin L_p$, and let c be isotopic into X_p . By Lemma 4.2(3), c is a curve of R_p with $d_{R_p}(c, \partial X_p) \leq 1$. If c were also isotopic into X_q , then $i(c, \partial X_q) = 0$, and (K1) would give $d_{R_p}(c, y) \leq 2$ for $y \in \pi_{R_p}(\partial X_q)$, which is non-empty by Lemma 4.1. Then $d_{R_p}(\partial X_p, y) \leq 3$, contradicting (4.1) for $k = p$. $\square$

Lemma 4.4. *If a curve c is isotopic into X_i , then $M(c) \subset \text{st}(i)$.*

Proof. Let $m \in M(c)$ with $m \neq i$ and $m \not\sim i$; we derive a contradiction. Note that $i(c, \partial X_m) = 0$ and $i(c, \partial X_i) = 0$.

If $m < i$, then $m \notin L_i$. By Lemma 4.2(3), c is a curve of R_i with $d_{R_i}(c, \partial X_i) \leq 1$, and by (K1) $d_{R_i}(c, y) \leq 2$ for $y \in \pi_{R_i}(\partial X_m)$. So $d_{R_i}(\partial X_i, y) \leq 3$, contradicting (4.1) for $k = i$.

If $m > i$, then $i \notin L_m$. If $\pi_{R_m}(c)$ were non-empty, Lemma 4.2(1), applied to $k = m$ and $j = i$, would give $i(c, \partial X_i) > 0$. So $\pi_{R_m}(c) = \emptyset$, and by Lemma 4.2(2) the curve c is isotopic into $X_{i'}$ for some $i' \in L_m$; here $i' \neq i$. If $i \sim i'$, the tori X_i and $X_{i'}$ are disjoint, so c is parallel to the boundary of both by Lemma 2.2(a), which contradicts Lemma 2.2(g). If $i \not\sim i'$, we have a contradiction with Lemma 4.3. $\square$

Lemma 4.5. *If a curve c is not isotopic into any of the tori, then $M(c)$ is a clique of Γ .*

Proof. Let $m' < m$ be elements of $M(c)$ that are not adjacent, so $m' \notin L_m$. By Lemma 4.2(2), $\pi_{R_m}(c)$ is not empty. Since $i(c, \partial X_m) = 0$, Lemma 4.2(1) gives $i(c, \partial X_{m'}) > 0$, although c misses $X_{m'}$. $\square$

Proof of Theorem 1.1. The tori satisfy (R1) and (R2). Let c be a curve of S . By Lemmas 4.4 and 4.5, $M(c)$ is a clique or is contained in a star $\text{st}(i)$. A clique with at least two vertices is a join; a single vertex v lies in the join $\{v\} * \{w\}$ for a neighbour w of v ; and $\text{st}(i) = \{i\} * \text{lk}(i)$ is a join, because $\text{lk}(i) \neq \emptyset$. So $M(c)$ is empty or contained in a join. By the remark after Lemma 3.1, this is property (F). $\square$

Remark 4.6. Chordality is used at one place: the earlier neighbours of each vertex have pairwise disjoint tori, so that the complement of R_k is a disjoint union of tori and Lemma 4.2(2) holds. The genus bound $g \geq n + 2$ guarantees that the surfaces R_k have genus at least three; it is not claimed to be optimal. The graph need not be connected.

Remark 4.7 (comparison with [ABD⁺26]). Let u and w be non-adjacent vertices without a common neighbour, for instance vertices in different connected components of Γ . Then the boundary curves of the tori satisfy $d_S(\partial X_u, \partial X_w) \geq 3$. Indeed, $i(\partial X_u, \partial X_w) > 0$ by (b). Suppose that a curve c is disjoint from both boundary curves, up to isotopy. The set $\{u, w\}$ is not contained in a join, so by (F) the curve c does not miss both tori. If c is isotopic into X_u and misses X_w , then $w \in M(c) \subset \text{st}(u)$ by Lemma 4.4, which is false, and symmetrically; and c is not isotopic into both tori by Lemma 4.3.

So, for a disjoint union of complete graphs with at least two vertices each, the tori of Theorem 1.1 satisfy the hypotheses of [ABD⁺26, Theorem A]: the subgroup of each component is reducible, and boundary curves of supports from different components are at distance at least three. For this class of graphs the existence of an LPA embedding is therefore not new: it follows from [ABD⁺26], and from [Loa21] for two components. Note that in general the filling of S by two subsurfaces is weaker than distance at least three between their boundary curves.

5 Territories, and graphs with chordal complement

5.1 Territory systems

Definition 5.1. Let Γ be a graph with vertex set V and complement Λ , and let S be a closed orientable surface of genus at least two. A *territory system* for Γ on S is a family $\{U_v\}_{v \in V}$ of subsurfaces of S of genus at least two such that

- (T1) if $v \neq w$ are adjacent in Γ , then U_v and U_w have disjoint interiors; if $v \neq w$ are not adjacent in Γ , then U_v and U_w cannot be isotoped apart;
- (T2) for every non-empty set $Y \subset V$ that is not contained in a join of Γ , the family $\{U_y\}_{y \in Y}$ fills S .

The system is *pointed* if in addition

- (P) for every non-empty clique Q of Λ , the interiors of the territories U_q , $q \in Q$, have a common point.

The subsurfaces U_v are called territories. It is allowed that $U_v = S$, and that $U_v = U_w$ for $v \neq w$. Two territories that cannot be isotoped apart overlap or are nested; so a territory system satisfies everything that is required of an LPA realization, except that territories of non-adjacent vertices may be nested or equal. If Γ has no isolated vertex, all territories are proper subsurfaces, because each of them is disjoint from the interior of another one. The following lemma repairs nesting.

Lemma 5.2 (territory lemma). *Let Γ be a graph, S a closed orientable surface of genus at least two, and $\{U_v\}_{v \in V}$ subsurfaces of S of genus at least two that satisfy (T1). Then there are one-holed tori X_v in the interior of U_v , $v \in V$, that satisfy (R1) and (R2). If Γ has no isolated vertex and $\{U_v\}$ is a territory system, the tori can be chosen so that they also have property (F); they are then an LPA realization of Γ on S .*

Proof. Number the vertices $v_1, \dots, v_n$ arbitrarily, and write $U_k = U_{v_k}$. Put $D = 30$. We choose the tori $X_k = X_{v_k}$ inductively, and write $a_k = \partial X_k$. In step k , let X_k^0 be a one-holed torus in the interior of U_k . Its boundary is a curve of U_k , because U_k has genus at least two. Let B_k be the finite set of curves consisting of all boundary components of all territories and of the curves a_j with $j < k$. Let ρ_k be a homeomorphism of S that is the identity outside U_k and pseudo-Anosov on U_k (K3), and put $X_k = \rho_k^N(X_k^0)$, where N is so large that

$$d_{U_k}(a_k, x) \geq D \quad \text{for all } b \in B_k \text{ and all } x \in \pi_{U_k}(b); \quad (5.1)$$

this is possible by (K2). Since $\pi_{U_k}(a_k) = \{a_k\}$, two applications of (K1) give:

$$\text{if } c \text{ is a curve with } \pi_{U_k}(c) \neq \emptyset \text{ and } i(c, a_k) = 0, \text{ then } i(c, b) > 0 \text{ for every } b \in B_k \text{ with } \pi_{U_k}(b) \neq \emptyset. \quad (5.2)$$

Indeed, if $i(c, b) = 0$, then $d_{U_k}(x, y) \leq 2$ for $x \in \pi_{U_k}(c)$ and $y \in \pi_{U_k}(b)$, and $d_{U_k}(x, a_k) \leq 2$, so $d_{U_k}(a_k, y) \leq 4 < D$.

Step 1: (R1) and (R2). If v_j and v_k are adjacent, the tori X_j and X_k lie in the interiors of U_j and U_k , which are disjoint. Let $j \neq k$ be such that v_j and v_k are not adjacent. We show $i(a_j, a_k) > 0$; then X_j and X_k overlap, because otherwise their boundary curves could be made disjoint. By (T1) there are three cases.

The territories U_j and U_k overlap. By Lemma 2.2(e) there are a boundary component b of U_j with $\pi_{U_k}(b) \neq \emptyset$ and a boundary component b' of U_k with $\pi_{U_j}(b') \neq \emptyset$. By (5.1) for k and (K1), $i(a_k, b) > 0$; so a_k cuts U_j , that is, $\pi_{U_j}(a_k) \neq \emptyset$. Suppose $i(a_j, a_k) = 0$. Then $d_{U_j}(a_j, y) \leq 2$ for $y \in \pi_{U_j}(a_k)$ by (K1). The curve a_k lies in the interior of U_k and is disjoint from b' , so $d_{U_j}(y, z) \leq 2$ for $z \in \pi_{U_j}(b')$. Hence $d_{U_j}(a_j, z) \leq 4$, contradicting (5.1) for j .

One territory is isotopic to a subsurface of the other, and they are not isotopic. Let U_j be isotopic to a subsurface of U_k . By Lemma 2.2(d) some boundary component b of U_j is isotopic to a curve of U_k . By (5.1) for k and (K1), $i(a_k, b) > 0$, so $\pi_{U_j}(a_k) \neq \emptyset$. If $j > k$, then $a_k \in B_j$, and (5.1) for j with (K1) gives $i(a_j, a_k) > 0$. If $j < k$, then a_j is a curve of U_j , hence isotopic to a curve of U_k by Lemma 2.2(b), and $a_j \in B_k$; so (5.1) for k gives $d_{U_k}(a_k, a_j) \geq D$ and $i(a_j, a_k) > 0$.

The territories are isotopic. Let $j < k$. Then a_j is isotopic to a curve of U_k and lies in B_k , and (5.1) for k gives $i(a_j, a_k) > 0$.

Step 2: property (F). Now let Γ have no isolated vertex and assume (T2). Let Y be non-empty and not contained in a join, and suppose that a curve c misses X_y for all $y \in Y$. Let M be the set of all m such that c misses X_m . It contains Y (in the numbering), so it is not contained in a join; by Lemma 3.1, $\Lambda[M]$ is connected and $|M| \geq 2$. Let M_2 be the set of $m \in M$ with $\pi_{U_m}(c) \neq \emptyset$. We claim that M_2 is empty. Suppose $m \in M_2$, and let $m' \in M$ be a neighbour of m in Λ . Then $v_m, v_{m'}$ are not adjacent in Γ , and $i(c, a_m) = i(c, a_{m'}) = 0$. By (T1) we are in one of the following four cases.

Case 1: U_m and $U_{m'}$ overlap. Let b be a boundary component of $U_{m'}$ with $\pi_{U_m}(b) \neq \emptyset$ and b' a boundary component of U_m with $\pi_{U_{m'}}(b') \neq \emptyset$ (Lemma 2.2(e)). By (5.2) for m , $i(c, b) > 0$; so $\pi_{U_{m'}}(c) \neq \emptyset$. By (K1), $d_{U_m}(x, a_m) \leq 2$ for $x \in \pi_{U_m}(c)$, and by (5.1) for m the distance from a_m to every point of $\pi_{U_m}(b)$ is at least D . Hence $d_{U_m}(c, \partial U_{m'}) \geq D - 2 \geq 10$, and the Behrstock inequality (K4) gives $d_{U_{m'}}(c, \partial U_m) \leq 4$. By (5.1) for m' , $d_{U_{m'}}(a_{m'}, z) \geq D$ for $z \in \pi_{U_{m'}}(b')$. So $d_{U_{m'}}(a_{m'}, x') \geq D - 4 > 2$ for $x' \in \pi_{U_{m'}}(c)$, which contradicts (K1), because $i(c, a_{m'}) = 0$.

Case 2: $U_{m'}$ is isotopic to a subsurface of U_m , and they are not isotopic. A boundary component b of $U_{m'}$ is isotopic to a curve of U_m (Lemma 2.2(d)). By (5.2) for m , $i(c, b) > 0$, so $\pi_{U_{m'}}(c) \neq \emptyset$. If $m' < m$, then $a_{m'}$ is isotopic to a curve of U_m and lies in B_m , and (5.2) gives $i(c, a_{m'}) > 0$, a contradiction. If $m' > m$, then $i(a_m, b) > 0$ by (5.1) for m , so $\pi_{U_{m'}}(a_m) \neq \emptyset$, and $a_m \in B_{m'}$. Since $i(c, a_m) = 0$, (K1) gives $d_{U_{m'}}(x', y) \leq 2$ for $x' \in \pi_{U_{m'}}(c)$ and $y \in \pi_{U_{m'}}(a_m)$; with (5.1) for m' this gives $d_{U_{m'}}(a_{m'}, x') \geq D - 2 > 2$, contradicting (K1) and $i(c, a_{m'}) = 0$.

Case 3: U_m is isotopic to a subsurface of $U_{m'}$, and they are not isotopic. By Lemma 2.2(c), c cuts $U_{m'}$, so $m' \in M_2$. Exchanging the roles of m and m' we are in Case 2.

Case 4: U_m and $U_{m'}$ are isotopic. Then $m' \in M_2$ as well, and we may assume $m' < m$. The curve $a_{m'}$ is isotopic to a curve of U_m and lies in B_m , so (5.2) gives $i(c, a_{m'}) > 0$, a contradiction.

This proves the claim. So c misses the territory U_m for every $m \in M$, in particular for every $y \in Y$. This contradicts (T2). $\square$

5.2 Territories from a clique tree

Proposition 5.3. *Let Γ be a graph whose complement Λ is connected and chordal, and let r be the number of maximal cliques of Λ . Then Γ has a pointed territory system on every closed orientable surface of genus at least $2r$.*

Proof. Let T be a clique tree of Λ (C2), with nodes t corresponding to the maximal cliques K_t , and let T_v be the subtree of nodes t with $v \in K_t$. For every node t take a compact orientable surface P_t

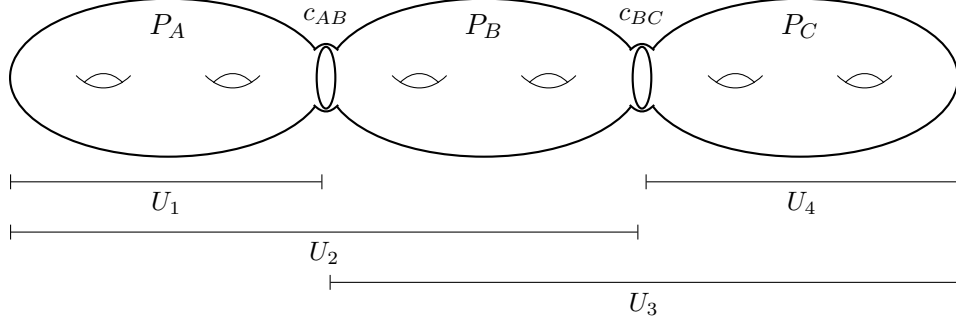


Figure 1: The territories of Proposition 5.3 for the path $\Lambda = P_4$ with vertices 1, 2, 3, 4. The maximal cliques are $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{3, 4\}$, the clique tree is the path $A - B - C$, and the surface is glued from three pieces of genus two. The territories $U_1 \subset U_2$ and $U_4 \subset U_3$ are nested, U_2 and U_3 overlap, and the other pairs have disjoint interiors. The only minimal vertex set that is not contained in a join of Γ is $\{2, 3\}$, and $U_2 \cup U_3 = S$.

of genus $g_t \geq 2$ with one boundary circle for each edge of T at t , and glue P_t to $P_{t'}$ along a circle $c_{tt'}$ for each edge tt' of T . The result is a closed orientable surface S of genus $\sum_t g_t$, which can be any integer $\geq 2r$. Each circle $c_{tt'}$ separates S into two parts that contain a piece of genus at least two, so it is an essential curve; and no two of these circles are isotopic, because they are disjoint and the region between them contains a piece. Let μ be the multicurve formed by them. Put

$$U_v = \bigcup_{t \in T_v} P_t.$$

Since T_v is a subtree, U_v is connected; it is a subsurface of genus at least two, bounded by the circles $c_{tt'}$ with $t \in T_v$ and $t' \notin T_v$ (and $U_v = S$ if v lies in all maximal cliques).

(T1). Two vertices $v \neq w$ are adjacent in Λ if and only if they lie in a common maximal clique, that is, if and only if T_v and T_w have a common node. If they have none, U_v and U_w have disjoint interiors. If t is a common node, then U_v and U_w contain the piece P_t , and P_t contains two curves a , b of P_t with $i(a, b) > 0$. If U_v and U_w could be isotoped apart, a and b could be made disjoint.

(P). A clique Q of Λ is contained in a maximal clique K_t , and then $P_t \subset U_q$ for all $q \in Q$.

(T2). Let Y be non-empty and not contained in a join of Γ . By Lemma 3.1, $\Lambda[Y]$ is connected and Y is dominating in Λ . We use two facts.

- (i) *Every maximal clique K_t meets Y .* Let T_Y be the union of the subtrees T_y , $y \in Y$. It is connected, because the subtrees of two vertices that are adjacent in Λ meet and $\Lambda[Y]$ is connected. Suppose that a node t is not in T_Y , and let t' be the neighbour of t on the path from t to T_Y . Every $u \in K_t$ lies outside Y and has a neighbour $y \in Y$ in Λ . The vertices u and y lie in a common clique K_s , where $s \in T_y \subset T_Y$. The subtree T_u contains t and s , hence t' . So $K_t \subset K_{t'}$, which contradicts the maximality of K_t .
- (ii) *For every edge tt' of T , some $y \in Y$ lies in $K_t \cap K_{t'}$.* Otherwise every subtree T_y , $y \in Y$, lies on one side of the edge tt' . Since T_Y is connected, all of them lie on the same side, and the clique of the node on the other side does not meet Y . This contradicts (i).

Now suppose that a curve c misses U_y for all $y \in Y$, and put c in minimal position with μ . If c is isotopic to a component $c_{tt'}$ of μ , take $y \in Y$ with $t, t' \in T_y$, by (ii). Then $c_{tt'}$ lies in the interior of U_y with the pieces P_t and $P_{t'}$ on its two sides; so it is a curve of U_y and cuts U_y (Lemma 2.2(a)), a contradiction. If c is not isotopic to a component of μ , let $y \in Y$. The boundary of U_y consists of

components of μ , which have intersection number zero with c ; so c is disjoint from ∂U_y . It does not lie in U_y , because it would then be parallel to a boundary component of U_y by Lemma 2.2(a). So c is disjoint from U_y for every $y \in Y$. But by (i) the territories U_y , $y \in Y$, cover S . $\square$

Proof of Theorem 1.2. Since Γ has no isolated vertex, Proposition 5.3 and the territory lemma give one-holed tori that form an LPA realization of Γ on every closed orientable surface of genus at least $2r$. $\square$

In the situation of Theorem 1.2 one has $r \geq 3$: if a connected graph has exactly two maximal cliques, a vertex in their intersection is adjacent to all other vertices.

5.3 Nice realizations of arbitrary graphs

For join components we need nice realizations without any filling property. Their existence is well known [CLM12, Section 2.4]; we need them on closed surfaces of every large genus.

Lemma 5.4. *For every graph Δ there is g_0 such that on every closed orientable surface of genus at least g_0 there are one-holed tori X_v , $v \in V(\Delta)$, satisfying (R1) and (R2) for Δ .*

Proof. Let E' be the set of pairs $e = \{v, w\}$ of distinct non-adjacent vertices. Take compact orientable surfaces Q_v (v a vertex), Q_e ($e \in E'$) and Q_0 , all of genus at least two, with boundary circles as follows: Q_v has one circle for each $e \in E'$ containing v and one more; Q_e has one circle for each of its two elements and one more; Q_0 has one circle for each vertex and one for each $e \in E'$. Glue a circle of Q_v to a circle of Q_e whenever $v \in e$, and the remaining circles of the Q_v and Q_e to the circles of Q_0 . The result is a closed orientable surface, and by varying the genus of Q_0 we obtain every sufficiently large genus. Every gluing circle is essential, since it has a piece of genus at least two on either side. Let U_v be the union of Q_v and the Q_e with $v \in e$. It is a subsurface of genus at least two. If $v \neq w$ are adjacent in Δ , then U_v and U_w have no common piece. If they are not adjacent, both contain $Q_{\{v,w\}}$, and they cannot be isotoped apart, by the argument given for (T1) in the proof of Proposition 5.3. So (T1) holds, and the first statement of the territory lemma gives the tori. $\square$

6 Complements of cycles

In this section subsurfaces are obtained as preimages of discs under a branched cover of the sphere; they intersect in the same pattern as the discs.

6.1 Totally ramified cyclic covers

Let $P \subset S^2$ be a finite set. We consider branched coverings $\pi: \tilde{S} \rightarrow S^2$ of degree d , where $\tilde{S}$ is a closed connected orientable surface, that are unbranched over $S^2 \setminus P$ and such that every point of P has exactly one preimage. We call such a covering *totally ramified over P* .

Lemma 6.1. *Let $d \geq 3$ be odd, let $P \subset S^2$ be finite with $|P| \geq 3$, and let $w: P \rightarrow \mathbb{Z}/d$, $p \mapsto w_p$, take values in the units of $\mathbb{Z}/d$ and satisfy $\sum_{p \in P} w_p = 0$. Then there is a branched covering $\pi: \tilde{S} \rightarrow S^2$ of degree d , totally ramified over P , such that $\tilde{S}$ has genus $(d-1)(|P|-2)/2$ and such that for every closed disc $D \subset S^2$ with $\partial D \cap P = \emptyset$ the preimage $\pi^{-1}(\partial D)$ has $\gcd(d, \sum_{p \in P \cap D} w_p)$ components.*

Here the sum is taken in $\mathbb{Z}/d$, and $\gcd(d, 0) = d$.

Proof. The group $\pi_1(S^2 \setminus P)$ is generated by loops γ_p around the points of P , subject to one relation which says that a product of the γ_p is trivial. So $\gamma_p \mapsto w_p$ defines a homomorphism to $\mathbb{Z}/d$, which is onto because the w_p are units. Let $\tilde{S}^\circ \rightarrow S^2 \setminus P$ be the corresponding connected covering of degree d . Since w_p generates $\mathbb{Z}/d$, the preimage of a small punctured disc around p is connected; so the covering extends to a branched covering $\pi: \tilde{S} \rightarrow S^2$ of a closed surface with exactly one point over each $p \in P$. The Riemann–Hurwitz formula gives $\chi(\tilde{S}) = 2d - |P|(d - 1)$. The boundary of a disc D as in the statement is freely homotopic in $S^2 \setminus P$ to a product of the loops γ_p , $p \in P \cap D$, so it is mapped to $s = \pm \sum_{p \in P \cap D} w_p$, and its preimage has $d/\text{ord}(s) = \gcd(d, s)$ components. $\square$

Lemma 6.2. *Let $d \geq 5$ be odd.*

- (a) *Every element of $\mathbb{Z}/d$ is a sum of two units.*
- (b) *For every $m \geq 5$ there are units $w_1, \dots, w_m$ of $\mathbb{Z}/d$ with $w_1 + \dots + w_m = 0$ and $w_{j-1} + w_j \neq 0$ for all j , indices taken modulo m .*

Proof. For $s \in \mathbb{Z}/d$ let $N(s)$ be the number of units a such that $s - a$ is a unit. By the Chinese remainder theorem, $N(s)$ is the product, over the prime powers ℓ^e exactly dividing d , of the number of residues modulo ℓ^e that are not congruent to 0 or to s modulo ℓ ; so $N(s) \geq \prod (\ell - 2)\ell^{e-1}$. This product is at least 3 for every odd $d \geq 5$: it is at least $\ell - 2 \geq 3$ if a prime $\ell \geq 5$ divides d , and equal to $3^{e-1} \geq 3$ if $d = 3^e$ with $e \geq 2$. This proves (a).

(b) Let $s = -(m-2)$. If $s \neq 0$ in $\mathbb{Z}/d$, put $w_1 = \dots = w_{m-2} = 1$, $w_{m-1} = a$ and $w_m = s - a$, where a and $s - a$ are units and $a \neq -1$, $a \neq s + 1$; such an a exists because $N(s) \geq 3$. The sums of cyclically consecutive terms are 2, $1 + a$, s and $s - a + 1$, all non-zero. If $s = 0$ in $\mathbb{Z}/d$, put $w_1 = \dots = w_{m-3} = 1$ and $(w_{m-2}, w_{m-1}, w_m) = (2, 1, -2)$. The sum is $m - 2 = 0$, and the consecutive sums are 2, 3, 3, -1 , -1 . $\square$

6.2 The lifting lemma

Lemma 6.3 (lifting). *Let $P \subset S^2$ be finite with $|P| \geq 4$, and let $\pi: \tilde{S} \rightarrow S^2$ be a branched covering of degree $d \geq 5$ that is totally ramified over P . Let $D_1, \dots, D_m \subset S^2$ be closed discs with $2 \leq |D_j \cap P| \leq |P| - 2$ whose boundary curves avoid P and, pairwise, are disjoint or are transverse and in minimal position in $S^2 \setminus P$. Put $\tilde{X}_j = \pi^{-1}(D_j)$.*

- (1) *Every component of $\pi^{-1}(\partial D_j)$ is an essential curve of $\tilde{S}$, and $\tilde{X}_j$ is a proper subsurface of $\tilde{S}$ with Euler characteristic $d - |D_j \cap P|(d - 1)$ and complexity $\xi(\tilde{X}_j) \geq 2$.*
- (2) *For $i \neq j$, every component of $\pi^{-1}(\partial D_i)$ and every component of $\pi^{-1}(\partial D_j)$ are disjoint or transverse and in minimal position.*
- (3) *If $D_i \cap D_j = \emptyset$, then $\tilde{X}_i \cap \tilde{X}_j = \emptyset$. If $\partial D_i \cap \partial D_j \neq \emptyset$, then $\tilde{X}_i$ and $\tilde{X}_j$ overlap.*
- (4) *Let $Y \subset \{1, \dots, m\}$ be such that every component of $S^2 \setminus \bigcup_{y \in Y} \partial D_y$ is an open disc that contains at most one point of P . Then every curve of $\tilde{S}$ has positive intersection number with a component of $\pi^{-1}(\partial D_y)$ for some $y \in Y$. In particular $\{\tilde{X}_y\}_{y \in Y}$ fills $\tilde{S}$.*
- (5) *The hypothesis of (4) is satisfied if for every partition $P = A \sqcup B$ with $|A|, |B| \geq 2$ there is $y \in Y$ such that the four sets $D_y \cap A$, $D_y \cap B$, $A \setminus D_y$, $B \setminus D_y$ are non-empty.*

Proof. We use the Riemann–Hurwitz formula in the following form. Let $R \subset S^2$ be a component of the complement of finitely many of the curves ∂D_j , let $k = |R \cap P|$, and let W be a component of

$\pi^{-1}(R)$. Then $W \rightarrow R$ is a branched covering of some degree $d' \leq d$. If $k \geq 1$, the unique preimage of a point of $R \cap P$ lies in the closure of every sheet over R , so $W = \pi^{-1}(R)$, $d' = d$ and

$$\chi(W) = d\chi(R) - k(d-1).$$

If $k = 0$, then $\chi(W) = d'\chi(R)$. In particular: *if R is an open disc with at most one point of P , then W is an open disc* (it is a connected surface that covers a disc, or a connected surface with $\chi(W) = d - (d-1) = 1$ that is not closed).

(1) Suppose that a component of $\pi^{-1}(\partial D_j)$ bounds a disc in $\tilde{S}$. Among the discs in $\tilde{S}$ bounded by components of $\pi^{-1}(\partial D_j)$ choose one, $\tilde{B}$, that is minimal with respect to inclusion. Its interior W is a component of $\pi^{-1}(E)$ for one of the two open discs E bounded by ∂D_j . But $k = |E \cap P| \geq 2$ gives $\chi(W) = d - k(d-1) \leq 2 - d < 1$. So all components of $\pi^{-1}(\partial D_j)$ are essential. The set $\tilde{X}_j$ is a compact surface with boundary $\pi^{-1}(\partial D_j)$. It is connected, because every point of it can be joined, by lifting a path in D_j , to the unique point over a given point of $D_j \cap P$. Its Euler characteristic is $d - k(d-1)$ with $k = |D_j \cap P| \geq 2$. Its boundary covers ∂D_j with degree d , so it has $b \leq d$ components. If h is its genus, then

$$\xi(\tilde{X}_j) = 3h - 3 + b = \frac{-3\chi(\tilde{X}_j) - b}{2} \geq \frac{3k(d-1) - 4d}{2} \geq d - 3 \geq 2.$$

Finally $\tilde{X}_j \neq \tilde{S}$.

(2) Let $\alpha = \partial D_i$ and $\beta = \partial D_j$, and suppose that a component of $\pi^{-1}(\alpha)$ and a component of $\pi^{-1}(\beta)$ are not in minimal position. They are transverse, so they form a bigon. Among all bigons formed by a component of $\pi^{-1}(\alpha)$ and a component of $\pi^{-1}(\beta)$ choose one, $\tilde{B}$, that is minimal with respect to inclusion. Its interior W does not meet $\pi^{-1}(\alpha \cup \beta)$: an arc of $\pi^{-1}(\alpha)$ in $\tilde{B}$ would have both end points on the side that lies on $\pi^{-1}(\beta)$ and would cut off a smaller bigon; a closed component in $\tilde{B}$ would be inessential, against (1); and the same holds for β . So W is a component of $\pi^{-1}(R)$ for a component R of $S^2 \setminus (\alpha \cup \beta)$, and $\chi(W) = 1$. This forces $\chi(R) = 1$, so R is an open disc; with $k = |R \cap P|$ we get $k = 0$ and $d' = 1$, or $k = 1$ and $d' = d$. The closure of R is a disc whose boundary consists alternately of arcs of α and of β ; so it has an even number, at least two, of corners. The boundary circle of $\tilde{B}$ covers the boundary circle of R with degree d' , and corners are mapped to corners. Hence $2 = d' \cdot (\text{number of corners of } R)$, so $d' = 1$, $k = 0$, and R is a bigon of α and β without points of P . This contradicts the minimal position of α and β in $S^2 \setminus P$.

(3) The first statement is clear. If ∂D_i and ∂D_j intersect, some component of $\pi^{-1}(\partial D_i)$ meets some component of $\pi^{-1}(\partial D_j)$, and by (2) these two boundary curves of $\tilde{X}_i$ and $\tilde{X}_j$ have positive intersection number. If $\tilde{X}_i$ and $\tilde{X}_j$ could be isotoped apart or were nested, their boundaries could be made disjoint.

(4) By the remark at the beginning of the proof, every component of the complement of $\bigcup_{y \in Y} \pi^{-1}(\partial D_y)$ in $\tilde{S}$ is an open disc. Let $\mathcal{A}$ be the set of all components of the multicurves $\pi^{-1}(\partial D_y)$, $y \in Y$. They are essential by (1), and any two of them are disjoint or transverse and in minimal position by (2). If a curve c of $\tilde{S}$ had intersection number zero with all of them, it would be isotopic to a curve disjoint from all of them by Lemma 2.3, hence to a curve in an open disc. For the last assertion, note that a curve that misses $\tilde{X}_y$ has intersection number zero with the boundary components of $\tilde{X}_y$.

(5) Let R be a component of $S^2 \setminus \bigcup_{y \in Y} \partial D_y$. If R contained two points p, q of P , no D_y would separate them, contradicting the hypothesis for $A = \{p, q\}$. Suppose that R is not simply connected. Then there is a simple closed curve $c \subset R \setminus P$ such that both components of $S^2 \setminus c$ contain points outside R . Each of them then contains a whole curve ∂D_y , because these curves do not meet c , and therefore contains one of the two discs bounded by this curve; so each side of c contains at least two

points of P . Let A and B be the sets of points of P on the two sides of c , and let D_y be as in the hypothesis. The curve ∂D_y lies on one side of c , say on the side of A . Then the closure of the other side is a connected set that does not meet ∂D_y , so the points of B are all inside or all outside D_y . This is a contradiction. $\square$

6.3 The planar picture and the proof of Theorem 1.3

Let $m \geq 5$. Let $s_1, \dots, s_m$ be points in this cyclic order on a round circle in S^2 , with indices modulo m , and $P = \{s_1, \dots, s_m\}$. Let e_j be the arc of the circle from s_{j-1} to s_j , and D_j a thin closed disc neighbourhood of e_j , so that

$$D_j \cap P = P_j := \{s_{j-1}, s_j\}$$

and $D_j \cap D_{j'} = \emptyset$ unless $j' \in \{j-1, j, j+1\}$; see Figure 2. The disc D_j belongs to the vertex j of the cycle $\Lambda = C_m$, in which j is adjacent to $j \pm 1$; the graph Γ is the complement of C_m .

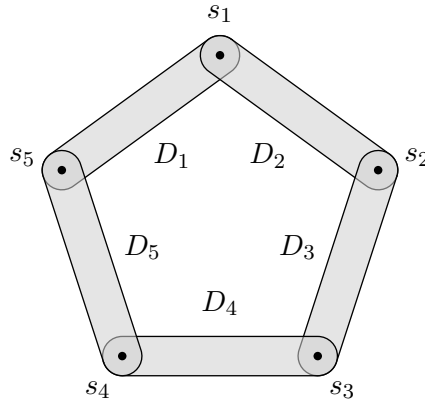


Figure 2: The planar picture for $m = 5$. The disc D_j is a neighbourhood of the arc from s_{j-1} to s_j (drawn straight) and contains the two punctures s_{j-1}, s_j . Consecutive discs share one puncture, and the other pairs are disjoint; so two discs are disjoint exactly when the corresponding vertices are adjacent in the pentagon Γ , the complement of the cycle $1 - 2 - 3 - 4 - 5$.

Lemma 6.4. *Let $m \geq 4$ and let Γ be the complement of the cycle C_m with vertices $1, \dots, m$. The minimal non-empty vertex sets that are not contained in a join of Γ are the m sets $Y_j = V \setminus \{j, j+1\}$, and every non-empty set that is not contained in a join contains one of them. Moreover, for every j and every partition $P = A \sqcup B$ with $|A|, |B| \geq 2$ there is $y \in Y_j$ such that P_y has one point in A and one in B .*

Proof. By Lemma 3.1 we have to determine the vertex sets that induce a connected subgraph of C_m and are dominating. A connected set is a set of consecutive vertices; it dominates itself and the two vertices next to it, so it is dominating if and only if at most two vertices are missing. This proves the first statement. The pairs $P_y, y \in Y_j$, are the $m-2$ pairs of consecutive points in the sequence $s_{j+1}, s_{j+2}, \dots, s_{j-1}$, which contains all points of P except s_j . If no P_y had one point in A and one in B , these $m-1$ points would all lie in A or all in B , and the other part would have at most one point. $\square$

Proposition 6.5. *Let $m \geq 5$, let Γ be the complement of C_m , let $d \geq 5$ be odd, and let $w_1, \dots, w_m$ be units of $\mathbb{Z}/d$ with sum zero. Let $\pi: \tilde{S} \rightarrow S^2$ be the covering of Lemma 6.1 for P and $w_{s_j} = w_j$. Then there are discs D'_j with $D'_j \cap P = P_j$, isotopic to D_j in $S^2 \setminus P$, such that the subsurfaces*

$X_j = \pi^{-1}(D'_j)$ form an LPA realization of Γ on $\tilde{S}$, with $X_j \cap X_{j'} = \emptyset$ for non-consecutive j, j' . The genus of $\tilde{S}$ is $(d-1)(m-2)/2$, and X_j has genus $(d-b_j)/2$ and b_j boundary components, where $b_j = \gcd(d, w_{j-1} + w_j)$. If $w_{j-1} + w_j \neq 0$ for all j , then all X_j have genus at least two, and $\{X_j\}$ is a pointed territory system for Γ .

Proof. Fix a complete hyperbolic metric of finite area on $S^2 \setminus P$. The curve ∂D_j is essential in $S^2 \setminus P$, because both sides contain at least two punctures. Let γ_j be the closed geodesic in its isotopy class, and let D'_j be the disc bounded by γ_j with $D'_j \cap P = P_j$. The geodesics γ_j are pairwise distinct, since they separate P in different ways, and distinct simple closed geodesics are in minimal position. If j and j' are not consecutive, then ∂D_j and $\partial D_{j'}$ are disjoint, so γ_j and $\gamma_{j'}$ are disjoint. Two discs in the sphere with disjoint boundary curves are disjoint, or nested, or cover the sphere; the discs D'_j and $D'_{j'}$ are not nested, because P_j and $P_{j'}$ are disjoint and non-empty, and they do not cover the sphere, because $m \geq 5$. So $D'_j \cap D'_{j'} = \emptyset$. For consecutive indices the four sets $P_j \cap P_{j+1} = \{s_j\}$, $P_j \setminus P_{j+1}$, $P_{j+1} \setminus P_j$ and $P \setminus (P_j \cup P_{j+1})$ are non-empty; so the curves γ_j and γ_{j+1} are not disjoint.

Now apply Lemma 6.3 to the discs D'_j ; note that $2 = |P_j| \leq m-2$. By (1) the X_j are proper subsurfaces of complexity at least two, by (3) they are disjoint for non-consecutive indices and overlap for consecutive ones. So (R1) and (R2) hold for Γ . Let Y be a non-empty set that is not contained in a join of Γ . By Lemma 6.4 it contains some Y_j , and for every partition $P = A \sqcup B$ with $|A|, |B| \geq 2$ there is $y \in Y_j$ such that $D'_y \cap P = P_y$ has one point in A and one in B ; then $A \setminus D'_y$ and $B \setminus D'_y$ are non-empty as well. By (5) and (4) of Lemma 6.3, $\{X_y\}_{y \in Y}$ fills $\tilde{S}$. This is (F).

The genus of $\tilde{S}$ and the number b_j of boundary components of X_j are given by Lemma 6.1, and the genus h_j of X_j follows from $2 - 2h_j - b_j = \chi(X_j) = 2 - d$. If $w_{j-1} + w_j \neq 0$, then b_j is a proper divisor of the odd number d , so $b_j \leq d/3$ and $h_j \geq d/3 > 1$. In that case the X_j form a territory system: (T1) and (T2) are (R1), (R2) and (F). It is pointed, because the cliques of C_m are the vertices and the edges $\{j, j+1\}$, and the point of $\tilde{S}$ over s_j lies in the interior of X_j and of X_{j+1} . $\square$

Proof of Theorem 1.3. Take units $w_1, \dots, w_m$ with sum zero (Lemma 6.2) and apply Proposition 6.5. For $m = 5$ and $d = 5$ we can take all $w_j = 1$; then $\tilde{S}$ has genus 6, the sums $w_{j-1} + w_j = 2$ are units, and every X_j has genus 2 and one boundary component. $\square$

Remark 6.6. Total ramification is what makes the lifting lemma work: a bigon of two curves on the sphere that contains one puncture lifts to a polygon with $2d$ corners, not to a bigon. The degree is odd so that units with sum zero exist for every m , and at least five so that the preimage of a disc with two punctures has complexity at least two. For $m = 4$ the curves ∂D_1 and ∂D_3 are isotopic; the 4-cycle is treated differently in Section 7.

7 Gluing along cliques

In this section we prove Theorem 1.4. The building blocks are pointed territory systems for complete graphs and for cycles, and they are combined by a gluing lemma. It is convenient to index territory systems by the complement graph: a *territory system for the complement of Λ* is a territory system for the graph Γ whose complement is Λ . In terms of Λ , condition (T1) says that territories of non-adjacent vertices have disjoint interiors and territories of adjacent vertices cannot be isotoped apart, and by Lemma 3.1 condition (T2) concerns the non-empty vertex sets Y such that $\Lambda[Y]$ is connected and Y is dominating in Λ .

7.1 The blocks

Lemma 7.1. *The following graphs Λ have pointed territory systems for their complements:*

- (1) the complete graph K_r , $r \geq 1$, on every closed orientable surface of genus at least 2;
- (2) the cycle C_m , $m \geq 5$, on the closed orientable surface of genus $e(m-2)$ for every integer $e \geq 2$;
- (3) the cycle C_4 , on every closed orientable surface of genus at least 4.

Proof. (1) Put $U_v = S$ for all v . Condition (T1) holds because any two vertices are adjacent in Λ , and (T2) and (P) are obvious.

(2) Let $d = 2e + 1$, choose units $w_1, \dots, w_m$ of $\mathbb{Z}/d$ with sum zero and without vanishing cyclically consecutive sums (Lemma 6.2(b)), and apply Proposition 6.5.

(3) Let Λ be the cycle $a_1 - b_1 - a_2 - b_2 - a_1$, and let S be a closed orientable surface of genus at least four. Let c_A be a separating curve of S such that both sides have genus at least two, let φ be a pseudo-Anosov mapping class of S , and let $c_B = \varphi^N(c_A)$ with N so large that $d_S(c_A, c_B) \geq 3$ (K2). Put the two curves in minimal position. Let U_{a_1} and U_{a_2} be the closures of the two components of $S \setminus c_A$, and U_{b_1}, U_{b_2} those of $S \setminus c_B$. They are subsurfaces of genus at least two. The territories U_{a_1}, U_{a_2} have disjoint interiors, and so have U_{b_1}, U_{b_2} . Since $i(c_A, c_B) > 0$, the territories U_{a_i} and U_{b_j} cannot be isotoped apart. This is (T1). A connected dominating vertex set of C_4 contains two adjacent vertices a_i, b_j . A curve that misses U_{a_i} and U_{b_j} is, up to isotopy, disjoint from or equal to c_A and to c_B , which is impossible because $d_S(c_A, c_B) \geq 3$. This is (T2). The non-empty cliques of C_4 are the vertices and the edges $\{a_i, b_j\}$, and near a point of $c_A \cap c_B$ all four sets $\text{int } U_{a_i} \cap \text{int } U_{b_j}$ are non-empty. This is (P). $\square$

7.2 The gluing lemma

Let Λ be the clique-sum of Λ_1 and Λ_2 along the non-empty clique K , with vertex sets V_1, V_2 , so that $V_1 \cap V_2 = K$. For $i = 1, 2$ let $\{U_v^i\}_{v \in V_i}$ be a pointed territory system for the complement of Λ_i on a closed surface S_i . By (P) we can choose a point $p_i \in S_i$ in the interior of all U_k^i , $k \in K$, and not on the boundary of any territory, and a closed disc Δ_i around p_i with the following properties: Δ_i lies in the interior of every territory that contains p_i and is disjoint from the other territories; and for every non-empty clique Q of Λ_i there is a common interior point of the U_q^i , $q \in Q$, outside Δ_i . Let S be the closed surface obtained from $S_1 \setminus \text{int } \Delta_1$ and $S_2 \setminus \text{int } \Delta_2$ by gluing along the boundary circles; let $c \subset S$ be the image of these circles, and let $S_i^\circ \subset S$ be the image of $S_i \setminus \text{int } \Delta_i$. Define

$$U_k = (U_k^1 \cap S_1^\circ) \cup (U_k^2 \cap S_2^\circ) \quad \text{for } k \in K, \quad U_v = U_v^i \cap S_i^\circ \quad \text{for } v \in V_i \setminus K.$$

See Figure 3. For $v \in V_i \setminus K$ the territory U_v is equal to U_v^i , or to U_v^i with the open disc $\text{int } \Delta_i$ removed.

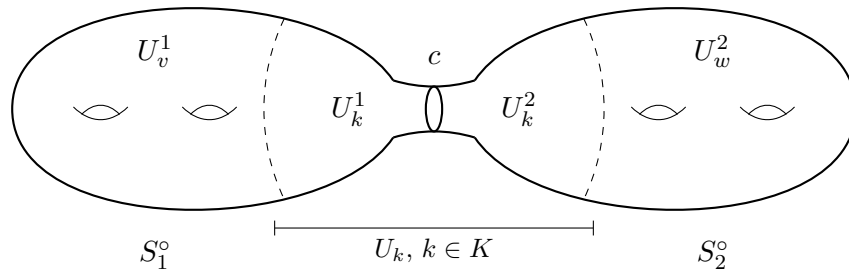


Figure 3: The gluing lemma, schematically. The surfaces S_1 and S_2 are cut open at discs that lie in the territories of the vertices of the clique K , and are glued along the curve c . The territories U_k^1 and U_k^2 of a vertex $k \in K$ are joined to one territory U_k ; a territory U_v^1 with $v \in V_1 \setminus K$ stays on its side.

Lemma 7.2 (gluing lemma). *The family $\{U_v\}_{v \in V_1 \cup V_2}$ is a pointed territory system for the complement of Λ on S .*

Proof. The surfaces S_i have genus at least two, so c is an essential curve of S . We need two remarks on simple closed curves α, β in the interior of S_1° ; they apply to S_2° as well.

(G1) If α is essential in S_1 , it is essential in S . If α is essential in S and not isotopic to c in S , it is essential in S_1 .

(G2) $i_S(\alpha, \beta) \geq i_{S_1}(\alpha, \beta)$.

For (G1): a disc in S bounded by α does not contain the essential curve c , so it lies in S_1° ; and a disc in S_1 bounded by α either lies in S_1° or contains Δ_1 , and in the second case α is parallel to c . For (G2): move α and β by an isotopy inside S_1° until they are transverse and do not form a bigon in S_1° . A bigon in S formed by them cannot contain c , so it would lie in S_1° . Hence they are in minimal position in S , and the number of their intersection points is at least $i_{S_1}(\alpha, \beta)$, because the isotopy took place in $S_1^\circ \subset S_1$.

The U_v are subsurfaces of genus at least two. They are compact and connected, and removing a disc or gluing two surfaces along a circle does not lower the genus. Their boundary components are boundary components of the old territories, which are essential in S by (G1), and possibly the curve c .

(T1). Let $v \neq w$ be non-adjacent in Λ . If $v \in V_1 \setminus K$ and $w \in V_2 \setminus K$, then U_v and U_w lie on different sides of c . Otherwise $v, w \in V_i$ for some i , and the interiors of U_v^i and U_w^i are disjoint; then the interiors of U_v and U_w are disjoint, also if one of the two vertices lies in K , because in that case the other territory does not contain p_i . Now let $v \neq w$ be adjacent in Λ . Then $v, w \in V_i$ for some i , say $i = 1$. Since U_v^1 and U_w^1 cannot be isotoped apart in S_1 , there are simple closed curves α in the interior of U_v^1 and β in the interior of U_w^1 with $i_{S_1}(\alpha, \beta) > 0$. (If the two territories overlap, take for β a boundary component of U_w^1 that cuts U_v^1 , pushed into U_w^1 , and for α a curve of U_v^1 at distance at least 3 from $\pi_{U_v^1}(\beta)$. If U_v^1 is isotopic to a subsurface of U_w^1 , take two curves of U_v^1 with positive intersection number, and move one of them into the interior of U_w^1 by an isotopy.) Push α and β off Δ_1 inside the interiors of the territories. By (G2), $i_S(\alpha, \beta) > 0$, with $\alpha \subset U_v$ and $\beta \subset U_w$. So U_v and U_w cannot be isotoped apart in S .

(P). A non-empty clique Q of Λ lies in V_1 or in V_2 , because there is no edge between $V_1 \setminus K$ and $V_2 \setminus K$. By the choice of Δ_i , the territories U_q^i , $q \in Q$, have a common interior point outside Δ_i , and it is a common interior point of the U_q .

(T2). Let Y be a non-empty vertex set such that $\Lambda[Y]$ is connected and Y is dominating in Λ . Then:

- (i) Y meets K . If $Y \subset V_1 \setminus K$, a vertex of $V_2 \setminus K$ is not in Y and has no neighbour in Y ; the same holds with the indices exchanged. And if Y met both $V_1 \setminus K$ and $V_2 \setminus K$ but not K , then $\Lambda[Y]$ would be disconnected.
- (ii) $Y_i = Y \cap V_i$ induces a connected subgraph of Λ_i and is dominating in Λ_i , for $i = 1, 2$. A path in $\Lambda[Y]$ between two vertices of Y_1 leaves V_1 only through K , and each of its excursions into $V_2 \setminus K$ can be replaced by an edge of the clique K , or omitted. A vertex of $V_1 \setminus K$ outside Y has all its neighbours in V_1 , hence a neighbour in Y_1 ; and a vertex of K outside Y is adjacent to the vertices of $Y \cap K$, which is non-empty by (i).

Suppose that a curve c' of S misses U_y for all $y \in Y$, and let $k \in Y \cap K$. The curve c lies in the interior of U_k . It is not parallel in U_k to a boundary component δ of U_k : the annulus between them would lie in some S_i° , and together with Δ_i it would be a disc in S_i bounded by δ . So c is a curve of

U_k , and by Lemma 2.2(a) c' is not isotopic to c . Moreover $i_S(c', c) = 0$, because c' misses U_k . After an isotopy, c' is disjoint from c and lies in the interior of S_1° , say. By (G1), c' is essential in S_1 . We show that c' misses U_y^1 in S_1 for every $y \in Y_1$; by (ii) this contradicts (T2) for Λ_1 .

Let $y \in Y_1$. Every boundary component δ of U_y^1 lies in S_1° and is, up to a small isotopy, contained in U_y ; so $i_S(c', \delta) = 0$, and $i_{S_1}(c', \delta) = 0$ by (G2). Hence c' is isotopic in S_1 to a curve c'' disjoint from ∂U_y^1 . If c'' lies outside U_y^1 or is parallel to a boundary component of U_y^1 , then c' misses U_y^1 . Otherwise c'' is a curve of U_y^1 (this includes the case $U_y^1 = S_1$), and there is a curve ε of U_y^1 with $i_{S_1}(c'', \varepsilon) > 0$. Push ε off Δ_1 inside the interior of U_y^1 . Then $\varepsilon \subset U_y$, and $i_S(c', \varepsilon) \geq i_{S_1}(c', \varepsilon) > 0$ by (G2). This contradicts the assumption that c' misses U_y . $\square$

Proof of Theorem 1.4. By induction on the construction of a graph in $\mathcal{K}$, using Lemma 7.1 for complete graphs and cycles and Lemma 7.2 for clique-sums, every graph in $\mathcal{K}$ has a pointed territory system for its complement on some closed orientable surface. If the complement Γ has no isolated vertex, the territory lemma gives an LPA realization of Γ by one-holed tori. $\square$

Remark 7.3 (genus). The surface produced for a graph $\Lambda \in \mathcal{K}$ is glued from the surfaces of the blocks, and its genus is the sum of their genera: any integer ≥ 2 for a complete block, any integer ≥ 4 for a block C_4 , and $e(m-2)$ with any integer $e \geq 2$ for a block C_m , $m \geq 5$. Let N be the least common multiple of the numbers $m-2$ over the blocks C_m with $m \geq 5$ used in the construction ($N = 1$ if there are none). Then every sufficiently large multiple of N occurs as the genus of a closed surface on which Γ has an LPA realization as in Theorem 1.4.

8 Further graphs: twins, cycle complements with one more vertex, the hexagon

This section contains the constructions behind Theorem 1.6(1)–(3).

8.1 Twins

Let v be a vertex of a graph Γ . Recall that a false twin of v is a new vertex w with $\text{lk}(w) = \text{lk}(v)$, and a true twin of v is a new vertex t with $\text{lk}(t) = \text{lk}(v) \cup \{v\}$. In the complement graph the roles are exchanged: w is adjacent to v in Λ , and t is not.

Proposition 8.1 (false twins). *Let $\{U_u\}_{u \in V}$ be a territory system for Γ on S , and let Γ^+ be obtained from Γ by adding a false twin w of a vertex v . Then $\{U_u\}_{u \in V}$ together with $U_w := U_v$ is a territory system for Γ^+ on S . If the given system is pointed, so is the new one.*

Proof. (T1): the vertex w is adjacent to the neighbours u of v , and $U_w = U_v$ and U_u have disjoint interiors; it is not adjacent to v , and to the vertices $u \neq v$ that are not adjacent to v , and $U_w = U_v$ cannot be isotoped apart from U_v or from such a U_u .

(T2): let Y be a non-empty set of vertices of Γ^+ that is not contained in a join of Γ^+ ; by Lemma 3.1, Y induces a connected subgraph of the complement Λ^+ of Γ^+ and is dominating in Λ^+ . In Λ^+ the vertex w is adjacent to v and has otherwise the same neighbours as v . Let Y' be Y with w replaced by v if $w \in Y$, and $Y' = Y$ otherwise. Then $\Lambda[Y']$ is connected, because a path through w can be led through v , and Y' is dominating in Λ , because a vertex with the neighbour w in Λ^+ is equal or adjacent to v . So Y' is not contained in a join of Γ , and $\{U_y\}_{y \in Y} = \{U_y\}_{y \in Y'}$ fills S .

(P): if Q is a clique of Λ^+ that contains w , then $(Q \setminus \{w\}) \cup \{v\}$ is a clique of Λ with the same set of territories. $\square$

For true twins the territories of v and of its twin are equal although the two vertices are adjacent, so (T1) fails for this pair. The following variant of the territory lemma places two disjoint tori in one territory.

Lemma 8.2 (territory lemma with partners). *Let Γ^+ be a graph without isolated vertices, and let $\mathcal{P}$ be a set of pairwise disjoint pairs $\{v, t\}$ of adjacent vertices of Γ^+ . Let S be a closed orientable surface of genus at least two and $\{U_u\}_{u \in V(\Gamma^+)}$ subsurfaces of genus at least two such that*

(T1') $U_v = U_t$ for $\{v, t\} \in \mathcal{P}$; for all other pairs of distinct vertices, U_u and $U_{u'}$ have disjoint interiors if u and u' are adjacent, and cannot be isotoped apart if they are not;

(T2) for every non-empty set Y that is not contained in a join of Γ^+ , the family $\{U_y\}_{y \in Y}$ fills S .

Then there are one-holed tori X_u in the interior of U_u that form an LPA realization of Γ^+ on S .

Proof. We repeat the proof of Lemma 5.2 with the following changes. The vertices are numbered so that the two vertices of each pair of $\mathcal{P}$ are consecutive, v_k and v_{k+1} . For such a pair the two tori are chosen in the same step: let X_k^0 and X_{k+1}^0 be disjoint one-holed tori in the interior of $U_k = U_{k+1}$. Their boundary curves are curves of U_k , because the complement of each torus in U_k contains the other one and so is not an annulus. With a homeomorphism ρ_k as before put $X_k = \rho_k^N(X_k^0)$ and $X_{k+1} = \rho_k^N(X_{k+1}^0)$, where N is so large that (5.1) holds for $a_k = \partial X_k$ and for $a_{k+1} = \partial X_{k+1}$ with the same set $B_{k+1} := B_k$; for $l > k + 1$ the sets B_l are defined as before and contain a_k and a_{k+1} . Thus $a_k \notin B_{k+1}$, and (5.2) holds for k and for $k + 1$.

The tori X_k and X_{k+1} are disjoint, as required for the adjacent vertices v_k, v_{k+1} . In the rest of the proof of Lemma 5.2 the conditions (5.1) and (5.2) for an index k are applied to curves $a_j \in B_k$ only for pairs (j, k) of non-adjacent vertices, in Step 1 as well as in Step 2, where m and m' are adjacent in the complement graph. Such a pair is never a pair of $\mathcal{P}$, so $a_j \in B_k$ holds for all pairs that occur, and both steps go through word for word, with (T1') in place of (T1). $\square$

Proposition 8.3 (true twins). *Let $\{U_u\}_{u \in V}$ be a territory system for a graph Γ without isolated vertices on S . Let Γ^+ be obtained from Γ by adding true twins $t_1, \dots, t_s$ of pairwise distinct vertices $v_1, \dots, v_s$ of Γ , one after the other (so every vertex of Γ receives at most one twin, and no twin of a twin is added). Then Γ^+ has an LPA realization on S by one-holed tori.*

Proof. In Γ^+ the vertex t_i is adjacent to v_i , to the neighbours of v_i in Γ , and to t_j exactly when v_i and v_j are adjacent; this does not depend on the order in which the twins are added. Put $U_{t_i} := U_{v_i}$ and $\mathcal{P} = \{\{v_i, t_i\}\}$. Let $r: V(\Gamma^+) \rightarrow V$ be the map that sends t_i to v_i and is the identity on V . Two vertices $x \neq y$ of Γ^+ that do not form a pair of $\mathcal{P}$ satisfy $r(x) \neq r(y)$, and they are adjacent in Γ^+ if and only if $r(x)$ and $r(y)$ are adjacent in Γ . Since $U_x = U_{r(x)}$, condition (T1') follows from (T1) for Γ .

For (T2) let Y be a non-empty set of vertices of Γ^+ that is not contained in a join of Γ^+ . We claim that $r(Y)$ is not contained in a join of Γ . Suppose $r(Y) \subset J_1 * J_2$, and let $J_i^+ = r^{-1}(J_i)$. Then J_1^+ and J_2^+ are disjoint and non-empty, and every vertex of J_1^+ is adjacent to every vertex of J_2^+ , by the description of adjacency above; so Y is contained in the join $J_1^+ * J_2^+$ of Γ^+ , a contradiction. Hence $\{U_y\}_{y \in Y} = \{U_u\}_{u \in r(Y)}$ fills S . The graph Γ^+ has no isolated vertex, and Lemma 8.2 applies. $\square$

8.2 Complements of cycles with one more vertex

We return to the situation of Section 6. The following lemma is a local version of part (4) of the lifting lemma.

Lemma 8.4. *In the situation of Lemma 6.3, let $i \neq j$ be such that ∂D_i and ∂D_j intersect and every component of $\text{int } D_i \setminus \partial D_j$ is an open disc that contains at most one point of P . Let c be a curve of $\tilde{S}$ that is isotopic to a simple closed curve in $\tilde{X}_i$. Then c has positive intersection number with some component of $\pi^{-1}(\partial D_j)$; in particular c cuts $\tilde{X}_j$.*

Proof. Suppose that c has intersection number zero with all components of $\pi^{-1}(\partial D_j)$. It also has intersection number zero with the components of $\pi^{-1}(\partial D_i) = \partial \tilde{X}_i$. By Lemma 6.3(1),(2) and Lemma 2.3 we may assume that c is disjoint from $\pi^{-1}(\partial D_i \cup \partial D_j)$. Then c lies in a component W of the preimage of a component R of $S^2 \setminus (\partial D_i \cup \partial D_j)$. If $R \subset D_i$, then R is an open disc with at most one point of P and W is an open disc, by the remark at the beginning of the proof of Lemma 6.3; this is impossible for an essential curve. So c is disjoint from $\tilde{X}_i$. Since c is isotopic to a simple closed curve in $\tilde{X}_i$, that curve is parallel to a boundary component δ_0 of $\tilde{X}_i$ by Lemma 2.2(a), so c is isotopic to δ_0 . But δ_0 covers ∂D_i , which meets ∂D_j ; so δ_0 meets a component δ of $\pi^{-1}(\partial D_j)$, and $i(c, \delta) = i(\delta_0, \delta) > 0$ by Lemma 6.3(2). This is a contradiction. $\square$

Proposition 8.5. *Let $m \geq 5$, let Γ be the complement of the cycle C_m with vertices $1, \dots, m$, and let I be a non-empty set of pairwise non-consecutive vertices of the cycle with $2|I| < m$. Let Γ_I be the graph obtained from Γ by adding a vertex q that is adjacent exactly to the vertices in I . Then, for every integer $e \geq 2$, the graph Γ_I has a pointed territory system on the closed orientable surface of genus $e(m-2)$. Hence Γ_I has an LPA realization by one-holed tori on this surface.*

For $m = 5$ the possible sets I are a single vertex, which gives the pentagon with a pendant vertex, and two vertices that are adjacent in the pentagon Γ , which gives the pentagon with a triangle attached along an edge.

Proof. Let $d = 2e + 1$, and let $\pi: \tilde{S} \rightarrow S^2$, the discs D'_j and the subsurfaces $X_j = \pi^{-1}(D'_j)$ be as in Proposition 6.5, for units w_j with sum zero and without vanishing consecutive sums. The X_j form a pointed territory system for Γ , and X_j has $b_j \leq d/3$ boundary components. The discs D'_i , $i \in I$, are pairwise disjoint. Put

$$U_j = X_j \ (1 \leq j \leq m), \quad U_q = \pi^{-1}\left(S^2 \setminus \bigcup_{i \in I} \text{int } D'_i\right) = \tilde{S} \setminus \bigcup_{i \in I} \text{int } X_i.$$

The set $S^2 \setminus \bigcup_{i \in I} \text{int } D'_i$ is connected and contains $m - 2|I| \geq 1$ points of P , so U_q is connected. It is a compact surface whose boundary components are boundary components of the X_i , $i \in I$; there are $b = \sum_{i \in I} b_i \leq |I|d/3$ of them, and

$$\chi(U_q) = d(2 - |I|) - (m - 2|I|)(d - 1).$$

A short computation gives the genus $h = (2 - \chi(U_q) - b)/2$ of U_q : for $|I| = 1$ we get $-\chi(U_q) \geq 3(d - 1) - d$ and $h \geq (5d/3 - 1)/2 > 1$; for $|I| \geq 2$ we get $-\chi(U_q) \geq (d - 1) + d(|I| - 2)$ and $h \geq (1 + d(2|I|/3 - 1))/2 \geq (1 + d/3)/2 > 1$. So U_q is a subsurface of genus at least two.

(T1). For $i \in I$ the territories U_q and U_i have disjoint interiors. Let $j \notin I$. If j is consecutive to some $i \in I$, the boundary curves of U_j and of U_i have positive intersection number, and the boundary of U_i is part of the boundary of U_q ; so U_j and U_q cannot be isotoped apart. Otherwise U_j is disjoint from all U_i , $i \in I$, so $U_j \subset U_q$, and a curve of U_j is a curve of U_q and cuts U_q (Lemma 2.2(a),(b)); again they cannot be isotoped apart. The pairs not involving q are covered by Proposition 6.5.

(T2). Let Λ_I be the complement of Γ_I : the cycle C_m together with the vertex q , which is adjacent to all $j \notin I$. Let Y be a non-empty vertex set that induces a connected subgraph of Λ_I and is dominating in Λ_I , and suppose that a curve c misses U_y for all $y \in Y$. If $q \notin Y$, then Y induces a connected subgraph of C_m and is dominating in C_m , and the U_y fill by Proposition 6.5. Let $q \in Y$.

Since c misses U_q , it is isotopic to a simple closed curve in X_i for some $i \in I$. The only neighbours of i in Λ_I are $i - 1$ and $i + 1$. If $i \notin Y$, one of them lies in Y , because Y is dominating; if $i \in Y$, one of them lies in Y , because $\Lambda_I[Y]$ is connected and contains $q \neq i$. So Y contains a vertex $j = i \pm 1$. The geodesics $\partial D'_i$ and $\partial D'_j$ are in minimal position and intersect in exactly two points: they are not disjoint, their number of intersection points is even, and the boundaries of the thin discs D_i, D_j of Section 6, which are isotopic to them, meet in two points. Hence $\text{int } D'_i \setminus \partial D'_j$ has two components, the lens $D'_i \cap D'_j$ and its complement in D'_i , and each of them is an open disc with exactly one point of P . By Lemma 8.4, c cuts $X_j = U_j$. This is a contradiction.

(P). The non-empty cliques of Λ_I are those of C_m , the sets $\{q\}$ and $\{q, j\}$ with $j \notin I$, and the sets $\{q, j, j + 1\}$ with $j, j + 1 \notin I$. Let $j \notin I$. The disc D'_j is connected, the discs $D'_i, i \in I$, are pairwise disjoint, and none of them contains both s_{j-1} and s_j ; so D'_j is not contained in the union of these discs, and $\text{int } U_j \cap \text{int } U_q$ is not empty. If $j, j + 1 \notin I$, the point s_j lies in no D'_i with $i \in I$, and its preimage is a common interior point of U_j, U_{j+1} and U_q .

The last statement follows from the territory lemma, since Γ_I has no isolated vertex. $\square$

8.3 The hexagon

The complement of the hexagon C_6 is the triangular prism. It has no clique whose removal disconnects it, and it is neither complete nor a cycle; so the hexagon is not covered by Theorem 1.4. We realize it by another planar picture.

Let a, b, c be the vertices of a triangle in the plane, t_I a point inside the triangle and t_O a point outside, and let $P = \{a, b, c, t_I, t_O\} \subset S^2$. Consider six arcs: the sides

$$\alpha_1 = [c, a], \quad \alpha_3 = [a, b], \quad \alpha_5 = [b, c]$$

of the triangle, and three arcs $\alpha_2, \alpha_4, \alpha_6$ from t_I to t_O that are pairwise disjoint except for their end points, such that α_2 crosses the side α_5 once and is disjoint from the other two sides, α_4 crosses α_1 once and is disjoint from the other two sides, and α_6 crosses α_3 once and is disjoint from the other two sides. Let D_j be a thin closed disc neighbourhood of α_j , the six neighbourhoods having different widths, so that their boundary curves are in general position; see Figure 4. Each D_j contains exactly two points of P , the end points of α_j .

Lemma 8.6. *The discs $D_1, \dots, D_6$ have the following properties, with indices modulo 6.*

- (H1) $D_j \cap D_{j+1} = \emptyset$ for all j .
- (H2) For each of the nine other pairs $\{j, k\}$ the curves ∂D_j and ∂D_k are transverse, they intersect, and every component of $S^2 \setminus (\partial D_j \cup \partial D_k)$ that is bounded by one arc of each curve contains a point of P . Hence the two curves are in minimal position in $S^2 \setminus P$.
- (H3) For each of the sets $Y = \{1, 4\}, \{2, 5\}, \{3, 6\}, \{1, 3, 5\}, \{2, 4, 6\}$, every component of $S^2 \setminus \bigcup_{y \in Y} \partial D_y$ is an open disc that contains at most one point of P .

Proof. (H1) The arcs α_j and α_{j+1} are a side of the triangle and an arc from t_I to t_O through a different side. They are disjoint, and so are their thin neighbourhoods.

(H2) There are three types of pairs. *Two sides of the triangle*, say D_1 and D_3 , with the common vertex a : the boundary curves meet in two points, and the complement of their union has four components: the lens $D_1 \cap D_3$, which contains a ; the rest of D_1 , which contains c ; the rest of D_3 , which contains b ; and the complement of $D_1 \cup D_3$, which contains t_I and t_O . *Two arcs from t_I to t_O* , say D_2 and D_4 : the boundary curves meet in four points, two near t_I and two near t_O , and there are six components: two lenses, which contain t_I and t_O ; the middle parts of D_2 and of D_4 , which contain

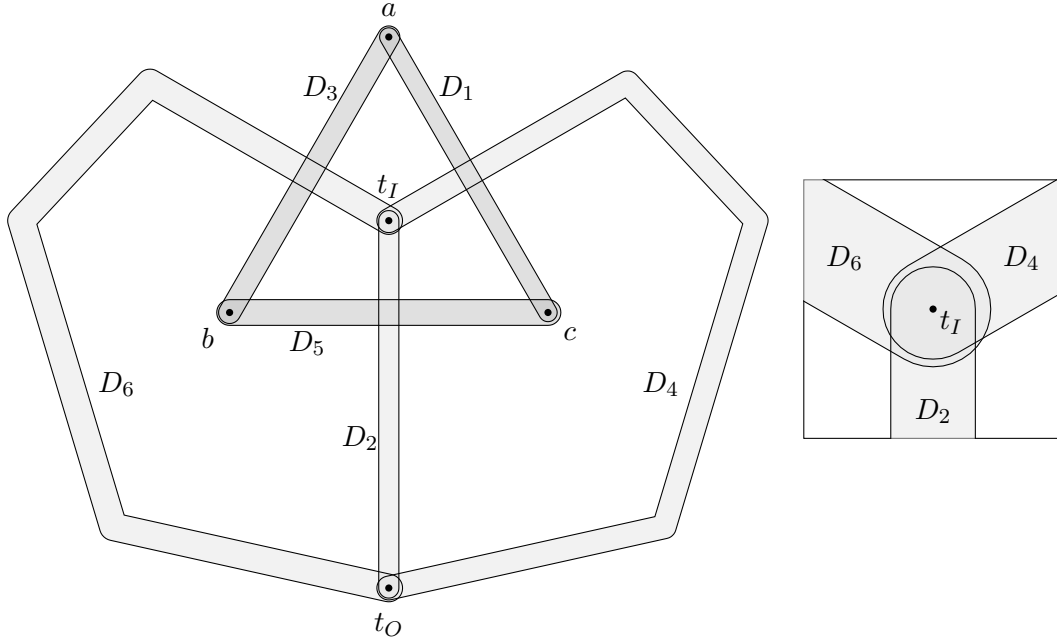


Figure 4: The planar picture for the hexagon: five punctures and six discs. The discs D_1 , D_3 , D_5 are neighbourhoods of the sides of the triangle abc , and D_2 , D_4 , D_6 are neighbourhoods of three arcs from t_I to t_O . The disc D_j is disjoint from $D_{j\pm 1}$ (indices modulo 6) and meets the other three discs. The square on the right shows a neighbourhood of t_I , enlarged. The coordinates are those of the computer checks of Section 11.

no puncture and are bounded by two arcs of each curve; and the two components of the complement of $D_2 \cup D_4$. The latter are separated by the closed curve $\alpha_2 \cup \alpha_4$, which crosses the triangle at one point of each of the sides α_5 and α_1 and so separates their common vertex c from a and b . So one component contains c and the other contains a and b . *A side and the arc that crosses it*, say D_1 and D_4 : the intersection $D_1 \cap D_4$ is a square around the crossing point, the boundary curves meet in its four corners, and there are six components: the square, which contains no puncture and is bounded by two arcs of each curve; the two ends of D_1 , which contain c and a ; the two ends of D_4 , which contain t_I and t_O ; and the complement of $D_1 \cup D_4$, which contains b and is bounded by two arcs of each curve. In all cases the components bounded by one arc of each curve contain a puncture. If the two curves were not in minimal position in $S^2 \setminus P$, they would form a bigon there, and a bigon whose interior does not meet the two curves would be the closure of a component without punctures bounded by one arc of each curve.

(H3) For $Y = \{1, 4\}$ the components were listed above: the square, four ends with one puncture each, and the outer component with the puncture b ; they are open discs. The sets $\{2, 5\}$ and $\{3, 6\}$ are of the same type. For $Y = \{1, 3, 5\}$ the union of the three discs is a neighbourhood of the triangle. Its complement has two components, the inner one with the puncture t_I and the outer one with t_O ; inside the union there are three lenses, which contain a , b and c , and the three middle parts of the discs. All eight components are open discs. For $Y = \{2, 4, 6\}$ the union of the three discs is a neighbourhood of the graph $\alpha_2 \cup \alpha_4 \cup \alpha_6$, which has two vertices and three edges and cuts the sphere into three discs, each containing one of the points a , b , c . So the complement of the union has three components with one puncture each. The other components lie in the union of the discs. The three boundary curves meet in twelve points, so there are fourteen components in all, and each of them is an open disc, because the union of the three curves is connected. The points t_I and t_O lie in different

components, since $D_2 \cap D_4$ consists of two lenses around t_I and t_O . So no component contains two punctures. $\square$

The statements of Lemma 8.6 concern an explicit finite configuration in the plane. They were also checked by four computer programs (Section 11), which compute the components, their corners and their punctures for the coordinates used in Figure 4.

Theorem 8.7. *Let $d \geq 5$ be odd. The hexagon C_6 has an LPA realization on the closed orientable surface of genus $3(d-1)/2$ by six subsurfaces of genus at least two, which is at the same time a territory system for C_6 . For $d = 5$ the surface has genus 6, and the six subsurfaces have genus 2 and one boundary component.*

Proof. Choose units $w_a = w_b = w_c = 1$ and w_{t_I}, w_{t_O} of $\mathbb{Z}/d$ with $w_{t_I} + w_{t_O} = -3$ (Lemma 6.2(a)); for $d = 5$ all five can be taken equal to 1. Let $\pi: \tilde{S} \rightarrow S^2$ be the covering of Lemma 6.1; its genus is $3(d-1)/2$. The discs D_j satisfy the hypotheses of Lemma 6.3: $|P| = 5$, each disc contains two points of P , and the boundary curves are pairwise disjoint or transverse and in minimal position by (H1) and (H2). Let $X_j = \pi^{-1}(D_j)$. By Lemma 6.3(1),(3) and (H1), (H2) the X_j are proper subsurfaces of complexity at least two, X_j and X_{j+1} are disjoint, and X_j and X_k overlap for the other pairs. Since the hexagon has the edges $\{j, j+1\}$, this is (R1) and (R2).

The complement of the hexagon is the prism, in which j is adjacent to $j+2$, $j+3$ and $j+4$. A connected dominating vertex set of the prism contains one of the five sets of (H3): the pairs $\{j, j+3\}$ and the two triangles $\{1, 3, 5\}$, $\{2, 4, 6\}$ are connected and dominating; a connected set that contains no pair $\{j, j+3\}$ lies in one of the two triangles; and a proper subset of a triangle is not dominating. So, by Lemma 3.1, every non-empty set Y that is not contained in a join of the hexagon contains one of these five sets, and $\{X_y\}_{y \in Y}$ fills $\tilde{S}$ by (H3) and Lemma 6.3(4). This is (F).

By Lemma 6.1, X_j has $b_j = \gcd(d, s_j)$ boundary components, where s_j is the sum of the exponents of the two punctures in D_j , that is, $s_j = 2$ for odd j and $s_j = -3$ for even j . Both are non-zero in $\mathbb{Z}/d$, so $b_j \leq d/3$, and the genus $(d - b_j)/2$ of X_j is at least two. For $d = 5$ we get $b_j = 1$ and genus 2. Finally (T1) and (T2) for the family $\{X_j\}$ are (R1), (R2) and (F). $\square$

Remark 8.8. The territory system of Theorem 8.7 is not pointed: the discs D_1, D_3, D_5 have no common point, although $\{1, 3, 5\}$ is a clique of the prism. So it cannot be used in the gluing lemma as it stands. Propositions 8.1 and 8.3 do apply to it. The partition criterion of Lemma 6.3(5) does not hold for this picture, for instance for $Y = \{2, 4, 6\}$ and the partition $\{t_I, t_O\} \sqcup \{a, b, c\}$; this is why part (4) of the lifting lemma is formulated in terms of the complementary regions.

9 Closure properties and small graphs

9.1 Proof of Theorem 1.5

If Γ is a single vertex, then $A(\Gamma) = \mathbb{Z}$, there are no joins, every non-trivial element is loxodromic, and a homomorphism that sends the generator to a pseudo-Anosov class of a closed surface is an LPA embedding.

Let Γ have at least two vertices and satisfy the hypothesis of the theorem, and let Γ' be the graph of Proposition 3.6: the disjoint union of Γ^- and of one path P_4 for each isolated vertex of Γ . By Proposition 3.6 and Lemma 3.4 it suffices to find an LPA embedding of $A(\Gamma')$, and by Proposition 3.2 it suffices to find proper subsurfaces of a closed surface that satisfy (R1), (R2) and (F) for Γ' .

The connected components of Γ' have at least two vertices. Let Γ_0 be the union of the chordal components (the paths are among them), and let $\Gamma_1, \dots, \Gamma_t$ be the other components; each of them is of kind (b) or (c). Each of these graphs has subsurfaces with (R1), (R2) and (F) on closed orientable surfaces of the following genera:

- Γ_0 , if it is not empty: every genus $\geq |V(\Gamma_0)| + 2$ (Theorem 1.1);
- a component of kind (b): every sufficiently large multiple of an integer N that depends on the component (Theorem 1.4 and Remark 7.3);
- a component that is a join: every sufficiently large genus (Lemma 5.4; property (F) is empty for a join).

So there is a genus G on which all of them are realized. Applying Proposition 3.7 t times, to $\Gamma_0 \sqcup \cdots \sqcup \Gamma_{i-1}$ and Γ_i , gives subsurfaces of the closed surface of genus G with (R1), (R2) and (F) for Γ' .

It remains to check the statement about graphs with at most five vertices. Let K be a connected component of Γ^- that is not chordal; it has four or five vertices and contains an induced cycle of length four or five. If K contains an induced 5-cycle, then $K = C_5$, whose complement is $C_5 \in \mathcal{K}$. Otherwise K contains an induced 4-cycle $a - b - c - d - a$. If K has four vertices, it is this cycle, which is a join. If K has a fifth vertex x , then x is adjacent to at least one vertex of the cycle. If x is adjacent only to a , the complement of K consists of the triangle $\{x, b, d\}$ and the path $x - c - a$; if x is adjacent exactly to a and b , the complement of K is the path $a - c - x - d - b$. Both complements are connected and chordal, hence in $\mathcal{K}$. In the remaining cases (x adjacent exactly to a and c , to three, or to all four vertices of the cycle) the complement of K is disconnected, so K is a join. Up to the symmetries of the cycle these are all cases. $\square$

9.2 The classes $\mathcal{T}$ and $\mathcal{T}_0$; proof of Theorem 1.6

For $m \geq 5$ and a set I as in Proposition 8.5, let $\Lambda_{m,I}$ be the complement of the graph Γ_I considered there: the cycle C_m together with one more vertex that is adjacent to all vertices of the cycle outside I .

Definition 9.1. Let $\mathcal{T}$ be the smallest class of graphs that contains the complete graphs, the cycles of length at least four and the graphs $\Lambda_{m,I}$, and that is closed under clique-sums and under adding true twins. Let $\mathcal{T}_0$ be the union of $\mathcal{T}$ and of the class of graphs obtained from the triangular prism by adding true twins repeatedly.

Thus $\mathcal{K} \subset \mathcal{T} \subset \mathcal{T}_0$, and all graphs in $\mathcal{T}_0$ are connected.

Lemma 9.2. *For every graph $\Lambda \in \mathcal{T}_0$ there is an integer $N \geq 1$ such that the complement of Λ has a territory system on the closed orientable surface of genus g for every sufficiently large multiple g of N . For $\Lambda \in \mathcal{T}$ these territory systems can be chosen pointed.*

Proof. For complete graphs and cycles this is Lemma 7.1, with $N = 1$ for complete graphs and for C_4 , and $N = m - 2$ for C_m with $m \geq 5$. For $\Lambda_{m,I}$ it is Proposition 8.5, with $N = m - 2$. A true twin in Λ is a false twin in the complement, so Proposition 8.1 gives a (pointed) territory system on the same surface. For a clique-sum the gluing lemma gives a pointed territory system on a surface whose genus is the sum of the genera of the two given surfaces; so the least common multiple of the two integers can be taken for N . For the prism the statement is Theorem 8.7, with $N = 3$, and for the prism with true twins added it follows again from Proposition 8.1. $\square$

Proposition 9.3. *Let Γ be a graph that is not a join, and suppose that every connected component K of Γ^- is of one of the following kinds:*

- (a) K is chordal;
- (c) K is a join;

- (d) K is obtained from a graph K_0 without isolated vertices, whose complement lies in $\mathcal{T}_0$, by adding one true twin to each of some pairwise distinct vertices of K_0 (possibly to none).

Then $A(\Gamma)$ has an LPA embedding into the mapping class group of a closed surface.

A component of kind (b) of Theorem 1.5 is of kind (d), with $K_0 = K$.

Proof. A component K of kind (d) has an LPA realization by one-holed tori on the closed surface of genus g for every sufficiently large multiple g of an integer N : by Lemma 9.2 the graph K_0 has a territory system on these surfaces, and Proposition 8.3 applies (the territory lemma, if no twins are added). With this information the proof of Theorem 1.5 goes through. $\square$

Proof of Theorem 1.6. Part (1) is Theorem 8.7 for $d = 5$, and part (2) is Proposition 8.5. For part (3), the graph Γ has no isolated vertex and has a territory system on a closed surface, by the proof of Theorem 1.4, by Theorem 8.7, or by Proposition 8.5. Adding false twins repeatedly gives graphs with territory systems on the same surface (Proposition 8.1), and Proposition 8.3 gives the LPA realization after true twins have been added (the territory lemma, if there are none). Part (4) follows from Proposition 9.3 and the enumeration described below: every graph with at most six vertices that is not a join satisfies the hypothesis of Proposition 9.3. $\square$

9.3 The enumeration

The graphs with at most seven vertices were enumerated up to isomorphism by computer; there are 1, 2, 4, 11, 34, 156, 1044 graphs with $1, \dots, 7$ vertices, of which 1, 1, 2, 6, 21, 112, 853 are not joins. For each of them the hypotheses of Theorem 1.5 and of Proposition 9.3 were tested component by component. Chordality and the join property are tested directly. Membership of a connected graph in $\mathcal{K}$ is tested by Lemma 2.6: the graph is split along clique separators until none is left, and the atoms are inspected. Membership in $\mathcal{T}$ is tested by an exhaustive search over all ways of obtaining the graph by a clique-sum or by adding a true twin, and kind (d) by an exhaustive search over the sets of true twins. The results are in Table 1. Column (i) of that table counts the graphs all of whose components are chordal, joins, have a connected chordal complement, or are complements of cycles of length at least five.

All graphs with at most five vertices that are not joins satisfy the hypothesis of Theorem 1.5, in accordance with the argument given above. Of the 112 graphs with six vertices that are not joins, 105 satisfy the hypothesis of Theorem 1.5; 93 of these are covered without Theorem 1.4, and twelve need it. The remaining seven graphs are connected. They are shown in Figure 5 and listed in Table 2, and all of them satisfy the hypothesis of Proposition 9.3. Of the 853 graphs with seven vertices that are not joins, 578 satisfy the hypothesis of Theorem 1.5 and 758 that of Proposition 9.3. The 95 others are connected and have between 7 and 13 edges; their list is part of the output of the programs (Section 11). Four of them are doubles along stars (Remark 3.9). This leaves 91 graphs with seven vertices for which we have no answer.

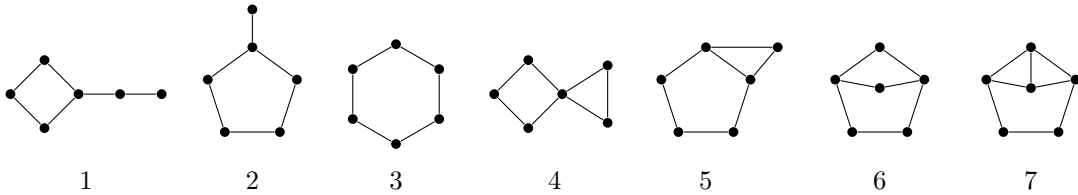


Figure 5: The seven graphs with six vertices that are not joins and do not satisfy the hypothesis of Theorem 1.5.

no.	graph	edges	covered by
1	C_4 with a pendant path of length 2	04 05 13 14 23 24	Theorem 1.6(3): a false twin added to the path P_5
2	pentagon with a pendant vertex	01 04 12 15 23 34	Theorem 1.6(2) with $m = 5$, $ I = 1$
3	hexagon	01 05 12 23 34 45	Theorem 1.6(1)
4	C_4 and a triangle with a common vertex	02 05 13 14 23 34 35	Theorem 1.6(3): a false twin added to the triangle with a pendant path of length 2
5	pentagon with a triangle on an edge	02 05 12 13 23 34 45	Theorem 1.6(2) with $m = 5$, $ I = 2$
6	pentagon with a false twin	01 04 05 12 23 34 35	Theorem 1.6(3): a false twin added to the pentagon
7	pentagon with a true twin	01 04 05 12 15 23 25 34	Theorem 1.6(3): a true twin added to the pentagon

Table 2: The graphs of Figure 5, with vertices $0, \dots, 5$ and edges written as pairs of digits. In the rows 1 and 4 the graph to which the twin is added has its complement in $\mathcal{K}$: a 4-cycle and a triangle with a common edge, and a 4-cycle with a pendant vertex, respectively.

10 Remarks and open questions

The general problem. We have no answer for the graphs that are reached neither by Proposition 9.3 nor by Proposition 3.8. Of the 95 graphs with seven vertices that do not satisfy the hypothesis of Proposition 9.3, four are doubles along stars (Remark 3.9). For the other 91, all connected, we have no answer; subgroups of finite index other than doubles along stars were not examined. Among these 91 graphs are the heptagon C_7 , the hexagon with a pendant vertex, the pentagon with a pendant path of length two, the pentagon with pendant vertices at two different vertices, and the 4-cycle and the pentagon with a common edge. We know of no reason to expect a negative answer for any of them.

Reductions. For homomorphisms of the kind of [CLM12] the results of this note reduce the problem as follows. By the territory lemma it suffices to construct territory systems: nesting of the subsurfaces does not matter. By the gluing lemma it suffices to construct pointed territory systems for the complements of graphs Λ that have no clique separator. One such graph that is neither complete nor a cycle is the triangular prism, the complement of the hexagon; the territory system of Theorem 8.7 is not pointed. The prism is not the smallest graph of this kind: the complete bipartite graph $K_{2,3}$ has no clique separator either. Its complement $K_2 \sqcup K_3$ is chordal and is covered by Theorem 1.1, but we did not construct a pointed territory system for it.

Problem 10.1. *Does the hexagon have a pointed territory system? Does every graph without isolated vertices have a territory system on some closed surface?*

A positive answer to the second question would answer Problem 2.28 positively, by the territory lemma and Proposition 3.6.

Limits of planar pictures. In Sections 6 and 8 the territories are preimages of connected regions of the sphere, and two territories have disjoint interiors exactly when the regions have. The complement graph Λ is then the intersection graph of a family of connected open subsets of the sphere, hence a string graph. Not every graph is a string graph [Sin66]; an example is the graph obtained from K_5 by subdividing every edge once. So planar pictures of this kind cannot treat all graphs.

Genus and powers. The genus has to grow with the graph: for every surface S there are graphs Γ such that $A(\Gamma)$ does not embed in $\text{Mod}(S)$ at all [KK14b, BCG18]. For the complement Γ of the path P_m , Katayama and Kuno [KK18, Theorem 1.3] proved that $A(\Gamma)$ embeds in the mapping class group of the closed surface of genus $g \geq 2$ if and only if $m \leq 2g + 1$; for $m \geq 4$, Theorem 1.2 gives an LPA realization of this graph in genus $2m - 2$, about four times the least genus in which the group embeds. We did not determine whether the nice realization of the pentagon in genus three of [CLM12, Section 2.4] has property (F), and we do not know the least genus of a closed surface S such that the group of the pentagon has an LPA embedding into $\text{Mod}(S)$. The powers of pseudo-Anosov maps that are needed in Proposition 3.2 come from Theorems 2.4 and 2.5 and are not explicit. Explicit powers for which the generators generate a right-angled Artin group are given by Runnels [Run21]; see also [Seo21]. Of [Run21] we read parts of the arXiv version (version 1, the only one there), whose main theorem concerns generation and undistortedness, and the abstract of the published version. That abstract states in addition, for a given collection of pure mapping classes, that “each element is pseudo-Anosov on the largest possible subsurface”. The introduction of [CLM12] describes Theorem 2.5 in the same words, so we take this to be a statement of the same kind, on the type of the image of each element for given supports. A statement of this kind does not answer Problem 2.28, which asks whether the supports can be chosen so that every loxodromic element becomes pseudo-Anosov on the whole surface. We did not use [Run21].

11 Computations and reproducibility

The package that accompanies this note contains the programs mentioned below, written in Python (some use `networkx`, `numpy` and `scipy`), with their outputs and a description. Computations enter the note at three places.

1. *The enumeration of Section 9.3.* The numbers in Table 1, the list in Table 2, and therefore Theorem 1.6(4), rest on it. Columns (i) and (ii) of Table 1 and the list of the seven graphs were computed by five programs that share no code: `coverage.py` with `coverage_atlas.py` and `coverage_E.py`, which use the list of graphs of `networkx`; `v2_coverage.py`; `lead_coverage.py`; `c_enum.py`; and `r4_enum.py`. The last four enumerate the graphs themselves. Column (iii), the list of the 95 graphs with seven vertices that do not satisfy the hypothesis of Proposition 9.3, and the four doubles of Remark 3.9 among them were computed by the last three of these programs; their lists agree graph by graph up to isomorphism. The four doubles were also confirmed by `lead_star_multiples.py`, which uses the isomorphism test of `networkx`.
2. *Lemma 8.6.* The statements (H1)–(H3) are proved in the text. They were checked for the coordinates of Figure 4 by `v2_run_planar.py`, which computes the arrangement of the six boundary curves in exact rational arithmetic, and by `lead_hexagon_planar.py`, which determines the complementary regions on a fine grid. The first program also builds the d -fold cyclic cover for $d = 5, 7, 9, 15$ as a cell complex and confirms the conclusions of Theorem 8.7: the genus, the six subsurfaces, the absence of bigons, and that the lifted curves of each of the five minimal sets cut the surface into discs, while those of six sets inside joins do not. Two further programs, `c_hexagon.py` and `r4_hexagon.py`, check (H1)–(H3) in exact rational arithmetic for the polygons printed in Figure 4; the first one also treats other coordinates and builds the cyclic covers of eight degrees $d \leq 25$. For all 63 non-empty sets of vertices they find that the hypothesis of Lemma 6.3(4) holds exactly for the 39 sets that are not contained in a join of the hexagon.
3. *Consistency checks of statements proved in the text.* Lemma 3.1 for all graphs with at most six vertices; the combinatorial steps of the proof of Theorem 1.1 and the facts (i), (ii) in the proofs

of Proposition 5.3 and of the gluing lemma for all graphs with at most seven vertices to which they apply; Lemma 6.4 for $m \leq 12$; Lemma 6.2(b) for $d \leq 151$ and $m \leq 200$. The program `v2_cover.py` builds the covers of Proposition 6.5 for $m \leq 8$ and $d = 5, 7, 9, 15$ as cell complexes and confirms the genus, the boundary curves, the minimal position of the lifted curves and the filling property. The combinatorial statements in the proofs of Propositions 8.1, 8.3 and 8.5 were checked for small graphs by `c_twins.py`, `c_cycle.py` and `r4_small.py`.

Verification. The claims were checked as follows.

1. The statements quoted from [CLM12] and from [KMT17] were read in the arXiv versions and in the published versions; for [CLM12] the two numberings were compared. The statements quoted from [KK14a, KK13, MM99, MM00, Man10, Loa21, ABD⁺26, KK18, BCG18] were read in the arXiv versions, and [Beh06, Theorem 4.3] was located there. Problem 2.28 and its remarks were read in the authors' preliminary version of [BKR26].
2. All programs were run again for this note, the last time from the source archive of the package; the outputs agree with the recorded ones, up to lines that report running times.
3. A first independent verification run, AI-assisted, examined a working write-up of Section 3, of the proof of Theorem 1.1 and of Section 5 line by line. It compared [CLM12] in both versions, re-derived the filling property for chordal graphs in the form of Lemmas 4.3 to 4.5, proved the facts of Lemma 2.2, and gave a direct proof of the Behrstock inequality for single curves. It found no mathematical error. The corrections it requested concern citations and definitions, among them the numbering of [CLM12] and the exclusion of the identity in Definition 2.1; they are incorporated.
4. A second independent verification run, AI-assisted, examined the working write-up of Sections 6 and 7 and of the proof of Theorem 1.5 line by line, and re-derived the lifting lemma and the gluing lemma. It found no mathematical error. It observed that part (4) of the lifting lemma only uses the complementary regions, found the picture of Figure 4, and proposed the constructions of Section 8. At its request the result of Loa [Loa21] is cited, and the description of the open cases was rewritten.
5. The results of Section 8 and their consequences (Theorem 1.6, Proposition 9.3, column (iii) of Table 1) were written out after the second run, and the programs `lead_coverage.py` and `lead_hexagon_planar.py` were written for them. A third independent verification run, AI-assisted, examined the finished manuscript. It re-derived the proofs of Sections 8 and 9 and of the lemmas added in Section 2 (Lemma 2.2(d),(e), Lemmas 2.3 and 2.6). With its own programs, written from the text, it reproduced Table 1 in all columns, Table 2 and the list of the 95 graphs, and it confirmed Lemma 8.6 and the conclusions of Theorem 8.7 in exact arithmetic, with the cyclic covers of eight degrees. It found no mathematical error and asked for three corrections of wording, which are incorporated: the number 95 refers to Proposition 9.3; the prism is not the smallest graph without clique separator that is neither complete nor a cycle; and every vertex receives at most one true twin.
6. A fourth independent verification run, AI-assisted, read the final text completely. With programs of its own it reproduced Tables 1 and 2, Figure 5, the 95 graphs and the four doubles, and it checked (H1)–(H3) of Lemma 8.6 in exact arithmetic for the polygons printed in Figure 4. It ran the package again from its source archive, compared the cited statements with the sources, and verified the DOIs of the references through Crossref. It found no mathematical error. The changes that it asked for concern wording and credit and are incorporated: the numbers for

seven vertices in the abstract and in Section 1, the sentences on [Run21] and on [MT16], the wording of the problem in the source, and this paragraph.

Scope and priority. The results are partial: Problem 2.28 of [BKR26] remains open for general graphs. They concern LPA embeddings into mapping class groups of closed surfaces, in the sense of Definition 2.1; for graphs that are not connected this definition extends the terminology of [KMT17], as explained in Section 2.2, and Table 1 gives the numbers for connected graphs separately. The proofs rely on published theorems whose proofs were not re-derived here: Theorems 2.2 and 6.1 of [CLM12], the results (K1) and (K2) of Masur and Minsky, the Behrstock inequality (K4), and [KMT17, Theorem 2.2] for Proposition 3.8; Remark 3.9 relies on Lemma 50 of the arXiv version of [KK13].

Of the published versions we read those of [CLM12] and [KMT17]. The other articles were read in their arXiv versions, where they exist: the statements used from [KK13, KK14a, MM99, MM00, Man10, Beh06, Loa21, ABD⁺26, KK18, BCG18] and parts of [MT16, Run21]; the number of the proposition of [MM99] in the journal is taken from the citation in [CLM12]. Of [KK14b, Seo21] only the abstracts were read, and of the published version of [Run21] only the abstract, on the publisher’s page; what this abstract says beyond the arXiv version is quoted in Section 10 and is not used. We did not consult [CP01, Kob12], which are cited as in the remarks of the problem, the books [FM12, LS01], the sources [Dir61, FG65, Gav74, Bun74, Tar85] for the classical facts on chordal graphs, or [Thu88, Pen88, Sin66, FM02]; these are quoted in their standard forms.

Literature was searched in the arXiv listings, in Crossref, OpenAlex and zbMATH, in particular for papers citing [KMT17], and by web searches, most recently on October 8, 2026. We found no source that answers Problem 2.28, and none that gives a positive answer for connected chordal graphs, graphs with chordal complement, complements of cycles, clique-sums, or the hexagon. The positive answer for disjoint unions of complete graphs or of joins is due to Loa [Loa21] and to Aougab, Bray, Dowdall, Hoganson, Maloni and Whitfield [ABD⁺26]; Theorem 1.1 overlaps with their results for disjoint unions of complete graphs (Remark 4.7). The use of a branched cover for the pentagon goes back to [CLM12, Section 2.4], and a nice realization of the complements of even cycles is in [MT16]; the filling property is not considered in either place. MathSciNet and Google Scholar were not searched. This negative search is not a proof of priority, and we claim none.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [ABD⁺26] T. Aougab, H. Bray, S. Dowdall, H. Hoganson, S. Maloni, and B. Whitfield. Constructing reducibly geometrically finite subgroups of the mapping class group. *Groups, Geometry, and Dynamics*, 2026. Published online; arXiv:2501.13234. doi:10.4171/ggd/937.
- [BCG18] E. A. Bering IV, G. Conant, and J. Gaster. On the complexity of finite subgraphs of the curve graph. *Osaka Journal of Mathematics*, 55:795–808, 2018. arXiv:1609.02548.
- [Beh06] J. A. Behrstock. Asymptotic geometry of the mapping class group and Teichmüller space. *Geometry & Topology*, 10(3):1523–1578, 2006. arXiv:math/0502367. doi:10.2140/gt.2006.10.1523.

- [BKR26] R. İ. Baykur, R. C. Kirby, and D. Ruberman, editors. *K3: A New Problem List in Low-Dimensional Topology*, volume 295 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2026. doi:10.1090/surv/295.
- [Bun74] P. Buneman. A characterisation of rigid circuit graphs. *Discrete Mathematics*, 9(3):205–212, 1974. doi:10.1016/0012-365X(74)90002-8.
- [CLM12] M. T. Clay, C. J. Leininger, and J. Mangahas. The geometry of right-angled Artin subgroups of mapping class groups. *Groups, Geometry, and Dynamics*, 6(2):249–278, 2012. arXiv:1007.1129. doi:10.4171/GGD/157.
- [CP01] J. Crisp and L. Paris. The solution to a conjecture of Tits on the subgroup generated by the squares of the generators of an Artin group. *Inventiones Mathematicae*, 145(1):19–36, 2001. doi:10.1007/s002220100138.
- [Dir61] G. A. Dirac. On rigid circuit graphs. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 25:71–76, 1961. doi:10.1007/BF02992776.
- [FG65] D. R. Fulkerson and O. A. Gross. Incidence matrices and interval graphs. *Pacific Journal of Mathematics*, 15(3):835–855, 1965. doi:10.2140/pjm.1965.15.835.
- [FM02] B. Farb and L. Mosher. Convex cocompact subgroups of mapping class groups. *Geometry & Topology*, 6(1):91–152, 2002. doi:10.2140/gt.2002.6.91.
- [FM12] B. Farb and D. Margalit. *A Primer on Mapping Class Groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012. doi:10.1515/9781400839049.
- [Gav74] F. Gavril. The intersection graphs of subtrees in trees are exactly the chordal graphs. *Journal of Combinatorial Theory, Series B*, 16(1):47–56, 1974. doi:10.1016/0095-8956(74)90094-X.
- [KK13] S.-H. Kim and T. Koberda. Embedability between right-angled Artin groups. *Geometry & Topology*, 17(1):493–530, 2013. arXiv:1105.5056. doi:10.2140/gt.2013.17.493.
- [KK14a] S.-H. Kim and T. Koberda. The geometry of the curve graph of a right-angled Artin group. *International Journal of Algebra and Computation*, 24(2):121–169, 2014. arXiv:1305.4363. doi:10.1142/S021819671450009X.
- [KK14b] S.-H. Kim and T. Koberda. An obstruction to embedding right-angled Artin groups in mapping class groups. *International Mathematics Research Notices*, 2014(14):3912–3918, 2014. arXiv:1207.5498. doi:10.1093/imrn/rnt064.
- [KK18] T. Katayama and E. Kuno. The RAAGs on the complement graphs of path graphs in mapping class groups. Preprint, arXiv:1804.03470, 2018.
- [KMT17] T. Koberda, J. Mangahas, and S. J. Taylor. The geometry of purely loxodromic subgroups of right-angled Artin groups. *Transactions of the American Mathematical Society*, 369(11):8179–8208, 2017. arXiv:1412.3663. doi:10.1090/tran/6933.
- [Kob12] T. Koberda. Right-angled Artin groups and a generalized isomorphism problem for finitely generated subgroups of mapping class groups. *Geometric and Functional Analysis*, 22(6):1541–1590, 2012. arXiv:1007.1118. doi:10.1007/s00039-012-0198-z.

- [Loa21] C. Loa. Free products of abelian groups in mapping class groups. Preprint, arXiv:2103.05144, 2021.
- [LS01] R. C. Lyndon and P. E. Schupp. *Combinatorial Group Theory*. Classics in Mathematics. Springer, Berlin, 2001. Reprint of the 1977 edition. doi:10.1007/978-3-642-61896-3.
- [Man10] J. Mangahas. Uniform uniform exponential growth of subgroups of the mapping class group. *Geometric and Functional Analysis*, 19(5):1468–1480, 2010. arXiv:0805.0133. doi:10.1007/s00039-009-0038-y.
- [MM99] H. A. Masur and Y. N. Minsky. Geometry of the complex of curves I: Hyperbolicity. *Inventiones Mathematicae*, 138(1):103–149, 1999. arXiv:math/9804098. doi:10.1007/s002220050343.
- [MM00] H. A. Masur and Y. N. Minsky. Geometry of the complex of curves II: Hierarchical structure. *Geometric and Functional Analysis*, 10(4):902–974, 2000. arXiv:math/9807150. doi:10.1007/PL00001643.
- [MT16] J. Mangahas and S. J. Taylor. Convex cocompactness in mapping class groups via quasi-convexity in right-angled Artin groups. *Proceedings of the London Mathematical Society* (3), 112(5):855–881, 2016. arXiv:1306.5278. doi:10.1112/plms/pdw009.
- [Pen88] R. C. Penner. A construction of pseudo-Anosov homeomorphisms. *Transactions of the American Mathematical Society*, 310(1):179–197, 1988. doi:10.1090/S0002-9947-1988-0930079-9.
- [Run21] I. Runnels. Effective generation of right-angled Artin groups in mapping class groups. *Geometriae Dedicata*, 214(1):277–294, 2021. arXiv:2004.13585. doi:10.1007/s10711-021-00615-0.
- [Seo21] D. Seo. Powers of Dehn twists generating right-angled Artin groups. *Algebraic & Geometric Topology*, 21(3):1511–1533, 2021. arXiv:1909.03394. doi:10.2140/agt.2021.21.1511.
- [Sin66] F. W. Sinden. Topology of thin film RC circuits. *Bell System Technical Journal*, 45(9):1639–1662, 1966. doi:10.1002/j.1538-7305.1966.tb01713.x.
- [Tar85] R. E. Tarjan. Decomposition by clique separators. *Discrete Mathematics*, 55(2):221–232, 1985. doi:10.1016/0012-365X(85)90051-2.
- [Thu88] W. P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bulletin of the American Mathematical Society (N.S.)*, 19(2):417–431, 1988. doi:10.1090/S0273-0979-1988-15685-6.
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Version 1.6.0; record KP-2.28.