

Asymmetric Hyperbolic Knots Detect Mapping Classes of 3-Manifolds: A Written Proof for a Problem of the K3 List

Alper Ferudun*

October 3, 2026

Abstract

Problem 1.19 of the list *K3: A New Problem List in Low-Dimensional Topology* (Baykur, Kirby and Ruberman, eds., 2026) asks: if $Y = Y_1 \# Y_2$ is a connected sum of 3-manifolds, neither of them S^3 , and the twist Φ about the connected-sum sphere is not isotopic to the identity, is there a knot $K \subset Y$ such that K and $\Phi(K)$ are not isotopic? The question goes back to Aceto, Bregman, Davis, Park and Ray. They proved the analogous statement for prime 3-manifolds, conjectured it in general, and recorded, in all versions of their paper since 2020, that Etnyre and Margalit have a proof of the general statement. That proof has not appeared, and to our knowledge no written proof exists. This note supplies one, for closed orientable 3-manifolds. It may well coincide with the argument of Etnyre and Margalit, with whom we have not been in contact, and we claim no priority. Let K be a knot in a closed, connected, orientable 3-manifold Y whose complement is hyperbolic with trivial isometry group; such knots exist by a theorem of Kawauchi. We show that every diffeomorphism f of Y for which $f(K)$ is isotopic to K is isotopic to the identity. Hence every diffeomorphism that is not isotopic to the identity changes the isotopy class of a knot, and the answer to Problem 1.19 is yes. The proof is the classical description of the symmetries of a hyperbolic knot (Mostow–Prasad, Waldhausen, Hatcher). Using a theorem of Chen and Tshishiku on finite group actions, we also show that a twist about a connected-sum sphere which is not isotopic to the identity changes the isotopy class of *every* knot with hyperbolic complement. This is an unrefereed note.

1 Introduction

Manifolds, embeddings and isotopies in this note are smooth, unless we say otherwise. Let Y be a closed, connected, oriented 3-manifold. Following Aceto, Bregman, Davis, Park and Ray [ABD⁺26], two knots in Y are *equivalent* if an orientation-preserving diffeomorphism of Y takes one to the other, and *isotopic* if this diffeomorphism can be chosen isotopic to the identity. They proved that if Y is prime and f is an orientation-preserving diffeomorphism of Y which is not isotopic to the identity, then K and $f(K)$ are not isotopic for some knot $K \subset Y$ [ABD⁺26, Theorem 1.1]. The published version proves the same for connected sums of copies of $S^1 \times S^2$ [ABD⁺26, Corollary 4.4]. The first version of [ABD⁺26] (arXiv, July 2020) conjectures that in every oriented 3-manifold equivalent knots are isotopic only if the orientation-preserving mapping class group is trivial (Conjecture 1.8), and a note there adds that J. Etnyre and D. Margalit have a proof of this conjecture by different methods. The published version (2026) says in a note that Etnyre and Margalit communicated to the authors a different proof of Theorem 1.1, which extends to the non-prime case and has not yet appeared. Its Remark 1.6 explains that the main remaining case is that of twists along separating 2-spheres. This case became a problem of the list K3 [BKR26].

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
Unrefereed preprint. CC BY 4.0.

Problem 1.1 ([BKR26, Problem 1.19], in paraphrase). *Let $Y = Y_1 \# Y_2$ be a connected sum of 3-manifolds with $Y_i \neq S^3$ for $i = 1, 2$, and let $\Phi: Y \rightarrow Y$ be a Dehn twist about the connected-sum sphere. Assume that Φ is not isotopic to the identity. Is there a knot $K \subset Y$ such that K and $\Phi(K)$ are not ambiently isotopic?*

This is Problem 1.19 of the list [BKR26] of 2026, not Problem 1.19 of Kirby’s list of 1997, which is a different problem; in the UnsolvedMath dataset [ula26] it is record KP-1.19. To our knowledge, the proof of Etnyre and Margalit has not appeared, and there is no other written proof of the general statement. This note supplies one, for closed, connected, orientable 3-manifolds. It may well coincide with the argument of Etnyre and Margalit, which we do not know; we have not been in contact with them, and we claim no priority.

A knot $K \subset Y$ is *hyperbolic* if $Y \setminus K$ admits a complete hyperbolic metric of finite volume. By Mostow–Prasad rigidity this metric is unique up to isometry, and its isometry group is finite. We call a hyperbolic knot *asymmetric* if the identity is the only isometry of this metric; orientation-reversing isometries are included. We regard knots as subsets: two knots $K, K' \subset Y$ are isotopic if $h(K) = K'$ for a diffeomorphism h of Y that is isotopic to the identity. No orientation of the knots is fixed.

Theorem A. *Let Y be a closed, connected, orientable 3-manifold and let $K \subset Y$ be an asymmetric hyperbolic knot. If f is a diffeomorphism of Y such that $f(K)$ is isotopic to K , then f is isotopic to the identity. Moreover, Y contains infinitely many asymmetric hyperbolic knots with pairwise non-diffeomorphic exteriors (Kawauchi [Kaw92]).*

So a single knot detects all mapping classes of Y , orientation-reversing ones included: the map $[f] \mapsto [f(K)]$ from $\pi_0 \text{Diff}(Y)$ to the set of isotopy classes of knots is injective.

Corollary 1.2. *Let Y be a closed, connected, oriented 3-manifold.*

- (a) *For every diffeomorphism f of Y that is not isotopic to the identity there is a knot $K \subset Y$ such that $f(K)$ is not isotopic to K .*
- (b) *Y contains a pair of equivalent knots that are not isotopic if and only if the group $\pi_0 \text{Diff}^+(Y)$ of isotopy classes of orientation-preserving diffeomorphisms is non-trivial.*
- (c) *The answer to Problem 1.1 is yes.*

Part (a), for orientation-preserving f , is [ABD⁺26, Theorem 1.1] without the hypothesis that Y is prime. Part (b) is Conjecture 1.8 of the first version of [ABD⁺26], for closed manifolds and in the smooth category. For twists about spheres more is true, and no asymmetry is needed.

Theorem B. *Let $Y = Y_1 \# Y_2$ be a closed, connected, oriented 3-manifold with $Y_i \neq S^3$, and let Φ be the twist about the connected-sum sphere. If Φ is not isotopic to the identity, then $\Phi(K)$ is not isotopic to K for every hyperbolic knot $K \subset Y$.*

Hyperbolic knots exist in every closed, connected, orientable 3-manifold [Mye82, Thu82]. If Y is not prime and not a connected sum of lens spaces and copies of $S^1 \times S^2$, then Theorem 4.4 extends Theorem B to all orientation-preserving diffeomorphisms of Y that induce inner automorphisms of $\pi_1(Y)$.

About the proofs. Both theorems rest on Proposition 3.3: if K is a hyperbolic knot and $f(K)$ is isotopic to K , then f is isotopic to a diffeomorphism $\hat{g}$ of finite order which acts on the exterior of K as an isometry. This is the classical description of the symmetries of a hyperbolic knot, by Mostow–Prasad rigidity, Waldhausen’s isotopy theorem and Hatcher’s theorem on the solid torus. If

K is asymmetric, then $\hat{g}$ is the identity. For Theorem B, $\hat{g}$ induces an inner automorphism of $\pi_1(Y)$, and a theorem of Chen and Tshishiku [CT26] shows that $\hat{g}$ is the identity unless Y is a connected sum of lens spaces and copies of $S^1 \times S^2$; for those Y the twist Φ is isotopic to the identity. None of the ingredients is new, and the use of hyperbolic knots without symmetries to detect diffeomorphisms is a natural idea. Our contribution is to write down the proof that these results settle Problem 1.1 and the conjecture of [ABD⁺26] for closed manifolds, and Theorem B. Experts may regard Theorem A as an easy consequence of known results; we did not find it stated.

2 Conventions and results used

Conventions. $\text{Diff}(W)$ is the group of diffeomorphisms of a manifold W , with the C^∞ topology, and $\text{Diff}^+(W)$ is the subgroup of orientation-preserving ones. Two diffeomorphisms are *isotopic* if they lie in the same path component of $\text{Diff}(W)$. For compact W , $\text{Diff}(W \text{ rel } \partial W)$ and $\text{Homeo}(W \text{ rel } \partial W)$ are the groups of diffeomorphisms and of homeomorphisms of W that restrict to the identity on ∂W . Groups of homeomorphisms carry the compact-open topology, and homeomorphisms are isotopic if a path of homeomorphisms joins them. Isotopic diffeomorphisms of a connected manifold W induce the same element of $\text{Out}(\pi_1(W))$, the group of automorphisms of $\pi_1(W)$ modulo inner automorphisms. A diffeomorphism *acts trivially on $\pi_1(W)$* if this element is trivial. Knots and their isotopy are defined in the introduction. In [ABD⁺26] a knot is an embedding of the oriented circle, and knots are isotopic if they are joined by a path of embeddings; by the isotopy extension theorem, their images are then isotopic in our sense. So knots that are *not* isotopic in our sense are not isotopic in the sense of [ABD⁺26].

Theorem 2.1 (Mostow [Mos73], Prasad [Pra73]; see [Thu80, Theorem 5.7.2 and Corollary 5.7.4]). *Let M be a complete hyperbolic 3-manifold of finite volume, and let $\text{Isom}(M)$ be the group of all its isometries. Every automorphism of $\pi_1(M)$ is induced by an isometry of M , and this isometry is determined by the class of the automorphism in $\text{Out}(\pi_1(M))$. So the natural map $\text{Isom}(M) \rightarrow \text{Out}(\pi_1(M))$ is an isomorphism, and both groups are finite.*

Theorem 2.2 (Waldhausen). *Let W be a compact, connected, orientable, irreducible 3-manifold with non-empty incompressible boundary, and let x be a point in the interior of W . If v is an orientation-preserving diffeomorphism of W with $v(x) = x$ which induces the identity of $\pi_1(W, x)$, then v is isotopic to the identity.*

This is part of [HW10, Proposition 2.1]: for such W , also with empty boundary, the map from the group of isotopy classes of orientation-preserving diffeomorphisms of W that fix x to the automorphism group of $\pi_1(W, x)$ is injective. The diffeomorphisms and the isotopies need not fix ∂W . The result is attributed there to Waldhausen [Wal68, Theorem 7.1]. The hypothesis on the orientation is needed. If F is a closed orientable surface of genus at least one, the reflection $(y, t) \mapsto (y, 1 - t)$ of $F \times [0, 1]$ fixes the points with $t = \frac{1}{2}$ and induces the identity of π_1 , but it reverses the orientation and is not isotopic to the identity; see Remark 3.4.

Theorem 2.3 (Cerf, Hatcher). *(a) For every compact 3-manifold W , the inclusion of the group $\text{Diff}(W \text{ rel } \partial W)$ into the group $\text{Homeo}(W \text{ rel } \partial W)$ is a weak homotopy equivalence. In particular it is a bijection on path components.*

(b) The group $\text{Diff}(S^1 \times D^2 \text{ rel } \partial)$ is contractible. In particular it is path-connected.

Cerf derived (a) from the statement that $\text{Diff}(D^3 \text{ rel } \partial D^3)$ is contractible [Cer59, Théorème 5, Corollaire 1], and this statement is Hatcher's theorem [Hat83]. Part (a) is statement (3) of the appendix of [Hat83], in the topological form mentioned there, and part (b) is statement (9). In

[Cer59] the diffeomorphisms are required to be tangent to the identity to infinite order along ∂W . We use only two consequences, which hold with either reading: (i) diffeomorphisms of a closed 3-manifold that are isotopic through homeomorphisms are isotopic; (ii) every homeomorphism h of a solid torus N that is the identity on ∂N is isotopic to the identity in $\text{Homeo}(N \text{ rel } \partial N)$. For (ii), by (a) h is isotopic in this group to a diffeomorphism, and by (b) this diffeomorphism is isotopic to the identity.

Theorem 2.4 (Kawauchi [Kaw92, Main Theorem]). *Let Y be a closed, connected, oriented 3-manifold and let $L \subset Y$ be a knot. Then there are a constant C^+ and an infinite family of knots $L^* \subset Y$ with the following properties. For every L^* in the family, $Y \setminus L^*$ admits a complete hyperbolic metric of volume less than C^+ , and the isometry group of this metric is trivial. Moreover, C^+ is the supremum of these volumes.*

In [Kaw92] the theorem is stated for *good* $(3, 1)$ -manifold pairs (M, L) : M is a compact, connected, oriented 3-manifold, and L is a compact proper 1-submanifold which meets every 2-sphere in ∂M in at least three points. A knot in a closed manifold is such a pair. A compact 3-manifold E bounded by tori is called hyperbolic there if its interior has a complete metric of constant curvature -1 , and $\text{Vol } E$ and $\text{Isom } E$ are the volume and the isometry group of this metric; by p. 299 of [Kaw92], $\text{Isom } E$ is isomorphic to $\text{Out}(\pi_1(E))$, so it is the full isometry group. The Main Theorem gives an infinite family of pairs (M, L^*) , called almost identical imitations of (M, L) , with hyperbolic exteriors $E(L^*, M)$. If L is a knot, then so is L^* , because the imitation map restricts to a diffeomorphism from a neighbourhood of L^* onto a neighbourhood of L [Kaw92, Section 2]. Part (1) of the theorem says that $\text{Vol } E(L^*, M) < C^+$ and that C^+ is the supremum of these volumes. Part (2) is a statement on the isometry groups of branched coverings, which contains the assertion $\text{Isom } E(L^*, M) = \{1\}$. See also [Kaw93, Theorem 1.1]. The volumes are smaller than their supremum, so infinitely many of them are different, and the volume is an invariant of the exterior by Theorem 2.1. Hence the family contains infinitely many asymmetric hyperbolic knots with pairwise non-diffeomorphic exteriors. Hyperbolic knots, not necessarily asymmetric, exist in every closed, connected, orientable 3-manifold Y . By Myers [Mye82, Theorem 6.1], Y contains a knot whose exterior is irreducible and boundary-irreducible and contains no essential annulus or torus, and by Thurston's hyperbolization theorem [Thu82] the complement of such a knot is hyperbolic.

The exterior of a hyperbolic knot. Let $K \subset Y$ be a hyperbolic knot. Fix a complete hyperbolic metric of finite volume on $M = Y \setminus K$, and write $M = \mathbb{H}^3/\Gamma$. The manifold M has one end. So by [Thu80, Proposition 5.11.1] there is a closed set $C \subset M$, the quotient of a horoball $\tilde{B} \subset \mathbb{H}^3$ by its stabiliser $P \cong \mathbb{Z}^2$ in Γ , such that $M_0 = M \setminus \text{int } C$ is compact. The boundary $T = \partial C = \partial M_0$ is a flat torus. We put $N = C \cup K$ and $X = M_0 = Y \setminus \text{int } N$. Then N is a closed tubular neighbourhood of K , and X is the exterior of K . This is standard; here is the argument. Let $N_1 \subset \text{int } N$ be a closed tubular neighbourhood of K . Then ∂N_1 lies in a region $T \times [0, 1] \subset C$ between T and a horospherical torus inside N_1 , and it separates the two boundary components of this region. The manifold $T \times [0, 1]$ is irreducible, so a compressible torus in it bounds a solid torus or lies in a ball [Hat00, Section 1.2], and then it does not separate them. So ∂N_1 is incompressible. By [Hat00, Proposition 1.11], applied to a product Seifert fibration of $T \times [0, 1]$, it is isotopic to a vertical torus, and hence to a level torus. So the region between ∂N_1 and T is a collar, and N is a tubular neighbourhood as well. Let $\mu \subset T$ be a meridian of N .

The inclusion $X \rightarrow M$ is a homotopy equivalence. We identify $\pi_1(X)$ with Γ in such a way that the image of $\pi_1(T)$ is P . The map $\pi_1(T) \rightarrow \Gamma$ is injective, so T is incompressible in X . Moreover X is irreducible, because its interior is covered by $\mathbb{H}^3$; see [Hat00, Theorem 1.1 and Proposition 1.6]. The translates of $\tilde{B}$ under Γ are pairwise disjoint or equal, and two horoballs with the same centre intersect. So P is the stabiliser in Γ of the centre ξ of $\tilde{B}$. Every non-trivial element of P is parabolic,

with ξ as its only fixed point at infinity. This gives the following property.

$$\text{If } 1 \neq a \in P, \delta \in \Gamma \text{ and } \delta a \delta^{-1} \in P, \text{ then } \delta \in P. \quad (*)$$

Indeed, $\delta a \delta^{-1}$ fixes $\delta \xi$ only, so $\delta \xi = \xi$.

Every isometry σ of M maps C to itself, because the cusp neighbourhoods of M are nested and σ preserves the volume. So $\sigma(X) = X$ and $\sigma(T) = T$. Let $E: T \times (-\epsilon, \epsilon) \rightarrow M$ be the normal exponential map of T , with $E(T \times [0, \epsilon)) \subset C$. It is a bicollar of T in Y , and $\sigma(E(x, s)) = E(\sigma(x), s)$.

3 Mapping classes that fix a hyperbolic knot

We keep the notation of Section 2 for a hyperbolic knot $K \subset Y$.

Lemma 3.1. *Let v be a diffeomorphism of X which acts trivially on $\pi_1(X)$. Then $v|_T$ induces the identity of $H_1(T; \mathbb{Z})$, and v preserves the orientation of X .*

Proof. Let $x_0 \in T$ and let β be a path in T from $v(x_0)$ to x_0 . The automorphism $a \mapsto [\bar{\beta} \cdot (v \circ a) \cdot \beta]$ of $\pi_1(X, x_0) = \Gamma$ represents the outer class of v , so it is the conjugation by some $\delta \in \Gamma$. If $a \in P$, then $v \circ a$ and β lie in T , so $\delta a \delta^{-1} \in P$. By $(*)$, $\delta \in P$. Since P is abelian, $[\bar{\beta} \cdot (v \circ a) \cdot \beta] = a$ for all $a \in P$, and this holds in $\pi_1(T, x_0)$ as well. Hence $v|_T$ is the identity on $H_1(T; \mathbb{Z})$. In particular $v|_T$ preserves the orientation of T , and so does v on X , as it maps inward normals to inward normals. $\square$

Lemma 3.2. *Let σ be an isometry of M such that $(\sigma|_T)_*[\mu] = \pm[\mu]$ in $H_1(T; \mathbb{Z})$. Then there is a diffeomorphism $\hat{g}$ of Y which agrees with σ on a neighbourhood of X in Y , satisfies $\hat{g}(N) = N$, and has the same order as σ . If $\sigma = \text{id}$, then $\hat{g} = \text{id}$.*

Proof. Write $T = \mathbb{R}^2/\Lambda$ for a lattice Λ . Then $\sigma|_T$ is a map $x \mapsto Ax + b$ with $A \in O(2)$ and $A\Lambda = \Lambda$. Let $m \in \Lambda$ be a primitive vector which represents $[\mu]$. Then $Am = \varepsilon m$ with $\varepsilon = \pm 1$. Put $c = |m|$ and $e = m/c$, and let $D^2 \subset \mathbb{C}$ be the unit disc. The circle $\mathbb{R}/c\mathbb{Z}$ acts freely on $T \times D^2$ by $s \cdot (x, z) = (x + se, e^{-2\pi i s/c} z)$. The quotient V is a solid torus, and $x \mapsto [x, 1]$ identifies T with ∂V . The discs $\{[x, z] : z \in D^2\}$ are meridian discs of V . They are bounded by the orbits of $\mathbb{R}/c\mathbb{Z}$ on T , which are closed geodesics in the class $\pm[\mu]$. Every point $[x, z]$ with $z \neq 0$ is $[x', r]$ with a unique $x' \in T$ and $r = |z|$; this defines a collar of ∂V .

Let $\tilde{\tau}(x, z) = (Ax + b, z)$ if $\varepsilon = 1$, and $\tilde{\tau}(x, z) = (Ax + b, \bar{z})$ if $\varepsilon = -1$. Then $\tilde{\tau}(s \cdot (x, z)) = (\varepsilon s) \cdot \tilde{\tau}(x, z)$, so $\tilde{\tau}$ induces a diffeomorphism τ of V . It restricts to $\sigma|_T$ on $\partial V = T$, and in the collar it is $\tau[x', r] = [\sigma(x'), r]$. If $(\sigma|_T)^k = \text{id}$, then $A^k = 1$, hence $\varepsilon^k = 1$ and $\tau^k = \text{id}$.

The solid tori V and N have the same boundary T and the same meridian class $\pm[\mu]$. So the identity of T extends to a diffeomorphism $\theta: V \rightarrow N$. By the uniqueness of collars we may assume that $\theta[x', r] = E(x', 1 - r)$ for r close to 1. Let $\hat{g} = \sigma$ on X and $\hat{g} = \theta\tau\theta^{-1}$ on N . For small $s \geq 0$ we have $\hat{g}(E(x, s)) = E(\sigma(x), s) = \sigma(E(x, s))$. So $\hat{g}$ agrees with σ on a neighbourhood of X , and it is a diffeomorphism of Y . If $\sigma^k = \text{id}$, then $\tau^k = \text{id}$ and $\hat{g}^k = \text{id}$; conversely $\hat{g}^k = \text{id}$ implies $\sigma^k = \text{id}$ on X , hence on M . If $\sigma = \text{id}$, then $\tau = \text{id}$ and $\hat{g} = \text{id}$. $\square$

Proposition 3.3. *Let K be a hyperbolic knot in a closed, connected, orientable 3-manifold Y , and let f be a diffeomorphism of Y such that $f(K)$ is isotopic to K . Then there is an isometry σ of $M = Y \setminus K$ with $(\sigma|_T)_*[\mu] = \pm[\mu]$ such that f is isotopic to the diffeomorphism $\hat{g}$ of Lemma 3.2. In particular, f is isotopic to a diffeomorphism of finite order. If K is asymmetric, then f is isotopic to the identity.*

Proof. Step 1: a normal form. Let h be a diffeomorphism of Y , isotopic to the identity, with $h(f(K)) = K$. Then $hf(N)$ is a closed tubular neighbourhood of K . By the uniqueness of tubular neighbourhoods [Hir76, Chapter 4], there is a diffeomorphism ρ of Y , isotopic to the identity, with $\rho(hf(N)) = N$. Put $g = \rho h f$. Then g is isotopic to f , and $g(N) = N$, $g(X) = X$, $g(T) = T$. Since $g|_N$ is a diffeomorphism of the solid torus N , we have $(g|_T)_*[\mu] = \pm[\mu]$.

Step 2: the isometry. By Theorem 2.1 there is an isometry σ of M which induces the same element of $\text{Out}(\Gamma)$ as $g|_X$. Then $u = (\sigma|_X)^{-1} \circ g|_X$ is a diffeomorphism of X which acts trivially on $\pi_1(X)$. By Lemma 3.1, u preserves the orientation, and $(\sigma|_T)_* = (g|_T)_*$ on $H_1(T; \mathbb{Z})$. So $(\sigma|_T)_*[\mu] = \pm[\mu]$. Let $\hat{g}$ be the diffeomorphism of Lemma 3.2, and put $k = \hat{g}^{-1}g$. Then $k(X) = X$, $k(N) = N$ and $k|_X = u$.

Step 3: u is isotopic to the identity in $\text{Diff}(X)$. Let x be a point in the interior of X . There is a diffeomorphism d of X , isotopic to the identity, with $d(u(x)) = x$. The automorphism of $\pi_1(X, x)$ induced by du is inner. Pushing the point x once around a suitable loop gives a diffeomorphism d' of X , isotopic to the identity, with $d'(x) = x$, such that $v = d' du$ induces the identity of $\pi_1(X, x)$. The diffeomorphism v is isotopic to u , preserves the orientation and fixes x . As recalled in Section 2, X is irreducible and its boundary is incompressible. By Theorem 2.2, v is isotopic to the identity. So there is a path $(u_t)_{t \in [0,1]}$ in $\text{Diff}(X)$ with $u_0 = \text{id}$ and $u_1 = u$. The next two steps use only that this is a path of homeomorphisms of X .

Step 4: extension to Y . Let $c: T \times [0, 1] \rightarrow N$ be a collar of ∂N , with $c(x, 0) = x$. For $t \in [0, 1]$ define a homeomorphism λ_t of N by $\lambda_t(c(x, s)) = c(u_{t+(1-t)s}u^{-1}(x), s)$ on the collar, and by the identity elsewhere. Then $\lambda_1 = \text{id}$, and $\lambda_t = u_t u^{-1}$ on T . Let $\kappa_t = u_t$ on X and $\kappa_t = \lambda_t \circ k$ on N . The two definitions agree on T . So (κ_t) is a path of homeomorphisms of Y , from κ_0 to $\kappa_1 = k$, and κ_0 is the identity on X .

Step 5: the solid torus. The restriction of κ_0 to N is a homeomorphism of the solid torus N which is the identity on ∂N . By consequence (ii) of Theorem 2.3, it is isotopic to the identity in $\text{Homeo}(N \text{ rel } \partial N)$. We extend this isotopy by the identity on X . Together with Step 4 this shows that k is isotopic to the identity through homeomorphisms of Y . Hence $g = \hat{g}k$ and $\hat{g}$ are isotopic through homeomorphisms. By consequence (i) of Theorem 2.3, they are isotopic. So f is isotopic to $\hat{g}$. By Theorem 2.1, σ has finite order, and so has $\hat{g}$. If K is asymmetric, then $\sigma = \text{id}$ and $\hat{g} = \text{id}$. $\square$

Proof of Theorem A and Corollary 1.2. The first assertion of Theorem A is the last sentence of Proposition 3.3. For the second, orient Y and apply Theorem 2.4 and the remarks after it to any knot $L \subset Y$.

Let K be an asymmetric hyperbolic knot in Y . Part (a) of Corollary 1.2 follows from Theorem A, with this K for every f . For (b), if $\pi_0 \text{Diff}^+(Y)$ is trivial, then equivalent knots are isotopic. If not, let f be an orientation-preserving diffeomorphism which is not isotopic to the identity. Then K and $f(K)$ are equivalent and not isotopic; by Section 2 this also holds for oriented knots in the sense of [ABD⁺26]. For (c), Φ is a diffeomorphism of the closed, connected, oriented manifold $Y = Y_1 \# Y_2$. If it is not isotopic to the identity, then K and $\Phi(K)$ are not isotopic. $\square$

Remark 3.4. Waldhausen's theorem is commonly stated for homeomorphisms which are homotopic to the identity through maps of pairs $(W, \partial W) \rightarrow (W, \partial W)$. The reflection described after Theorem 2.2 is homotopic to the identity, but not through maps of pairs, since it exchanges the two boundary components. For the map u of Step 3 the condition holds, and this can be checked without Theorem 2.2. The manifold X is aspherical and u acts trivially on $\pi_1(X)$. So there is a homotopy H from the identity to u . Let α be the track of a point $x_0 \in T$, and let β be a path in T from $u(x_0)$ to x_0 . As in the proof of Lemma 3.1, the class of the loop $\alpha \cdot \beta$ conjugates P to itself, so it lies in P by (*). Thus α is homotopic, relative to its end points, to a path in T . The covering $X_P \rightarrow X$ with group P deformation retracts onto a lift $\tilde{T}$ of T . The restriction of H to $T \times [0, 1]$ lifts to X_P , and the lift maps $T \times \{0, 1\}$ into $\tilde{T}$. So this restriction deforms into T , relative to $T \times \{0, 1\}$, and H can be replaced by a homotopy of pairs.

4 Twists about spheres

Let $S \subset Y$ be an embedded 2-sphere with a product neighbourhood $U \cong S^2 \times [0, 1]$. Let $\varphi: [0, 1] \rightarrow SO(3)$ be a smooth loop which is the identity near 0 and 1 and represents the generator of $\pi_1(SO(3)) \cong \mathbb{Z}/2$. The *twist* T_S about S is the diffeomorphism of Y given by $T_S(x, t) = (\varphi(t)x, t)$ on U and by the identity outside U . Its isotopy class depends only on S , and T_S^2 is isotopic to the identity by an isotopy with support in U ; see [CT26, Section 2.1] and [HW10, Section 2]. This is the Dehn twist of Problem 1.1, called a Gluck twist in [ABD⁺26] and a sphere twist in [CT26]. For a connected sum $Y = Y_1 \# Y_2$ let $\Phi = T_S$, where S is the connected-sum sphere. A *lens space* is an $L(p, q)$ with $p \geq 2$.

Lemma 4.1 ([CT26, Lemma 2.1]). *Let $*$ $\in Y \setminus U$. Then T_S induces the identity of $\pi_1(Y, *)$. In particular T_S acts trivially on $\pi_1(Y)$.*

Proof. A loop at $*$ which is transverse to ∂U is a concatenation of paths in $Y \setminus \text{int } U$, which T_S fixes pointwise, and of paths in U with end points on ∂U . The image of such a path has the same end points, and it is homotopic to the path relative to them, since U is simply connected. $\square$

Lemma 4.2. *Let $Y = Y_1 \# Y_2$ with connected-sum sphere S . If every prime summand of Y_1 is a lens space or $S^1 \times S^2$, then T_S is isotopic to the identity. Consequently, if T_S is not isotopic to the identity, then each of Y_1 and Y_2 has a prime summand which is neither a lens space nor $S^1 \times S^2$, and Y is not a connected sum of lens spaces and copies of $S^1 \times S^2$.*

Proof. (a) Let Q be $S^1 \times S^2$ or a lens space. Then Q carries an action R_θ of the circle $\mathbb{R}/2\pi\mathbb{Z}$ with a fixed point x_0 at which the differential of R_θ is the rotation by the angle θ about an axis. On $S^1 \times S^2$ we rotate the factor S^2 about an axis. On $L(p, q)$, the quotient of $S^3 \subset \mathbb{C}^2$ by $(z, w) \mapsto (e^{2\pi i/p}z, e^{2\pi i q/p}w)$, we put $R_\theta[z, w] = [z, e^{i\theta}w]$ and $x_0 = [1, 0]$. Choose an invariant metric and concentric geodesic balls $D \subset D'$ around x_0 . In polar coordinates $D' \setminus \text{int } D \cong S^2 \times [0, 1]$, with $\partial D = S^2 \times \{0\}$, we have $R_\theta(y, t) = (\rho_\theta y, t)$, where $\rho_\theta \in SO(3)$ is the rotation by θ about a fixed axis. The loop $\theta \mapsto \rho_\theta$, $0 \leq \theta \leq 2\pi$, generates $\pi_1(SO(3))$. Let $\lambda: [0, 1] \rightarrow [0, 1]$ be smooth, equal to 0 near 0 and to 1 near 1. For $r \in [0, 1]$ let H_r be the identity on D , let $H_r(y, t) = (\rho_{2\pi r \lambda(t)} y, t)$ on $D' \setminus \text{int } D$, and let $H_r = R_{2\pi r}$ outside D' . This is an isotopy of Q , fixed on D , from the identity to a twist about a sphere parallel to ∂D . So in $Q \setminus \text{int } D$ this twist is isotopic to the identity relative to the boundary.

(b) Let $W \subset \mathbb{R}^3$ be a closed round ball minus $n \geq 1$ disjoint open round balls with centres on an axis. The same construction, with the rotations about this axis and with radial collars of the $n + 1$ boundary spheres, gives an isotopy of W , fixed near ∂W , from the identity to the composition of the twists about the $n + 1$ spheres parallel to the boundary; see also [HW10, Section 2] and [CT26, Lemma 2.5]. Since twists have order two up to isotopy, the twist about the outer sphere is isotopic, relative to ∂W , to the composition of the twists about the inner spheres.

(c) Let $Y_1 = Q_1 \# \dots \# Q_n$ be a prime decomposition; $n \geq 1$ because $Y_1 \neq S^3$. The complement of an open ball in Y_1 is the union of W and of the manifolds $Q_i \setminus \text{int } D_i$, glued along the inner boundary spheres of W , and S is the outer boundary sphere of W . We may take T_S with support in a collar of the outer sphere in W . By (b), T_S is isotopic to the composition of the twists about spheres parallel to the ∂D_i , and by (a) each of them is isotopic to the identity. The last assertion follows from the uniqueness of the prime decomposition [Mil62]. $\square$

Theorem 4.3 (Chen–Tshishiku). *Let Y be a closed, connected, oriented 3-manifold which is not prime and which is not a connected sum of lens spaces and copies of $S^1 \times S^2$. Let g be an orientation-preserving diffeomorphism of Y of finite order which acts trivially on $\pi_1(Y)$. Then $g = \text{id}$.*

This is a special case of [CT26, Lemma 6.1]: if a closed, oriented 3-manifold M is reducible and not a connected sum of lens spaces, then the kernel of the map from $\text{Diff}(M)$ to $\text{Out}(\pi_1(M))$ is

torsion-free. See also [CT26, Theorem 1.4]; its additional hypothesis, that M is not a connected sum of copies of $\mathbb{R}P^3$, holds for our Y . In [CT26] the manifold $S^1 \times S^2 = L(0,1)$ is counted as a lens space, acting trivially on π_1 has the same meaning as here [CT26, Remark 2.2], and *reducible* is used in the usual sense (some embedded 2-sphere does not bound a ball), so a manifold that is not prime is reducible. We refer to the published version of [CT26]. In the second arXiv version, Theorems 1.4 and 2.4 and Lemma 2.5 are Theorems 1.5 and 2.3 and Lemma 2.4; the other numbers that we cite are the same.

Theorem 4.4. *Let Y be a closed, connected, oriented 3-manifold which is not prime and which is not a connected sum of lens spaces and copies of $S^1 \times S^2$. Let f be an orientation-preserving diffeomorphism of Y which acts trivially on $\pi_1(Y)$ and is not isotopic to the identity. Then $f(K)$ is not isotopic to K , for every hyperbolic knot $K \subset Y$.*

Proof. Suppose that $f(K)$ is isotopic to K . By Proposition 3.3, f is isotopic to a diffeomorphism $\hat{g}$ of finite order. Then $\hat{g}$ preserves the orientation and acts trivially on $\pi_1(Y)$, because isotopic diffeomorphisms induce the same element of $\text{Out}(\pi_1(Y))$. By Theorem 4.3, $\hat{g} = \text{id}$. So f is isotopic to the identity, a contradiction. $\square$

Proof of Theorem B. The manifold Y is not prime, since $Y_1, Y_2 \neq S^3$. The twist Φ preserves the orientation, and it acts trivially on $\pi_1(Y)$ by Lemma 4.1. If Φ is not isotopic to the identity, then Y is not a connected sum of lens spaces and copies of $S^1 \times S^2$, by Lemma 4.2. So Theorem 4.4 applies. $\square$

Remark 4.5. (1) By [CT26, Theorem 2.4 and its proof], the twist Φ of $Y_1 \# Y_2$ is not isotopic to the identity if and only if each of Y_1 and Y_2 has a prime summand which is neither a lens space nor $S^1 \times S^2$. Lemma 4.2 is the easy direction. (2) The isotopy classes of orientation-preserving diffeomorphisms which act trivially on $\pi_1(Y)$ are the products of twists about spheres; see [CT26, Section 1]. So if Y is as in Theorem 4.4, every hyperbolic knot in Y detects every non-trivial element of the twist group of Y . For a connected sum of lens spaces and copies of $S^1 \times S^2$ the argument gives nothing: every non-trivial element of its twist group contains a diffeomorphism of order two [CT26, Theorem 5.1]. (3) In Theorem B the knots must be hyperbolic. A knot which is disjoint from S , or meets S transversally in two points, is invariant under a suitable twist about S . By Theorem B, such knots are not hyperbolic if Φ is not isotopic to the identity.

Verification. The proofs above do not depend on computation. The following checks were made.

1. The statements quoted from [Kaw92], [Kaw93], [Hat83], [Cer59], [HW10] (fourth arXiv version), [CT26] (published version), [Thu80], [Mye82], [Hat00] and from the first, the third and the published version of [ABD⁺26] were read in the sources, and Problem 1.1 in the authors' preliminary version of [BKR26].
2. A program of the author, `lead_checks.py`, tests numerically the formulas in the proof of Lemma 3.2 and the local form of the circle actions in the proof of Lemma 4.2. It checks $(*)$ in exact arithmetic for all reduced words of length at most 7 in the two standard generators of the figure-eight knot group. These are sanity checks of formulas, not proofs.
3. A first independent verification run, AI-assisted, examined a working write-up of Proposition 3.3 and of Section 4, re-derived every step and found no mathematical error. It read Kawauchi's theorem in the original and confirmed that it gives Theorem A, a route which the write-up had only mentioned. It asked for Theorem A to be stated first, for an account of the proof of Etnyre and Margalit and for a check of the classical theorems against the sources. The text was then rewritten.

4. A second independent verification run, AI-assisted, examined the rewritten text line by line, including the parts that were new: Lemma 3.1, the argument that $C \cup K$ is a tubular neighbourhood, and Steps 3 to 5 of the proof of Proposition 3.3, which it compared with the statements of [HW10], [Hat83] and [Cer59]. It read the sources of item 1 again, repeated the literature search, and found no mathematical error. Its own program, `run2_checks.py`, tests in exact arithmetic the formulas of Lemma 3.2 for the symmetries of six lattices that preserve a primitive class up to sign, the collar formula of Step 4, and $(*)$ in the same group for all reduced words of length at most 10. At its request the account of the proof of Etnyre and Margalit was made more precise, Remark 4.5(2) was restricted to the manifolds of Theorem 4.4, and an appendix with a variant proof of Theorem 4.3 was removed, because the equivariant sphere theorem and the results of Smith theory on which it relied could not be read in the originals.

Scope and priority. The results concern closed, connected, orientable 3-manifolds in the smooth category. Problem 1.1 speaks of 3-manifolds without further specification; its remarks refer to [ABD⁺26], where the manifolds are closed and oriented. Manifolds with boundary and non-orientable manifolds are not treated. Theorem A answers Problem 1.1, and it proves Conjecture 1.8 of the first version of [ABD⁺26] for closed manifolds in the smooth category; that version works in the topological category, which we do not treat.

All versions of [ABD⁺26] since 2020 record that Etnyre and Margalit have a proof of the general statement; the published version of 2026 says that it has not yet appeared. We found no written proof, and this note supplies one. It may well coincide with their argument, which we do not know; we have not been in contact with them, and we claim no priority. The ingredients are known: Proposition 3.3 is the standard description of the symmetries of a hyperbolic knot, and Theorem B rests on the theorem of Chen and Tshishiku, quoted from the published version of [CT26].

We read the statement of Kawauchi’s theorem and the definitions, not its proof. We did not see [Wal68, Mos73, Pra73, Thu82, Mil62, Hir76]; they are quoted in their standard forms, and Waldhausen’s theorem in the form of [HW10, Proposition 2.1]. The formulation in Remark 3.4 is the one commonly quoted; we did not check it against [Wal68]. We searched arXiv, most recently on October 3, 2026, for joint work and recent papers of Etnyre and Margalit, for papers on twists about spheres, on equivalent and isotopic knots, on knots and mapping class groups of 3-manifolds, on hyperbolic knots without symmetries, and on the list [BKR26]. We searched Crossref, OpenAlex, OpenCitations, Semantic Scholar and zbMATH, in particular for papers that cite [ABD⁺26] or [CT26], and we ran web searches; an earlier search covered MathOverflow. We found no solution of Problem 1.1 and no statement of Theorem A. MathSciNet and Google Scholar were not searched. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [ABD⁺26] P. Aceto, C. Bregman, C. W. Davis, J. Park, and A. Ray. Isotopy and equivalence of knots in 3-manifolds. *Journal of the London Mathematical Society* (2), 113(6):Paper No. e70600, 2026. arXiv:2007.05796; first version July 2020. doi:10.1112/jlms.70600.
- [BKR26] R. İ. Baykur, R. C. Kirby, and D. Ruberman, editors. *K3: A New Problem List in Low-Dimensional Topology*, volume 295 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2026. doi:10.1090/surv/295.

- [Cer59] J. Cerf. Groupes d'automorphismes et groupes de difféomorphismes des variétés compactes de dimension 3. *Bulletin de la Société Mathématique de France*, 87:319–329, 1959. doi:10.24033/bsmf.1528.
- [CT26] L. Chen and B. Tshishiku. Nielsen realization for sphere twists on 3-manifolds. *Israel Journal of Mathematics*, 272(1):1–29, 2026. arXiv:2204.04820. doi:10.1007/s11856-025-2813-3.
- [Hat83] A. E. Hatcher. A proof of the Smale conjecture, $\text{Diff}(S^3) \simeq \text{O}(4)$. *Annals of Mathematics* (2), 117(3):553–607, 1983. doi:10.2307/2007035.
- [Hat00] A. Hatcher. Notes on basic 3-manifold topology. Lecture notes, <https://pi.math.cornell.edu/~hatcher/3M/3Mdownloads.html>, 2000.
- [Hir76] M. W. Hirsch. *Differential Topology*, volume 33 of *Graduate Texts in Mathematics*. Springer, New York, 1976. doi:10.1007/978-1-4684-9449-5.
- [HW10] A. Hatcher and N. Wahl. Stabilization for mapping class groups of 3-manifolds. *Duke Mathematical Journal*, 155(2):205–269, 2010. arXiv:0709.2173. doi:10.1215/00127094-2010-055.
- [Kaw92] A. Kawauchi. Almost identical imitations of (3,1)-dimensional manifold pairs and the branched coverings. *Osaka Journal of Mathematics*, 29(2):299–327, 1992. doi:10.18910/4930.
- [Kaw93] A. Kawauchi. Almost identical imitations of (3,1)-dimensional manifold pairs and the manifold mutation. *Journal of the Australian Mathematical Society (Series A)*, 55(1):100–115, 1993. doi:10.1017/S1446788700031955.
- [Mil62] J. Milnor. A unique decomposition theorem for 3-manifolds. *American Journal of Mathematics*, 84(1):1–7, 1962. doi:10.2307/2372800.
- [Mos73] G. D. Mostow. *Strong Rigidity of Locally Symmetric Spaces*, volume 78 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1973. doi:10.1515/9781400881833.
- [Mye82] R. Myers. Simple knots in compact, orientable 3-manifolds. *Transactions of the American Mathematical Society*, 273(1):75–91, 1982. doi:10.1090/S0002-9947-1982-0664030-0.
- [Pra73] G. Prasad. Strong rigidity of $\mathbf{Q}$ -rank 1 lattices. *Inventiones Mathematicae*, 21(4):255–286, 1973. doi:10.1007/BF01418789.
- [Thu80] W. P. Thurston. The geometry and topology of three-manifolds. Lecture notes, Princeton University; electronic version 1.1, MSRI, 2002, <https://library.slmath.org/books/gt3m/>, 1980.
- [Thu82] W. P. Thurston. Three dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bulletin of the American Mathematical Society (N.S.)*, 6(3):357–381, 1982. doi:10.1090/S0273-0979-1982-15003-0.
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Version 1.6.0; record KP-1.19.
- [Wal68] F. Waldhausen. On irreducible 3-manifolds which are sufficiently large. *Annals of Mathematics* (2), 87(1):56–88, 1968. doi:10.2307/1970594.