

Regularity of the Sylow p -Subgroups of $GL_n(\mathbb{Z}/p^m)$ at the Class- p Boundary: New Cases of a Question of Wehrfritz

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Abstract

In Problem 8.3 of the Kourovka Notebook (1982), B. A. F. Wehrfritz asked for which m and n the group $P_n(\mathbb{Z}/p^m)$ of all $n \times n$ matrices over $\mathbb{Z}/p^m$ with diagonal entries $\equiv 1$ and entries below the diagonal $\equiv 0 \pmod{p}$, a Sylow p -subgroup of $GL_n(\mathbb{Z}/p^m)$, is regular. It is known to be regular when $nm - 1 < p$ (by Merzlyakov's bound for the class) and when $n^2 < p$ (Kolesnikov), and the answer is complete for $m \leq 2$ (Yagzhev for $m = 1$; Kolesnikov, and Egorychev, Kolesnikov and Leontiev, for $m = 2$) and for $m = 3$ and $p \equiv 1 \pmod{3}$, where Kolesnikov and Leontiev proved irregularity for $n \geq (p + 2)/3$. Our result is a partial answer: $P_n(\mathbb{Z}/p^m)$ is regular in the boundary cases $nm - 1 = p$ for $(p, n, m) = (11, 4, 3), (17, 6, 3), (23, 8, 3), (29, 10, 3), (19, 5, 4), (23, 6, 4), (29, 6, 5)$, and $P_n(\mathbb{Z}/p^3)$ is irregular for $n = (p + 4)/3$ and $p \in \{11, 17, 23, 29\}$; hence for every prime $p \leq 29$, $P_n(\mathbb{Z}/p^3)$ is regular iff $3n - 1 \leq p$, and for $p = 11$ the answer is complete except for $n = 4, m \geq 4$; the question stays open in general. In particular, for $m = 3, 4, 5$ the group can be regular although neither of the known sufficient conditions $nm - 1 < p$ and $n^2 < p$ holds, whereas for $m \leq 2$ the condition $nm - 1 < p$ is necessary. All new cases rest on computer-assisted proofs. The irregularity witnesses are the pair of matrices used by Kolesnikov and Leontiev, for which we compute the agemo subgroup $\mathcal{O}_1(\langle a, b \rangle')$ exactly. For regularity, the Hall–Petresco identity and the Magnus embedding reduce the question, at the boundary $nm - 1 = p$, to a finite computation in linear algebra over $\mathbb{F}_p$ on the two-dimensional subspaces of an n -dimensional space, shortened by a torus symmetry. For $m = 3$ we prove by hand that the boundary case $n = (p + 1)/3$ is regular whenever $S = e_{21} + e_{32} + \cdots + e_{n, n-1}$ and any other element of the first subdiagonal, not a multiple of S , generate a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{F}_p)$ that contains e_{n1} ; we verify this condition by computer for $n = 4, 6, 8, 10$, which gives the new cases $(23, 8, 3)$ and $(29, 10, 3)$, and show that it fails for every odd $n \geq 5$. Three conjectures suggested by the computations are stated. No printed decision of the new cases was found in the sources searched (zbMATH, arXiv, Crossref, OpenAlex, Semantic Scholar, web searches); a search that finds nothing is not a proof of novelty. This is an unrefereed note.

1 Introduction

1.1 The question

Let p be a prime and $m, n \geq 1$. Problem 8.3 of the Kourovka Notebook [KM26], posed by B. A. F. Wehrfritz in the 8th issue (1982), concerns the group $P_n(\mathbb{Z}/p^m)$ of all $n \times n$ matrices (a_{ij}) over $\mathbb{Z}/p^m$ with $a_{ii} \equiv 1 \pmod{p}$ for all i and $a_{ij} \equiv 0 \pmod{p}$ for all $i > j$. It is the full preimage of the upper unitriangular group over $\mathbb{F}_p$, hence a Sylow p -subgroup of $GL_n(\mathbb{Z}/p^m)$, of order $p^{(m-1)n^2 + n(n-1)/2}$. The question asks for which m and n it is regular. We write $P_n(\mathbb{Z}/p^m)$ for this group. The entry reads

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the same in versions 48 and 49 of [KM26] (6 and 20 October 2026); it is not marked as solved, and its editors' comments (2001 and 2009) record the results of [Mer64, Yag94, Kol01, Kol06] described below.

Following P. Hall [Hal34], a finite p -group K is *regular* if for all $a, b \in K$ there are $s_1, \dots, s_r$ in the derived subgroup $\langle a, b \rangle'$ such that $(ab)^p = a^p b^p s_1^p \cdots s_r^p$; Hall showed that the same then holds with p replaced by any power of p , and [KL22] uses that form as the definition. With the agemo subgroup $\mathcal{U}_1(L) = \langle x^p : x \in L \rangle$ the condition reads

$$Z(a, b) := (a^p b^p)^{-1} (ab)^p \in \mathcal{U}_1(\langle a, b \rangle') \quad \text{for all } a, b \in K, \quad (1)$$

since the products of p -th powers of elements of L form the subgroup $\mathcal{U}_1(L)$. Subgroups and quotients of regular p -groups are regular, and p -groups of class less than p are regular [Hal34]; see also [Hup67, Chapter III].

1.2 What was known

- (K1) *Class.* For $n \geq 2$ and $p > 2$, $P_n(\mathbb{Z}/p^m)$ has nilpotency class $nm - 1$ [Mer64] (as quoted in [KL22, Kol13]; for Sylow subgroups of Chevalley groups see [Lev92]); the filtration of Section 2 shows that the class is at most $nm - 1$. Hence the group is regular if $nm - 1 < p$. (The Notebook states this as $mn < p$; the two conditions differ only when $mn = p$ and $n = 1$ or $m = 1$, where the group is abelian or has class $p - 1$.)
- (K2) $m = 1$. $P_n(\mathbb{F}_p)$ is regular if and only if $n \leq p$ [Yag94].
- (K3) $m = 2$. $P_n(\mathbb{Z}/p^2)$ is regular if and only if $2n - 1 < p$: the Notebook attributes the completion of the case $m = 2$ to Kolesnikov [Kol01], and Egorychev, Kolesnikov and Leontiev proved, for the Sylow p -subgroups of the elementary Chevalley groups of types A_n, B_n, C_n, D_n over $\mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$, that regularity holds if and only if $hm - 1 < p$, h the Coxeter number [EKL23, Theorem 1] (we know this statement from the zbMATH review of [EKL23] only; see “Conclusion, scope and priority” in Section 8). For type A_{n-1} , $h = n$ and the group is a Sylow p -subgroup of $SL_n(\mathbb{Z}/p^m)$, which we may take to be $P_n(\mathbb{Z}/p^m) \cap SL_n$; so its irregularity for $2n - 1 \geq p$ gives that of $P_n(\mathbb{Z}/p^2)$, and (K1) gives the converse. Since $P_n(\mathbb{Z}/p^m)$ maps onto $P_n(\mathbb{Z}/p^2)$, $P_n(\mathbb{Z}/p^m)$ is irregular for $n \geq (p + 1)/2$ and every $m \geq 2$; by (K2) this fails for $m = 1$.
- (K4) $n^2 < p$. $P_n(\mathbb{Z}/p^m)$ is regular for every m [Kol06].
- (K5) $p \equiv 1 \pmod{3}$. $P_n(\mathbb{Z}/p^m)$ is irregular for $n \geq (p + 2)/3$ and $m \geq 3$ [KL22, Theorem 2]. Kolesnikov and Leontiev derive a necessary condition for regularity from P. Hall's collection formula [KL22, Theorem 1, Corollary 1] (using [KLE20]) and evaluate it on the pair $E + A$, $E + pB$ with $A = e_{21} + e_{32} + \cdots + e_{n,n-1}$ and $B = e_{1n}$ [KL22, Section 4]. With (K1), the case $m = 3$ is thus settled for $p \equiv 1 \pmod{3}$: regular if and only if $3n - 1 < p$.
- (K6) *Necessary conditions.* For $m \geq 2$, $p \geq 5$ and $2 \leq n \leq (p - 1)/2$, Kolesnikov showed that $P_n(\mathbb{Z}/p^m)$ satisfies the classical necessary conditions of regularity [Kol13, Theorem 1 and Section 3]; they do not decide regularity, and (K5) shows irregularity in part of this range.

The Notebook states that [Yag94] is erroneous for $m > 1$. Its list for $m > 1$, as given in the zbMATH review (Zbl 0837.20027), would make $P_n(\mathbb{Z}/p^3)$ regular for $p > 7$ only when $n \leq 2$, which contradicts (K4) for $(p, n) = (11, 3)$ and also Theorem A; we use [Yag94] only for $m = 1$. zbMATH also lists a letter to the editor concerning [Yag94] (Mat. Zametki 60:4 (1996), 482), which we have not seen. Sylow subgroups of Chevalley groups of other types are also treated in [Kol06, KM11, KL22, EKL23, KP25]; of these papers we use only the results for type A quoted above.

Before this note the smallest open cases were $p = 11$ with $n \in \{4, 5\}$, $m \geq 3$, and $p = 13$ with $n = 4$, $m \geq 4$; for $p \leq 7$ the answer follows from (K1)–(K5).

1.3 Results

Theorem A (regularity at the boundary $nm - 1 = p$). *For each of the triples $(p, n, m) = (11, 4, 3)$, $(17, 6, 3)$, $(23, 8, 3)$, $(29, 10, 3)$, $(19, 5, 4)$, $(23, 6, 4)$, $(29, 6, 5)$, the group $P_n(\mathbb{Z}/p^m)$ is regular. Consequently $P_{n'}(\mathbb{Z}/p^{m'})$ is regular for all $n' \leq n$ and $m' \leq m$.*

In these seven cases $nm - 1 = p$, so the class is p by (K1), and $n^2 > p$, so (K4) does not apply. For $m = 3$, Theorem A follows from a reduction proved by hand and a computer check of its hypothesis. Let $S = e_{21} + e_{32} + \dots + e_{n,n-1}$ in $\mathfrak{gl}_n(\mathbb{F}_p)$. We say that $(L1_n)$ holds over $\mathbb{F}_p$ if:

for every $Y \in \text{span}(e_{21}, e_{32}, \dots, e_{n,n-1})$ that is not a scalar multiple of S , the Lie subalgebra of $\mathfrak{gl}_n(\mathbb{F}_p)$ generated by S and Y contains e_{n1} .

Theorem A' (the boundary for $m = 3$). *Let $p \geq 5$ be a prime with $p \equiv 2 \pmod{3}$, and $n = (p+1)/3$. If $(L1_n)$ holds over $\mathbb{F}_p$, then $P_n(\mathbb{Z}/p^3)$ is regular.*

Condition $(L1_2)$ is vacuous, and $(L1_n)$ holds over $\mathbb{F}_p$ for $(n, p) = (4, 11)$, $(6, 17)$, $(8, 23)$ and $(10, 29)$ (Proposition 6.6; a computer check after two reductions proved by hand). This proves the cases $m = 3$ of Theorem A; for $(11, 4, 3)$ and $(17, 6, 3)$ the finite check of Section 5 gives a second, independent proof. In Theorem A' the number $n = (p+1)/3$ is even; for odd $n \geq 5$, $(L1_n)$ fails (Remark 6.3).

Theorem B (irregularity). *Let $p \in \{11, 17, 23, 29\}$ and $n = (p+4)/3$, that is $n \in \{5, 7, 9, 11\}$. Then $P_n(\mathbb{Z}/p^3)$ is irregular. Consequently $P_{n'}(\mathbb{Z}/p^{m'})$ is irregular for all $n' \geq (p+4)/3$ and $m' \geq 3$.*

The witness is the pair of [KL22] described in (K5), transported to the group of the question (Section 4). Since $(p+4)/3 \leq (p-1)/2$, these groups have $P_n(\mathbb{Z}/p^2)$ regular, so Theorem B does not follow from (K3); and $p \equiv 2 \pmod{3}$, so (K5) does not apply.

Corollary 1 ($m = 3$). *For $p \in \{11, 17, 23, 29\}$, $P_n(\mathbb{Z}/p^3)$ is regular if and only if $3n - 1 \leq p$. Hence, for every prime $p \leq 29$, $P_n(\mathbb{Z}/p^3)$ is regular if and only if $3n - 1 \leq p$.*

The second sentence combines the first with the known results (K1)–(K5); the sources are listed prime by prime in Section 7. For $p \in \{11, 17, 23, 29\}$ both directions rest on the computer-assisted Theorems A and B, which are unrefereed.

Corollary 2 ($p = 11$). *Let $n, m \geq 1$ with $n \neq 4$ or $m \leq 3$. Then $P_n(\mathbb{Z}/11^m)$ is regular if and only if $m = 1$ and $n \leq 11$, or $m = 2$ and $n \leq 5$, or $m = 3$ and $n \leq 4$, or $m \geq 4$ and $n \leq 3$. The cases $n = 4$, $m \geq 4$ remain open.*

Corollary 3. *For $p \in \{17, 23, 29\}$, $P_n(\mathbb{Z}/p^m)$ is irregular whenever $(p+4)/3 \leq n < (p+1)/2$ and $m \geq 3$, that is for $n \in \{7, 8\}$, $\{9, 10, 11\}$ and $\{11, \dots, 14\}$ respectively.*

In the terms of the question, our result is a partial answer: $P_n(\mathbb{Z}/p^m)$ is regular in the boundary cases $nm - 1 = p$ for $(p, n, m) = (11, 4, 3)$, $(17, 6, 3)$, $(23, 8, 3)$, $(29, 10, 3)$, $(19, 5, 4)$, $(23, 6, 4)$, $(29, 6, 5)$, and $P_n(\mathbb{Z}/p^3)$ is irregular for $n = (p+4)/3$ and $p \in \{11, 17, 23, 29\}$; hence for every prime $p \leq 29$, $P_n(\mathbb{Z}/p^3)$ is regular iff $3n - 1 \leq p$, and for $p = 11$ the answer is complete except for $n = 4$, $m \geq 4$; the question stays open in general. In particular $P_4(\mathbb{Z}/11^3)$ is regular although $nm - 1 = p$ and $n^2 > p$: for $m = 3$, and by the other cases of Theorem A also for $m = 4$ and $m = 5$, the group can be regular although neither of the known sufficient conditions $nm - 1 < p$ (K1) and $n^2 < p$ (K4) holds, whereas for $m \leq 2$ the condition $nm - 1 < p$ is also necessary, by (K2) and (K3). (That $nm - 1 < p$ alone

is not necessary for $m \geq 3$ follows already from (K4): $P_2(\mathbb{Z}/5^m)$ is regular for every m .) The data suggest three conjectures, stated in Section 7.1: for $m = 3$, regularity holds if and only if $3n - 1 \leq p$ (now proved for $p \leq 29$; open for $p \equiv 2 \pmod{3}$, $p \geq 41$); $nm - 1 = p$ with $m \geq 3$ implies regularity; and (L1_n) holds over $\mathbb{F}_p$ for every even n and every prime $p > 2n$, which by Theorem A' would give the regular half of the first.

1.4 What is new, what is not, and what stays open

Theorems A and B are computer-assisted. Theorem B is proved by an exact computation of $\mathcal{U}_1(\langle a, b \rangle')$ for the pair of Kolesnikov and Leontiev (Section 4); this is a direct computation, not an application of their necessary condition [KL22, Corollary 1]. For Theorem A (Section 5), the Hall–Petresco identity reduces regularity to the last factor $c_p(a, b)$, which at the boundary $nm - 1 = p$ is a power of one central element given by a polynomial in the leading terms of a and b , and a graded lower bound for $\mathcal{U}_1(\langle a, b \rangle')$ depends only on the span of these leading terms. The computer part of Theorem A is finite linear algebra over $\mathbb{F}_p$; no computation in the group enters its proof. For $m = 3$, Theorem A' (Section 6) replaces the enumeration of subspaces by a proof by hand that needs only (L1_n), a statement about the Lie algebra of strictly lower triangular matrices over $\mathbb{F}_p$; (L1_n) is checked by computer for $n = 4, 6, 8, 10$, after two reductions proved by hand. The ingredients are classical, and we claim no novelty for them. No printed decision of the new cases was found in the sources we searched (listed in “Conclusion, scope and priority”, Section 8); a search that finds nothing is not a proof of novelty, and we claim no priority. What stays open is described in Section 7.3.

2 The setting

Throughout, $R = \mathbb{Z}/p^m$, E is the identity matrix and e_{ij} are the matrix units. For $0 \neq x \in R$ let $v_p(x)$ be the largest $k \leq m - 1$ with $p^k \mid x$.

2.1 The group

Let G be the group of all $g \in M_n(R)$ with $g_{ii} \equiv 1 \pmod{p}$ for all i and $g_{ij} \equiv 0 \pmod{p}$ for all $i < j$. The map $g \mapsto (g^T)^{-1}$ is an automorphism of $GL_n(R)$ (a composition of two anti-automorphisms) and maps $P_n(\mathbb{Z}/p^m)$ onto G . So $P_n(\mathbb{Z}/p^m) \cong G$, and we work in G , as [KL22] do; the pair of (K5) lies in G :

$$a = E + e_{21} + e_{32} + \cdots + e_{n,n-1}, \quad b = E + pe_{1n}. \quad (2)$$

In $P_n(\mathbb{Z}/p^m)$ it corresponds to the pair $(a^T)^{-1}, (b^T)^{-1}$.

Lemma 2.1. *$P_n(\mathbb{Z}/p^m)$ is isomorphic to a subgroup of $P_{n+1}(\mathbb{Z}/p^m)$ and to a quotient of $P_n(\mathbb{Z}/p^{m+1})$. Hence, if $P_n(\mathbb{Z}/p^m)$ is regular, so is $P_{n'}(\mathbb{Z}/p^{m'})$ for $n' \leq n$, $m' \leq m$; if it is irregular, so is $P_{n'}(\mathbb{Z}/p^{m'})$ for $n' \geq n$, $m' \geq m$.*

Proof. $g \mapsto \text{diag}(g, 1)$ is an embedding. Reduction modulo p^m maps $P_n(\mathbb{Z}/p^{m+1})$ onto $P_n(\mathbb{Z}/p^m)$: a matrix satisfying the congruences lifts entrywise to one satisfying them over $\mathbb{Z}/p^{m+1}$, which is invertible since its determinant is $\equiv 1 \pmod{p}$. Subgroups and quotients of regular p -groups are regular. $\square$

2.2 The weight filtration

For $X = (x_{ij}) \in M_n(R)$ put $w(X) = \min\{nv_p(x_{ij}) + i - j : x_{ij} \neq 0\}$ ($w(0) = \infty$) and $M_w = \{X : w(X) \geq w\}$. Then $w(p^k e_{ij}) = kn + i - j$ for $0 \leq k \leq m - 1$, and, since $(nv + i - l) + (nv' + l - j) =$

$$n(v + v') + i - j,$$

$$w(X + Y) \geq \min(w(X), w(Y)), \quad w(XY) \geq w(X) + w(Y), \quad w(pX) \geq w(X) + n. \quad (3)$$

So M_0 (the matrices with $x_{ij} \equiv 0 \pmod{p}$ for $i < j$) is a ring, each M_w is a two-sided ideal of M_0 , $G = E + M_1$, and $M_{mn} = 0$ (if $nv_p(x) + i - j \geq mn$ then $v_p(x) \geq m$, as $i - j \leq n - 1$). Let $G_w = E + M_w$ for $w \geq 1$.

Lemma 2.2. (a) Each G_w is a normal subgroup of G , $G_1 = G$, $G_{mn} = 1$ and $[G_v, G_w] \leq G_{v+w}$. (b) For $g = E + X \in G_v$ and $h = E + Y \in G_w$, $[g, h] = g^{-1}h^{-1}gh \in E + (XY - YX) + M_{v+w+1}$. (c) The map $E + X \mapsto X + M_{w+1}$ induces an isomorphism of $L_w = G_w/G_{w+1}$ onto the $\mathbb{F}_p$ -vector space $A_w = M_w/M_{w+1}$, which has the basis $\varepsilon(i, j, k) = p^k e_{ij} + M_{w+1}$ over the triples with $kn + i - j = w$ and $0 \leq k \leq m - 1$; the coordinate of X at $\varepsilon(i, j, k)$ is the digit of x_{ij} at p^k . For each w and each row i there is at most one such triple.

Proof. (a) For $X \in M_w$ we have $X^k \in M_{kw} = 0$ for $k \geq mn$, so $E + X$ has the inverse $\sum_{k < mn} (-X)^k \in E + M_w$, and $(E + X)(E + Y) = E + X + Y + XY$; thus G_w is a group, normal in G because M_w is an ideal of $M_0 \supseteq G$. (b) Since $gh - hg = XY - YX$, we have $[g, h] = E + g^{-1}h^{-1}(XY - YX)$ with $g^{-1}h^{-1} \in E + M_1$; this gives (b) and the last part of (a). (c) The map is a homomorphism because $XY \in M_{2w} \subseteq M_{w+1}$, its kernel is G_{w+1} , and A_w is an $\mathbb{F}_p$ -space because $pM_w \subseteq M_{w+n}$. An entry x_{ij} of $X \in M_w$ contributes to $X + M_{w+1}$ only through its digit at p^k with $kn + i - j = w$; for given w and i , the conditions $j \equiv i - w \pmod{n}$ and $1 \leq j \leq n$ determine j , and then k . $\square$

The graded algebra. Matrix multiplication induces bilinear maps $A_v \times A_w \rightarrow A_{v+w}$ which make $A = \bigoplus_{w=1}^{mn-1} A_w$ an associative graded $\mathbb{F}_p$ -algebra, with

$$\varepsilon(i, j, k) \varepsilon(j', l, k') = \delta_{jj'} \varepsilon(i, l, k + k') \text{ if } k + k' \leq m - 1, \text{ and } 0 \text{ otherwise.} \quad (4)$$

By Lemma 2.2, the Lie ring $\bigoplus_w L_w$ of the filtration (with the bracket induced by commutators) is A with the bracket $[x, y] = xy - yx$. For $g \in G_w$ we call the image of g in A_w its *leading term* in A_w (it is 0 if and only if $g \in G_{w+1}$). For a subgroup $K \leq G$ let $\text{gr}_w(K) \subseteq A_w$ be the image of $K \cap G_w$, a subspace, and $\text{gr}(K) = \bigoplus_w \text{gr}_w(K)$.

Slots and walks. A_1 has the basis $u_s = \varepsilon(s + 1, s, 0)$ ($1 \leq s \leq n - 1$) and $u_n = \varepsilon(1, n, 1)$, the images of $e_{s+1, s}$ and of pe_{1n} ; we write $z = \sum_s z_s u_s$ for $z \in A_1$. By (4), a product of slots $u_{s_1} u_{s_2} \cdots u_{s_w}$ can be nonzero only if each index is one less than the previous one modulo n , i.e. if it follows the walk on the rows $i \rightarrow i - 1 \rightarrow \cdots \rightarrow 1 \rightarrow n \rightarrow n - 1 \rightarrow \cdots$ starting at the row i of u_{s_1} ; it is then $\varepsilon(i, j, k)$ with $j \equiv i - w \pmod{n}$, where k is the number of factors u_n , if $k \leq m - 1$, and 0 otherwise. Every basis element $\varepsilon(i, j, k)$ of A_w is the product of the w slots of its walk from row i ; in particular A is generated by A_1 .

The top layer. The largest weight is $mn - 1$, attained only by $\varepsilon(n, 1, m - 1)$. Hence $G_{mn-1} = \langle t \rangle$ with $t = E + p^{m-1} e_{n1}$, a central subgroup of order p (by Lemma 2.2(a), $[t, G] \leq G_{mn} = 1$). We write $\bar{t} = \varepsilon(n, 1, m - 1)$ for its leading term.

Lemma 2.3 (p -th powers). *Let $w \geq 1$ with $(p - 1)w \geq n + 1$, and let $g = E + X \in G_w$. Then $g^p \in E + pX + M_{w+n+1}$. Hence $g^p \in G_{w+n}$, and its leading term in A_{w+n} is $\pi(x)$, where x is the leading term of g in A_w and $\pi : A_w \rightarrow A_{w+n}$ is the linear map $\varepsilon(i, j, k) \mapsto \varepsilon(i, j, k + 1)$ (read as 0 if $k = m - 1$). The map π is multiplication by the central element $\bar{p} = \sum_{i=1}^n \varepsilon(i, i, 1)$ of A . If $n \leq p - 2$, the hypothesis holds for every $w \geq 1$.*

Proof. $(E + X)^p = E + pX + \sum_{k=2}^{p-1} \binom{p}{k} X^k + X^p$. Since p divides $\binom{p}{k}$ for $1 \leq k \leq p-1$, (3) gives $w(\binom{p}{k} X^k) \geq n + kw \geq w + n + 1$ for $k \geq 2$, and $w(X^p) \geq pw \geq w + n + 1$. The digit of px_{ij} at p^{k+1} is the digit of x_{ij} at p^k , whence π ; and $\varepsilon(i, i, 1)\varepsilon(i, j, k) = \varepsilon(i, j, k+1) = \varepsilon(i, j, k)\varepsilon(j, j, 1)$ by (4). If $n \leq p-2$ then $(p-1)w \geq p-1 \geq n+1$. $\square$

3 Regularity, the Hall–Petresco identity and the agemo subgroup

Let F be the free group on X, Y . By the Hall–Petresco identity [Hal34, Pet54] (see also [Hup67, Chapter III]) there are $c_2, \dots, c_p \in F$ with $c_i \in \gamma_i(F)$ such that

$$X^p Y^p = (XY)^p c_2^{\binom{p}{2}} c_3^{\binom{p}{3}} \cdots c_{p-1}^{\binom{p}{p-1}} c_p. \quad (5)$$

For elements a, b of a group, $c_i(a, b)$ denotes the image of c_i under $X \mapsto a, Y \mapsto b$.

Lemma 3.1. *Let K be a finite p -group, $a, b \in K$ and $H = \langle a, b \rangle$. Then $Z(a, b) \in \mathcal{U}_1(H')$ if and only if $c_p(a, b) \in \mathcal{U}_1(H')$. Hence K is regular if and only if $c_p(a, b) \in \mathcal{U}_1(\langle a, b \rangle')$ for all $a, b \in K$.*

Proof. By (5), $Z(a, b)^{-1} = (ab)^{-p} a^p b^p = q c_p(a, b)$ with $q = \prod_{i=2}^{p-1} c_i(a, b)^{\binom{p}{i}}$. For $2 \leq i \leq p-1$, p divides $\binom{p}{i}$ and $c_i(a, b) \in \gamma_i(H) \leq H'$, so $c_i(a, b)^{\binom{p}{i}} = (c_i(a, b)^{\binom{p}{i}/p})^p \in \mathcal{U}_1(H')$. Thus $q \in \mathcal{U}_1(H')$, and the lemma follows with (1). $\square$

Lemma 3.2 (agemo recursion). *Let K be a finite p -group with $\gamma_p(K) = 1$, generated by a set S . Then $\mathcal{U}_1(K) = \langle s^p : s \in S \rangle \mathcal{U}_1(K')$.*

Proof. $N = \mathcal{U}_1(K')$ is characteristic in $K' \trianglelefteq K$, hence normal in K , so $M = \langle s^p : s \in S \rangle N$ is a subgroup, and $M \leq \mathcal{U}_1(K)$. For $x, y \in K$, (5) gives $x^p y^p = (xy)^p q c_p(x, y)$ with $q \in N$ as in the proof of Lemma 3.1 and $c_p(x, y) \in \gamma_p(K) = 1$; so $(xy)^p \in x^p y^p N$. By induction on the length of a word in $S \cup S^{-1}$ representing $g \in K$ we get $g^p \in M$: if $g = g' s^{\pm 1}$ then $g^p \in g'^p (s^{\pm 1})^p N \subseteq M$. $\square$

Lemma 3.3 (agemo of a filtered generating set). *Let $K \leq G_{w_0}$ be a subgroup of G with $pw_0 \geq mn$, and let $B \subseteq K$ be a finite set such that $K \cap G_v = \langle B \cap G_v \rangle$ for every $v \geq w_0$. Then $\mathcal{U}_1(K) = \langle b^p : b \in B \rangle$.*

Proof. Let $N = \langle b^p : b \in B \rangle \leq \mathcal{U}_1(K)$ and $K_v = K \cap G_v$. We show $\mathcal{U}_1(K_v) \leq N$ for all $v \geq w_0$ by downward induction on v ; for $v \geq mn$, $K_v = 1$. Let $w_0 \leq v < mn$. By Lemma 2.2(a), $\gamma_p(K_v) \leq G_{pv} = 1$ and $K'_v \leq K \cap G_{2v} = K_{2v}$; so Lemma 3.2 with $S = B \cap G_v$ gives $\mathcal{U}_1(K_v) = \langle (B \cap G_v)^p \rangle \mathcal{U}_1(K'_v) \leq N \mathcal{U}_1(K_{2v}) \leq N$, since $2v > v$. For $v = w_0$ this is the claim. $\square$

Sifting. Let $B = (b_1, \dots, b_r)$ be a list of elements of G with $b_i \in G_{w_i}$ and nonzero leading term $\beta_i \in A_{w_i}$, such that the β_i of the same weight have pairwise distinct pivots (the first nonzero coordinate, in the basis of Lemma 2.2(c) ordered by rows), with coefficient 1 there. To sift $h \in G$: while $h \neq E$, let w be the weight of $h - E$ and, for the b_i of weight w in the order of increasing pivots, replace h by hb_i^{-c} , where c is the current coordinate at the pivot of β_i of the leading term of h in A_w ; if the leading term in A_w is then still nonzero, stop and return it as the *residue*, otherwise continue. Since the leading term changes linearly and each β_i vanishes before its pivot, the residue is nonzero exactly when the leading term is not in the span of the β_i of weight w , and then it has zero coordinates at all their pivots.

Lemma 3.4 (sifting). *Suppose that every b_i^p and every commutator $[b_i, b_j]$ ($i < j$) sifts to E . Then the elements of G that sift to E form the subgroup $\langle B \rangle$, $\langle B \rangle \cap G_v = \langle b_i : w_i \geq v \rangle$ for every v , $|\langle B \rangle| = p^r$, and $\text{gr}_w(\langle B \rangle)$ is spanned by the β_i with $w_i = w$.*

Proof. Let S_v be the set of elements of G_v that sift to E and $N_v = \langle b_i : w_i \geq v \rangle$. Sifting an element of G_v uses only the b_i with $w_i \geq v$, so $S_v \subseteq N_v$. We show by downward induction on v that $S_v = N_v$ and $N_v \cap G_{v+1} = N_{v+1}$; for $v \geq mn$ all these sets are trivial. Assume both statements for $v+1$, and let $b_{i_1}, \dots, b_{i_k}$ be the elements of weight v . For b_i of weight v and b_j of weight $> v$, the commutator of the two in the order of the list lies in G_{v+1} and sifts to E , so it lies in $S_{v+1} = N_{v+1}$, and so does its inverse; hence $b_j^{b_i} = b_j[b_j, b_i] \in N_{v+1}$, conjugation by b_i maps the finite group N_{v+1} onto itself, and $N_{v+1} \trianglelefteq N_v$. In the same way the b_{i_l} commute modulo N_{v+1} (their commutators lie in G_{2v}) and $b_{i_l}^p \in G_{v+1}$ lies in N_{v+1} , so $|N_v/N_{v+1}| \leq p^k$. The leading-term map $N_v \rightarrow A_v$ is a homomorphism with image the span of the linearly independent $\beta_{i_1}, \dots, \beta_{i_k}$ and kernel $N_v \cap G_{v+1} \supseteq N_{v+1}$; so $p^k \leq |N_v/N_{v+1}| \leq p^k$, whence $N_v \cap G_{v+1} = N_{v+1}$ and $\text{gr}_v(N_v) = \text{span}(\beta_{i_1}, \dots, \beta_{i_k})$. Finally let $h \in N_v$. If $h \in G_{v+1}$, then $h \in N_{v+1} = S_{v+1}$. Otherwise its leading term lies in this span, and the pass at weight v produces an element of $N_v \cap G_{v+1} = N_{v+1} = S_{v+1}$; so h sifts to E , and $N_v \subseteq S_v$. The statements for $v = 1$ give the lemma. $\square$

The programs build such lists by sifting the generators of a subgroup and appending the residues (normalised so that the pivot coefficient is 1), together with the p -th powers and the commutators of the new elements with the old ones (and with given elements, for normal closures). An element that sifts to E still does so after the list is extended, because at each weight the coefficients are determined by the span; so the final list satisfies the hypothesis of Lemma 3.4. The check program of this note verifies this hypothesis again for every final list, after the construction (an end-of-run certificate).

4 Irregularity: proof of Theorem B

Let $p \in \{11, 17, 23, 29\}$, $n = (p+4)/3$ and $m = 3$; let $a, b \in G$ be the pair (2) and $H = \langle a, b \rangle$. By Lemma 2.1 it suffices to show that $Z(a, b) \notin \mathcal{U}_1(H')$. The program `check_note.py` (Part A) computes two lists B_1 and B_2 as in Section 3, by exact arithmetic in $GL_n(\mathbb{Z}/p^3)$, and verifies, for the final lists:

- (i) B_1 satisfies the hypothesis of Lemma 3.4; $[a, b]$ sifts to E with respect to B_1 ; and so do $[y, a]$ and $[y, b]$ for every $y \in B_1$;
- (ii) every element of B_1 lies in G_2 ;
- (iii) B_2 satisfies the hypothesis of Lemma 3.4, and y^p sifts to E with respect to B_2 for every $y \in B_1$;
- (iv) $Z(a, b)$ does not sift to E with respect to B_2 : the sifting stops at weight p with the residue given in Table 1.

By construction, the elements of B_1 are obtained from $[a, b]$ by products, powers and commutators with a and b , so $B_1 \subseteq H'$, and the elements of B_2 are obtained from the y^p , $y \in B_1$.

Proof of Theorem B. By (i) and Lemma 3.4, $\langle B_1 \rangle$ is a subgroup that contains $[a, b]$ and is normalised by a and b ; hence it contains the normal closure of $[a, b]$ in H , which is H' . So $\langle B_1 \rangle = H'$. By (ii), $H' \leq G_2$, and $2p \geq 3n = p+4$; by Lemma 3.4, $H' \cap G_v = \langle B_1 \cap G_v \rangle$ for all v . Thus Lemma 3.3 (with $w_0 = 2$) gives $\mathcal{U}_1(H') = \langle y^p : y \in B_1 \rangle$, and by (iii) and Lemma 3.4 this is $\langle B_2 \rangle$, the set of elements that sift to E with respect to B_2 . By (iv), $Z(a, b) \notin \mathcal{U}_1(H')$, and $G \cong P_n(\mathbb{Z}/p^3)$ is irregular. The rest follows from Lemma 2.1. $\square$

Only the verified properties (i)–(iv) enter the conclusion $Z(a, b) \notin \mathcal{U}_1(H')$: the subgroup $K = \langle B_1 \rangle$ contains H' by (i), and $\mathcal{U}_1(K) = \langle y^p : y \in B_1 \rangle \leq \langle B_2 \rangle$ by (ii), (iii) and Lemmas 3.3 and 3.4, so $\mathcal{U}_1(H') \leq \langle B_2 \rangle$, which does not contain $Z(a, b)$ by (iv). The inclusions $B_1 \subseteq H'$ and $B_2 \subseteq \mathcal{U}_1(H')$, which hold by construction, are used only for the equalities behind the data of Table 1.

(p, n)	$\log_p H $	$\log_p H' $	$\log_p \mathcal{U}_1(H') $	$w(Z)$, leading coords	residue at weight p	$\text{gr}_p(\mathcal{U}_1(H'))$
(11, 5)	35	32	8	7; (5, 6)	(0, 3, 10, 2)	$\langle (1, 9, 2, 10) \rangle$
(17, 7)	63	60	12	9; (8, 9)	(0, 11, 8, 2)	$\langle (1, 7, 10, 16) \rangle$
(23, 9)	99	96	16	11; (11, 12)	(0, 10, 15, 2)	$\langle (1, 14, 9, 22) \rangle$
(29, 11)	143	140	20	13; (14, 15)	(0, 24, 7, 2)	$\langle (1, 6, 23, 28) \rangle$

Table 1: Certificates for Theorem B ($m = 3$, the pair (2), $Z = Z(a, b)$). The leading term of Z has the two nonzero coordinates shown, at $p^2 e_{1,n-1}$ and $p^2 e_{2,n}$. The layer of weight p has the basis $p^2 e_{n-3,1}, p^2 e_{n-2,2}, p^2 e_{n-1,3}, p^2 e_{n,4}$; the residue and the spanning vector of $\text{gr}_p(\mathcal{U}_1(H'))$ are given in this basis.

The certificates. Table 1 lists the data. The sifting writes $Z = r u$ with $u \in \mathcal{U}_1(H')$ and $r \in G_p$, where the leading term of r is the residue; by Lemma 3.4, $\text{gr}_p(\mathcal{U}_1(H'))$ is spanned by the leading terms of the elements of B_2 of weight p (last column), and the residue, which has a zero coordinate at their pivot, is not in this span. Its class modulo $\text{gr}_p(\mathcal{U}_1(H'))$ does not depend on the order of the computation. The program also computes H ($\log_p |H|$ is not used in the proof), checks every final list with the end-of-run certificate of Section 3, and computes $\mathcal{U}_1(H')$ a second time along the derived series by Lemma 3.2, as the program with which the result was first obtained did; both give the same subgroup. It needs about seven seconds for the four cases. As controls, the same program finds that the pair (2) is a witness of irregularity in $P_3(\mathbb{Z}/7^3)$ and $P_5(\mathbb{Z}/13^3)$ (irregular by (K5)) and in $P_3(\mathbb{Z}/5^2)$ (irregular by (K3)), and that it is not a witness in $P_4(\mathbb{Z}/11^3)$ (Theorem A), $P_2(\mathbb{Z}/5^3)$ and $P_3(\mathbb{Z}/11^4)$ (regular by (K4)).

Credit. The pair (2) is the pair of Kolesnikov and Leontiev [KL22, Section 4], who use it with their necessary condition [KL22, Corollary 1] to prove (K5). Our proof does not use that condition: it computes $\mathcal{U}_1(\langle a, b \rangle')$ itself. For $p \equiv 2 \pmod{3}$ the number $(p+4)/3$ is the least integer n with $3n-1 > p$, so Theorem B is the analogue for $p \in \{11, 17, 23, 29\}$ of [KL22, Theorem 2], which covers $n \geq (p+2)/3$ for $p \equiv 1 \pmod{3}$.

5 Regularity at the boundary: proof of Theorem A

In this section $mn-1 = p$ and $n \leq p-2$ (both hold in the seven cases of Theorem A). Then $G_p = \langle t \rangle$ has order p , $G_{p+1} = 1$ and $A_p = \mathbb{F}_p \bar{t}$. Let $\mu_s = m$ for $1 \leq s \leq n-1$ and $\mu_n = m-1$, so that $\sum_s \mu_s = p$, and for $x, y, z \in A_1$ let

$$f(z) = \prod_{s=1}^n z_s^{\mu_s}, \quad \sigma(x, y) = f(x+y) - f(x) - f(y).$$

Lemma 5.1 (the forced walk). *For $z \in A_1$, $z^p = f(z) \bar{t}$.*

Proof. In $z^p = \sum z_{s_1} \cdots z_{s_p} u_{s_1} \cdots u_{s_p}$, a nonzero product of $p = mn-1$ slots is a basis element of weight $mn-1$, so it is $\bar{t}$ and its walk starts in row n . The walk of length p from row n is $n \rightarrow n-1 \rightarrow \cdots \rightarrow 1 \rightarrow n \rightarrow \cdots \rightarrow 1$, with m descents and $m-1$ steps $1 \rightarrow n$; it uses u_s ($s < n$) m times and u_n $m-1$ times. So exactly one word contributes, with coefficient $f(z)$. $\square$

Proposition 5.2 (the last Hall–Petresco factor). *Let $a, b \in G$ with leading terms $x, y \in A_1$ (where $x = 0$ if $a \in G_2$, and $y = 0$ if $b \in G_2$). Then $c_p(a, b) = t^{-\sigma(x, y)}$.*

Proof. Let $\Lambda = \mathbb{Z}\langle\langle \xi, \eta \rangle\rangle$ be the ring of formal power series in non-commuting variables and I_k the ideal of series without terms of degree $< k$. As in Lemma 2.2, the subgroups $1 + I_k$ of the group $1 + I_1$

satisfy $[1 + U, 1 + U'] \in 1 + (UU' - U'U) + I_{k+l+1}$ for $U \in I_k$, $U' \in I_l$. Let $\psi : F \rightarrow 1 + I_1$ be the Magnus homomorphism $X \mapsto 1 + \xi$, $Y \mapsto 1 + \eta$ [Mag35], and $\varphi : F \rightarrow G$ the homomorphism $X \mapsto a$, $Y \mapsto b$. For $u \in F$ with $\psi(u) \in 1 + I_k$ let $\lambda_k(u)$ be the component of degree k of $\psi(u) - 1$, and let θ be the ring homomorphism from the polynomials without constant term in ξ, η to A with $\xi \mapsto x$, $\eta \mapsto y$ (it maps degree d into A_d , and is 0 in degrees $\geq mn$).

Claim: for $u \in \gamma_k(F)$, $\psi(u) \in 1 + I_k$, $\varphi(u) \in G_k$, and the leading term of $\varphi(u)$ in A_k is $\theta(\lambda_k(u))$. For $k = 1$ both sides are additive in u and agree on X and Y . For $k \geq 2$ the set of $u \in \gamma_k(F)$ satisfying the claim is a subgroup (leading terms in A_k and components of degree k are additive), and $\gamma_k(F)$ is generated by the commutators $[u, z]$ with $u \in \gamma_{k-1}(F)$, $z \in F$. For these, $\psi([u, z]) \in 1 + [\lambda_{k-1}(u), \lambda_1(z)] + I_{k+1}$, and by Lemma 2.2(b) $\varphi([u, z]) \in G_k$ has the leading term $[\theta\lambda_{k-1}(u), \theta\lambda_1(z)] = \theta([\lambda_{k-1}(u), \lambda_1(z)])$.

Apply ψ to (5) and compute modulo the ideal $J = p\Lambda + I_{p+1}$. First, $(1 + \xi)^p(1 + \eta)^p \equiv (1 + \xi^p)(1 + \eta^p) \equiv 1 + \xi^p + \eta^p$ and $((1 + \xi)(1 + \eta))^p \equiv 1 + (\xi + \eta + \xi\eta)^p \equiv 1 + (\xi + \eta)^p$. Next, for $2 \leq i \leq p - 1$ write $\psi(c_i) = 1 + W$ with $W \in I_2$ and $N = \binom{p}{i}$; then $\psi(c_i^N) = \sum_k \binom{N}{k} W^k \equiv 1$, because p divides $\binom{N}{k}$ for $1 \leq k \leq p - 1$ (as $k \binom{N}{k} = N \binom{N-1}{k-1}$ and $p \mid N$, $p \nmid k$) and $W^k \in I_{2k} \subseteq I_{p+1}$ for $k \geq p$. Finally $\psi(c_p) \equiv 1 + \lambda_p(c_p)$ by the claim. Hence $1 + \xi^p + \eta^p \equiv (1 + (\xi + \eta)^p)(1 + \lambda_p(c_p)) \equiv 1 + (\xi + \eta)^p + \lambda_p(c_p)$, that is,

$$\lambda_p(c_p) \equiv \xi^p + \eta^p - (\xi + \eta)^p \pmod{p}.$$

Since A has characteristic p , the claim and Lemma 5.1 show that the leading term of $c_p(a, b)$ in A_p is $x^p + y^p - (x + y)^p = -\sigma(x, y)\bar{t}$. As $G_{p+1} = 1$ and t^c has leading term $c\bar{t}$, $c_p(a, b) = t^{-\sigma(x, y)}$. $\square$

The induction in the claim also shows that $\lambda_p(c_p)$ lies in the Lie subring generated by ξ and η ; so $(\xi + \eta)^p - \xi^p - \eta^p$ is congruent modulo p to a Lie polynomial, which is Jacobson's formula [Jac37]. If x and y are linearly dependent, then $\sigma(x, y) = 0$: $f(cz) = c^p f(z)$ and $(1 + \lambda)^p = 1 + \lambda^p$ in $\mathbb{F}_p$.

The graded lower bound. For a subspace $V \subseteq A_1$ let $\mathfrak{h}(V)$ be the smallest graded subspace of A that contains V and is closed under brackets and π ; let $\mathfrak{d}(V)$ be the smallest graded subspace that contains $[\mathfrak{h}(V), \mathfrak{h}(V)]$ and is closed under $[\cdot, \mathfrak{h}(V)]$ and π ; and let $\mathfrak{o}(V)$ be the smallest graded subspace that contains $\pi(\mathfrak{d}(V))$ and is closed under $[\cdot, \mathfrak{h}(V)]$ and π . Then $\mathfrak{h}(V) \supseteq \mathfrak{d}(V) \supseteq \mathfrak{o}(V)$.

Proposition 5.3. *Let $a, b \in G$ have leading terms $x, y \in A_1$ with $\text{span}(x, y) = V$, and let $H = \langle a, b \rangle$. Then $\mathfrak{o}(V) \subseteq \text{gr}(\mathcal{U}_1(H'))$. In particular, if $\bar{t} \in \mathfrak{o}(V)$ then $t \in \mathcal{U}_1(H')$.*

Proof. For subgroups $K, K' \leq G$, Lemma 2.2(b) gives $[\text{gr}_v(K), \text{gr}_w(K')] \subseteq \text{gr}_{v+w}([K, K'])$, and Lemma 2.3 gives $\pi(\text{gr}_w(K)) \subseteq \text{gr}_{w+n}(K)$, since $n \leq p - 2$. Now $V \subseteq \text{gr}_1(H)$, and $\text{gr}(H)$ is closed under brackets and π , so $\mathfrak{h}(V) \subseteq \text{gr}(H)$. Next, $\text{gr}(H')$ contains $[\text{gr}(H), \text{gr}(H)]$ and is closed under $[\cdot, \text{gr}(H)]$ (as $H' \trianglelefteq H$) and under π , so $\mathfrak{d}(V) \subseteq \text{gr}(H')$. Finally $\text{gr}(\mathcal{U}_1(H'))$ contains $\pi(\text{gr}(H'))$ (p -th powers of elements of H') and is closed under $[\cdot, \text{gr}(H)]$ ($\mathcal{U}_1(H')$ is characteristic in H' , hence normal in H) and under π , so $\mathfrak{o}(V) \subseteq \text{gr}(\mathcal{U}_1(H'))$. If $\bar{t} \in \mathfrak{o}(V)$, then $\mathcal{U}_1(H') \cap G_p \neq 1$, and $G_p = \langle t \rangle$ has order p . $\square$

Lemma 5.4. *Let $V \subseteq A_1$ have the basis (x, y) . The following are equivalent: (i) $\sigma(u, v) = 0$ for all $u, v \in V$; (ii) $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for all $\alpha, \beta \in \mathbb{F}_p$; (iii) the polynomial $\prod_s (x_s + \lambda y_s)^{\mu_s} \in \mathbb{F}_p[\lambda]$ has no terms of degree $1, \dots, p - 1$.*

Proof. (i) says that f is additive on V , hence $\mathbb{F}_p$ -linear, which is (ii). Since $f(\alpha z) = \alpha^p f(z) = \alpha f(z)$, (ii) is equivalent to $g(\lambda) = f(x) + \lambda f(y)$ for all $\lambda \in \mathbb{F}_p$, where $g(\lambda) = f(x + \lambda y)$ is the polynomial in (iii). It has degree $\leq p$, constant term $f(x)$ and coefficient $f(y)$ at λ^p ; as $\lambda^p = \lambda$ on $\mathbb{F}_p$ and a polynomial of degree $\leq p - 1$ vanishing on $\mathbb{F}_p$ is zero, this is equivalent to (iii). $\square$

Proposition 5.5 (finite criterion). *Let $mn-1 = p$ and $n \leq p-2$. Suppose that every two-dimensional subspace $V \subseteq A_1$ satisfies $\bar{t} \in \mathfrak{o}(V)$ or the conditions of Lemma 5.4. Then $P_n(\mathbb{Z}/p^m)$ is regular.*

Proof. Let $a, b \in G$ with leading terms x, y , and $H = \langle a, b \rangle$. By Lemma 3.1 we must show $c_p(a, b) \in \mathcal{U}_1(H')$, and by Proposition 5.2 $c_p(a, b) = t^{-\sigma(x, y)}$. If x, y are linearly dependent, then $\sigma(x, y) = 0$. Otherwise $V = \text{span}(x, y)$ is two-dimensional; if σ vanishes on $V \times V$, then $c_p(a, b) = 1$, and otherwise $\bar{t} \in \mathfrak{o}(V)$, so $t \in \mathcal{U}_1(H')$ by Proposition 5.3 and $c_p(a, b) \in \langle t \rangle \leq \mathcal{U}_1(H')$. So $G \cong P_n(\mathbb{Z}/p^m)$ is regular. $\square$

Both $\mathfrak{o}(V)$ and the conditions of Lemma 5.4 depend only on V , not on the basis (x, y) and not on the terms of a and b of higher weight; so the check over all two-dimensional V covers every pair (a, b) whose leading terms span V . This is why σ is required to vanish on all of $V \times V$.

Symmetry. The torus $(\mathbb{F}_p^\times)^n$ acts on $A_1 = \mathbb{F}_p^n$ by scaling the slot coordinates.

Lemma 5.6 (torus symmetry and normal forms). (a) *For $\tau \in (\mathbb{F}_p^\times)^n$ let $\phi_\tau : A \rightarrow A$ be the linear map that multiplies each basis element $\varepsilon(i, j, k)$ by the product of the τ_s over the slots u_s of its walk. Then ϕ_τ is an automorphism of the algebra A , $\phi_\tau \circ \pi = (\tau_1 \cdots \tau_n) \pi \circ \phi_\tau$, $\mathfrak{o}(\phi_\tau V) = \phi_\tau(\mathfrak{o}(V))$, $\phi_\tau(\bar{t}) \in \mathbb{F}_p^\times \bar{t}$, and $\sigma(\phi_\tau x, \phi_\tau y) = (\prod_s \tau_s^{\mu_s}) \sigma(x, y)$. Hence the hypothesis of Proposition 5.5 holds for V if and only if it holds for $\phi_\tau(V)$.* (b) *Every orbit of two-dimensional subspaces of $\mathbb{F}_p^n$ under column scaling contains a subspace with a basis (r_1, r_2) in reduced row echelon form, with pivots in columns $i < j$, such that the entries of r_1 between columns i and j are 0 or 1, each column after j is one of $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, \beta)$ with $\beta \neq 0$, and the first column of the last type has $\beta = 1$ (coordinates in the order $u_1, \dots, u_n$). The number of these normal forms is*

$$N(p, n) = \sum_{1 \leq i < j \leq n} 2^{j-i-1} \left(3^L + \frac{(p+2)^L - 3^L}{p-1} \right), \quad L = n - j.$$

Proof. (a) If a product of two basis elements is nonzero, its walk is the concatenation of their walks, so ϕ_τ is multiplicative; it is bijective. Each $\varepsilon(i, i, 1)$ is a closed walk of length n through every slot once, so $\phi_\tau(\bar{p}) = (\prod_s \tau_s) \bar{p}$, which gives the relation with π . As ϕ_τ preserves brackets and the grading, and closure under π is the same as closure under a nonzero multiple of π , ϕ_τ maps $\mathfrak{h}(V)$, $\mathfrak{d}(V)$, $\mathfrak{o}(V)$ onto $\mathfrak{h}(\phi_\tau V)$, $\mathfrak{d}(\phi_\tau V)$, $\mathfrak{o}(\phi_\tau V)$. On A_1 , ϕ_τ is the column scaling, and $f(\phi_\tau z) = \prod_s \tau_s^{\mu_s} f(z)$. (b) Column scaling preserves the pivot columns. Scaling column l by τ_l and renormalising the rows changes the entries $(r_1[l], r_2[l])$ to $(\tau_l r_1[l]/\tau_i, \tau_l r_2[l]/\tau_j)$. So each nonzero entry between i and j and each nonzero entry of a column after j with one nonzero entry can be made 1; in a column after j with two nonzero entries the first can be made 1, and the ratio τ_i/τ_j then multiplies all the remaining β by a common factor, which makes the first of them 1. Counting the choices column by column gives $N(p, n)$. $\square$

Proof of Theorem A for the five triples of Table 2. For each of these triples, $mn-1 = p$ and $n \leq p-2$. For every normal form V of Lemma 5.6(b), the programs described below compute $\mathfrak{h}(V)$, $\mathfrak{d}(V)$ and $\mathfrak{o}(V)$ by Gaussian elimination in A and test the conditions of Lemma 5.4; the counts are in Table 2, and no normal form fails both conditions. By Lemma 5.6, every two-dimensional $V \subseteq A_1$ satisfies the hypothesis of Proposition 5.5, so $P_n(\mathbb{Z}/p^m)$ is regular. The rest follows from Lemma 2.1. The triples $(23, 8, 3)$ and $(29, 10, 3)$ are proved in Section 6, where $(11, 4, 3)$ and $(17, 6, 3)$ get a second proof. $\square$

The computer part of this proof is finite linear algebra over $\mathbb{F}_p$ (closures of subspaces in an algebra of dimension at most 159, and a polynomial identity in one variable); no computation in the group is needed. For $(11, 4, 3)$ the check was also made over all 16226 subspaces, so that it does not depend on Lemma 5.6. The counts of Table 2 were obtained by three programs: the program with which the

(p, n, m)	$\dim A$	subspaces checked	number	$\bar{t} \in \mathfrak{o}(V)$	$\sigma \equiv 0$ on $V \times V$	neither
(11, 4, 3)	38	all	16226	15830	396	0
(11, 4, 3)	38	normal forms	44	26	18	0
(17, 6, 3)	87	normal forms	9891	7376	2515	0
(19, 5, 4)	85	normal forms	682	511	171	0
(23, 6, 4)	123	normal forms	20391	16436	3955	0
(29, 6, 5)	159	normal forms	36723	30968	5755	0

Table 2: The finite check of Proposition 5.5 for Theorem A (the triples (23, 8, 3) and (29, 10, 3) are treated in Section 6). “All” means all two-dimensional subspaces of $A_1 = \mathbb{F}_{11}^4$, without symmetry; “normal forms” are those of Lemma 5.6(b). The last three columns are exclusive: the column “ $\sigma \equiv 0$ on $V \times V$ ” counts the subspaces with $\bar{t} \notin \mathfrak{o}(V)$ on which σ vanishes identically. $\dim A = (m-1)n^2 + n(n-1)/2$.

results were first obtained, the program of the second independent verification run (which, unlike the original program, represents A_w by vectors indexed by rows and tests σ through the linearity of f on V), and `check_note.py`; the first verification run reproduced the counts for (11, 4, 3) (all subspaces) and (17, 6, 3) with a fourth program, and the third verification run reproduced all of them with a fifth (Section 8). All counts agree. For (29, 6, 5) the run of the original program ended only after the first version of this note had been completed (after about three hours); the four programs that treated this case give the same counts.

Validation. The method was run on boundary cases $mn - 1 = p$ with known answer, with the same hypothesis $n \leq p - 2$. For the irregular groups $P_3(\mathbb{Z}/5^2)$, $P_4(\mathbb{Z}/7^2)$ and $P_6(\mathbb{Z}/11^2)$ (irregular by (K3)) exactly one normal form, the subspace $V_{KL} = \text{span}(u_1 + \cdots + u_{n-1}, u_n)$ of the leading terms of the pair (2), fails both conditions (7, 40 and 3927 normal forms); over all subspaces, its orbit has 4 elements for (5, 3, 2) and 36 for (7, 4, 2). For $P_7(\mathbb{Z}/13^2)$, 24 of the 79117 normal forms fail (second verification run). So none of these groups is certified, as it must be. For the regular groups $P_2(\mathbb{Z}/5^3)$, $P_2(\mathbb{Z}/7^4)$, $P_3(\mathbb{Z}/11^4)$, $P_2(\mathbb{Z}/13^7)$, $P_2(\mathbb{Z}/11^6)$ (regular by (K4)), and, in the second verification run, also $(p, n, m) = (17, 3, 6)$, $(17, 2, 9)$, $(19, 4, 5)$, $(23, 4, 6)$, $(23, 3, 8)$, $(29, 5, 6)$, $(29, 3, 10)$, no normal form fails. A deliberately wrong rule, $\mathfrak{d}(V)$ in place of $\mathfrak{o}(V)$, certifies the irregular groups $P_3(\mathbb{Z}/5^2)$, $P_4(\mathbb{Z}/7^2)$ and $P_6(\mathbb{Z}/11^2)$; for V_{KL} the top lies in $\mathfrak{d}(V)$ but not in $\mathfrak{o}(V)$. Exact computations in the group, on random pairs with prescribed leading terms and random terms of higher weight, found no exception to Propositions 5.2 and 5.3 (Section 8).

Remark 5.7 (a mechanism, observed in the computed cases). For $(p, n, m) = (11, 4, 3)$, $(7, 4, 2)$ and $(11, 6, 2)$, every normal form V on which σ does not vanish identically, except V_{KL} , contains in $\mathfrak{d}(V)$ an element of weight $n-1$ with nonzero coordinate at $\varepsilon(n, 1, 0)$; its image under π^{m-1} is a nonzero multiple of $\bar{t}$, so $\bar{t} \in \mathfrak{o}(V)$. For V_{KL} the top is reached, for $m = 3$, through the element $\pi([u_n, u_1 + \cdots + u_{n-1}])$ of $\mathfrak{o}(V)$ and brackets with $\mathfrak{h}(V)$; for $m = 2$ this element is 0, because the bracket already has $k = 1$ and π raises k to m . So among these normal forms only V_{KL} needs this element, which is nonzero for $m \geq 3$ and zero for $m = 2$. This is consistent with the boundary $nm - 1 = p$ being irregular for $m = 2$ and regular in all computed cases with $m \geq 3$; it is an observation from the computed cases, not a general proof. For $m = 3$ both mechanisms are made explicit in the proof of Theorem A': Case 1 there produces the element of weight $n - 1$ for every V outside the torus orbit of V_{KL} on which σ does not vanish identically (given (L1_n)), and Case 2 shows that $\pi(\text{ad}(\omega)^{2n-2}(u_n))$, with $\omega = u_1 + \cdots + u_{n-1}$, is a nonzero multiple of $\bar{t}$.

6 The boundary for $m = 3$: proof of Theorem A'

In this section $m = 3$, $p \geq 5$ is a prime with $p \equiv 2 \pmod{3}$, and $n = (p + 1)/3$. Then $mn - 1 = 3n - 1 = p$ and $n \leq p - 2$ (as $n \geq 2$), so the results of Section 5 apply, and n is even, since $3n = p + 1$ is even. We show that the hypothesis of Proposition 5.5 holds if $(L1_n)$ holds over $\mathbb{F}_p$ (Section 6.1), and we verify $(L1_n)$ for $n = 4, 6, 8, 10$ (Section 6.2).

6.1 The reduction to $\mathfrak{gl}_n$

Let $\mathfrak{n} \subseteq \mathfrak{gl}_n(\mathbb{F}_p)$ be the Lie algebra of strictly lower triangular matrices, and $\mathfrak{n}_d = \text{span}(e_{i,i-d} : d < i \leq n)$ for $1 \leq d \leq n - 1$. Then $\mathfrak{n} = \bigoplus_d \mathfrak{n}_d$, $[\mathfrak{n}_c, \mathfrak{n}_d] \subseteq \mathfrak{n}_{c+d}$ (which is 0 if $c + d \geq n$), $\mathfrak{n}_{n-1} = \mathbb{F}_p e_{n1}$, and $S \in \mathfrak{n}_1$. A Lie monomial of degree 1 in two letters is a letter, and one of degree $d \geq 2$ is a bracket $[w_1, w_2]$ of Lie monomials of degrees $d_1 + d_2 = d$; it can be evaluated at any two elements of a Lie algebra. For $Y \in \mathfrak{n}_1$ let $\mathfrak{g}(S, Y)$ be the Lie subalgebra of $\mathfrak{gl}_n(\mathbb{F}_p)$ generated by S and Y . It is spanned by the values at (S, Y) of the Lie monomials, and a monomial of degree d takes its value in $\mathfrak{n}_d$. Hence $(L1_n)$ holds for Y if and only if some Lie monomial w of degree $n - 1$ has $w(S, Y) \neq 0$, and then $w(S, Y)$ is a nonzero multiple of e_{n1} .

The level-zero map $\rho : A \rightarrow \mathfrak{n}$ is the linear map with $\rho(\varepsilon(i, j, 0)) = e_{ij}$ and $\rho(\varepsilon(i, j, k)) = 0$ for $k \geq 1$. Let $\omega = u_1 + \cdots + u_{n-1}$, the leading term of a in (2); so $V_{KL} = \text{span}(\omega, u_n)$.

Lemma 6.1. (a) ρ is a homomorphism of algebras, hence of Lie algebras, with $\rho(A_w) \subseteq \mathfrak{n}_w$ ($\mathfrak{n}_w = 0$ for $w \geq n$), $\rho(u_s) = e_{s+1,s}$ for $s < n$, $\rho(u_n) = 0$ and $\rho(\omega) = S$. (b) For $x \in A_{n-1}$, $\pi^2(x) = c\bar{t}$, where c is the coefficient of e_{n1} in $\rho(x)$. (c) For $x \in A_{2n-1}$, $\pi(x) = c'\bar{t}$, where c' is the coordinate of x at $\varepsilon(n, 1, 1)$.

Proof. (a) By (4), the product of $\varepsilon(i, j, k)$ and $\varepsilon(j', l, k')$ is a multiple of $\varepsilon(i, l, k + k')$ or 0; its level $k + k'$ is 0 only if $k = k' = 0$, and then the product is $\delta_{jj'}\varepsilon(i, l, 0)$, which corresponds to $e_{ij}e_{j'l} = \delta_{jj'}e_{il}$. So $\rho(xy) = \rho(x)\rho(y)$. The rest is clear. (b), (c) The basis elements $\varepsilon(i, j, k)$ of A_w satisfy $kn + i - j = w$ with $0 \leq k \leq 2$. For $w = n - 1$ they are $\varepsilon(n, 1, 0)$ and the $\varepsilon(i, i + 1, 1)$ with $1 \leq i \leq n - 1$; for $w = 2n - 1$ they are $\varepsilon(n, 1, 1)$ and the $\varepsilon(i, i + 1, 2)$. The map π raises k by one and is 0 at $k = 2$, and $\varepsilon(n, 1, 2) = \bar{t}$. So π^2 maps $\varepsilon(n, 1, 0)$ to $\bar{t}$ and kills the $\varepsilon(i, i + 1, 1)$, and π maps $\varepsilon(n, 1, 1)$ to $\bar{t}$ and kills the $\varepsilon(i, i + 1, 2)$. Finally, the coordinate of $x \in A_{n-1}$ at $\varepsilon(n, 1, 0)$ is the coefficient of e_{n1} in $\rho(x)$. $\square$

Lemma 6.2. The coordinate of $\text{ad}(\omega)^{2n-2}(u_n) \in A_{2n-1}$ at $\varepsilon(n, 1, 1)$ is $(-1)^{n-1}\binom{2n-2}{n-1}$, which is nonzero in $\mathbb{F}_p$.

Proof. Left and right multiplication by ω commute, so

$$\text{ad}(\omega)^{2n-2}(u_n) = \sum_{j=0}^{2n-2} (-1)^j \binom{2n-2}{j} \omega^{2n-2-j} u_n \omega^j.$$

Expanding $\omega^a u_n \omega^b$ ($a + b = 2n - 2$) gives a sum of products of $2n - 1$ slots in which u_n is the factor in position $a + 1$. By the paragraph on slots and walks in Section 2, such a product is 0 or the basis element of its walk. The walk of $\varepsilon(n, 1, 1)$ is $n \rightarrow n - 1 \rightarrow \cdots \rightarrow 1 \rightarrow n \rightarrow \cdots \rightarrow 1$, and its only step $1 \rightarrow n$ (the slot u_n) is step n . So only $a = b = n - 1$ contributes, through the single product $u_{n-1} \cdots u_1 u_n u_{n-1} \cdots u_1 = \varepsilon(n, 1, 1)$, and the coordinate is $(-1)^{n-1}\binom{2n-2}{n-1}$. It is nonzero in $\mathbb{F}_p$, because $2n - 2 < p$. $\square$

Proof of Theorem A'. By Proposition 5.5 it suffices to show that every two-dimensional subspace $V \subseteq A_1$ satisfies $\bar{t} \in \mathfrak{o}(V)$ or the conditions of Lemma 5.4. Two consequences of the definitions of the closures are used: a Lie monomial of degree ≥ 2 evaluated at elements of V lies in $\mathfrak{d}(V)$, because

it is a bracket of two Lie monomials of positive degree, whose values lie in $\mathfrak{h}(V)$, so that it lies in $[\mathfrak{h}(V), \mathfrak{h}(V)] \subseteq \mathfrak{d}(V)$; and $\pi(\mathfrak{d}(V)) \subseteq \mathfrak{o}(V)$ and $\pi(\mathfrak{o}(V)) \subseteq \mathfrak{o}(V)$.

Step 1. If no element of V has all its slot coordinates nonzero, then σ vanishes on $V \times V$: for $u, v \in V$ each of $u, v, u+v$ has a zero coordinate, and f vanishes there, since every $\mu_s \geq 1$.

Step 2. Otherwise let $z \in V$ have all $z_s \neq 0$, and let $\tau_s = z_s^{-1}$. Then $\phi_\tau(z) = \omega + u_n$, and by Lemma 5.6(a) the hypothesis of Proposition 5.5 holds for V if it holds for $\phi_\tau(V)$. So we may assume that $\omega + u_n \in V$, and we show that then $\bar{t} \in \mathfrak{o}(V)$. Write $V = \text{span}(\omega + u_n, y)$ and $y = y' + y_n u_n$ with $y' \in \text{span}(u_1, \dots, u_{n-1})$, and let $Y = \rho(y) = \sum_{s < n} y_s e_{s+1, s}$.

Case 1: Y is not a multiple of S . Then $n \geq 3$, since $\mathfrak{n}_1 = \mathbb{F}_p S$ for $n = 2$. By (L1_n) there is a Lie monomial w of degree $n-1$ with $w(S, Y) = c e_{n1}$, $c \neq 0$. Let $x = w(\omega + u_n, y) \in A_{n-1}$. By Lemma 6.1(a), $\rho(x) = w(\rho(\omega + u_n), \rho(y)) = w(S, Y) = c e_{n1}$, and $x \in \mathfrak{d}(V)$ since $n-1 \geq 2$. By Lemma 6.1(b), $c\bar{t} = \pi^2(x) \in \mathfrak{o}(V)$.

Case 2: $Y = \lambda S$ with $\lambda \in \mathbb{F}_p$. Then $y' = \lambda\omega$, and $y - \lambda(\omega + u_n) = (y_n - \lambda)u_n$ is nonzero, as $y \notin \mathbb{F}_p(\omega + u_n)$. Hence $u_n \in V$ and $V = \text{span}(\omega, u_n) = V_{KL}$. The element $x = \text{ad}(\omega)^{2n-2}(u_n)$ is the value of a Lie monomial of degree $2n-1 \geq 3$ at elements of V , so $x \in \mathfrak{d}(V)$; by Lemma 6.1(c) and Lemma 6.2, $\pi(x) = (-1)^{n-1} \binom{2n-2}{n-1} \bar{t}$ with a nonzero coefficient, and $\pi(x) \in \mathfrak{o}(V)$.

In both cases $\bar{t} \in \mathfrak{o}(V)$. So the hypothesis of Proposition 5.5 holds, and $P_n(\mathbb{Z}/p^3)$ is regular. $\square$

The proof uses Propositions 5.2, 5.3 and 5.5 and Lemma 5.6(a), but no enumeration of subspaces. For $n = 2$ ($p = 5$) Case 1 does not occur, and the theorem gives the regularity of $P_2(\mathbb{Z}/5^3)$, a case of (K4). For $m = 2$ the element $\text{ad}(\omega)^{2n-2}(u_n)$ of Case 2 is a nonzero multiple of the top $\bar{t} = \varepsilon(n, 1, 1)$ itself, so $\bar{t} \in \mathfrak{d}(V_{KL})$, but π maps it to 0; this matches the validation in Section 5, where V_{KL} is the uncovered normal form and $\bar{t}$ lies in $\mathfrak{d}(V_{KL})$ but not in $\mathfrak{o}(V_{KL})$.

Remark 6.3 (odd n). (L1_n) fails over $\mathbb{F}_p$ for every odd $n \geq 5$ and every odd prime p . Let J be the antidiagonal permutation matrix and $\kappa(X) = -JX^T J$; this is an automorphism of the Lie algebra $\mathfrak{gl}_n(\mathbb{F}_p)$, with $\kappa(e_{ij}) = -e_{n+1-j, n+1-i}$. For $Y = \sum_{i=2}^n D(i)e_{i, i-1}$ we get $\kappa(S) = -S$, $\kappa(Y) = -\sum_i D(n+2-i)e_{i, i-1}$ and $\kappa(e_{n1}) = -e_{n1}$. If D is *palindromic*, that is $D(i) = D(n+2-i)$ for all i , then $\kappa(Y) = -Y$; so κ maps $\mathfrak{g}(S, Y)$ onto itself and multiplies the value of every Lie monomial of degree $n-1$ by $(-1)^{n-1}$. For odd n this factor is 1, while $\kappa = -1$ on $\mathfrak{n}_{n-1} = \mathbb{F}_p e_{n1}$; as p is odd, $\mathfrak{g}(S, Y) \cap \mathfrak{n}_{n-1} = 0$. For odd $n \geq 5$ there are palindromic D that are not constant (for instance $D(3) = D(n-1) = 1$ and $D(i) = 0$ otherwise), and (L1_n) fails for $Y = DS$. For $n = 3$ every palindromic D is constant, and (L1₃) holds, since $[S, Y] = (D(2) - D(3))e_{31}$. As $n = (p+1)/3$ is even, this obstruction does not arise in Theorem A'. In the computations of Section 6.2, the failures for $n = 5$ (over $\mathbb{F}_{11}$ and $\mathbb{F}_{19}$) and for $n = 7$ (over $\mathbb{F}_{13}$) are exactly the palindromic classes: one class, and $p+1 = 14$ classes.

6.2 The condition (L1_n) for $n = 4, 6, 8, 10$

For a function $P : \{d+1, \dots, n\} \rightarrow \mathbb{F}_p$ write $PS^d = \sum_{i=d+1}^n P(i)e_{i, i-d} \in \mathfrak{n}_d$; every element of $\mathfrak{n}_d$ has this form for exactly one P , and S^d is the case $P = 1$, so that $S^{n-1} = e_{n1}$. Let ∇P be the function $i \mapsto P(i) - P(i-1)$ on $\{d+2, \dots, n\}$. Every $Y \in \mathfrak{n}_1$ is $Y = DS$ for a function D on $\{2, \dots, n\}$, and $Y \in \mathbb{F}_p S$ if and only if D is constant. Since $\mathfrak{g}(S, DS)$ depends only on $\text{span}(S, DS)$, it suffices to check (L1_n) for one D in each class of nonconstant functions modulo constants and nonzero scalars; we take $D(2) = 0$ and the first nonzero value among $D(3), \dots, D(n)$ equal to 1. There are $(p^{n-2} - 1)/(p-1)$ classes.

Lemma 6.4. *Let P be a function on $\{d+1, \dots, n\}$ and D a function on $\{2, \dots, n\}$. Then $[S, PS^d] = -(\nabla P)S^{d+1}$ and $[DS, PS^d] = QS^{d+1}$ with $Q(i) = D(i)P(i-1) - P(i)D(i-d)$ for $d+2 \leq i \leq n$. Moreover, with $L_d = \mathfrak{g}(S, DS) \cap \mathfrak{n}_d$, we have $L_1 = \text{span}(S, DS)$ and $L_{d+1} = [S, L_d] + [DS, L_d]$.*

Proof. Since $e_{i,i-1}e_{j,j-d} = \delta_{i-1,j}e_{i,i-d-1}$, we have $(DS)(PS^d) = \sum_i D(i)P(i-1)e_{i,i-d-1}$ and $(PS^d)(DS) = \sum_i P(i)D(i-d)e_{i,i-d-1}$, summed over $d+2 \leq i \leq n$. This gives the second formula, and the first is the case $D = 1$. By the Jacobi identity, the Lie algebra generated by S and DS is spanned by the brackets $[x_1, [x_2, [\dots, [x_{k-1}, x_k] \dots]]$ with all $x_i \in \{S, DS\}$, which gives the last statement. $\square$

Lemma 6.5 (two reductions, proved by hand). *Let $n \geq 4$ and D a function on $\{2, \dots, n\}$. Put $\ell(D) = (\nabla^{n-2}D)(n) = \sum_{t=0}^{n-2} (-1)^t \binom{n-2}{t} D(n-t)$ and $c(D) = (\nabla^{n-3}D)(n)$. (a) $\text{ad}(S)^{n-2}(DS) = (-1)^{n-2}\ell(D)e_{n1}$. Hence (L1_n) holds for DS if $\ell(D) \neq 0$. (b) If $\ell(D) = 0$, then $[DS, \text{ad}(S)^{n-3}(DS)] = (-1)^{n-3}c(D)(D(n) - D(2))e_{n1}$. Hence (L1_n) holds for DS if $\ell(D) = 0$, $c(D) \neq 0$ and $D(n) \neq D(2)$.*

Proof. By Lemma 6.4, $\text{ad}(S)^k(DS) = (-1)^k(\nabla^k D)S^{k+1}$ for $0 \leq k \leq n-2$; for $k = n-2$ this is (a). If $\ell(D) = 0$, the function $\nabla^{n-3}D$ on $\{n-1, n\}$ takes equal values, as their difference is $\ell(D)$; so $\text{ad}(S)^{n-3}(DS) = (-1)^{n-3}c(D)S^{n-2}$, and Lemma 6.4 with $P = 1$ and $d = n-2$ gives $[DS, S^{n-2}] = (D(n) - D(2))e_{n1}$. $\square$

The functionals ℓ , c and $D \mapsto D(n) - D(2)$ are linear and vanish on the constants. On the functions with $D(2) = 0$, the pair (ℓ, c) and the pair $(\ell, D(n) - D(2))$ are linearly independent when $n-2 < p$: in ℓ and c the coefficients of $(D(n), D(n-1))$ are $(1, -(n-2))$ and $(1, -(n-3))$, and in ℓ the coefficient of $D(n-1)$ is nonzero. So the classes not settled by Lemma 6.5(a) form the hyperplane $H = \{\ell = 0\}$, with $(p^{n-3} - 1)/(p-1)$ classes, and those not settled by (a) and (b) lie in $A^* = \{\ell = c = 0\}$ or in $B^* = \{\ell = 0, D(n) = D(2)\}$, each with $(p^{n-4} - 1)/(p-1)$ classes.

Proposition 6.6 (computer-assisted). (L1₄) holds over $\mathbb{F}_{11}$, (L1₆) over $\mathbb{F}_{17}$, (L1₈) over $\mathbb{F}_{23}$ and (L1₁₀) over $\mathbb{F}_{29}$.

Computation. The programs compute the spaces L_d of Lemma 6.4, as spaces of functions, by Gaussian elimination over $\mathbb{F}_p$, or the values of the brackets $\text{ad}(x_1) \cdots \text{ad}(x_{n-3})([S, DS])$ with $x_i \in \{S, DS\}$, which span L_{n-1} , and test whether $L_{n-1} \neq 0$; one program of the fourth verification run computes $\mathbf{g}(S, DS)$ with $n \times n$ matrices, without Lemma 6.4. Table 3 lists the sets of classes checked; no class fails. For $n = 4, 6$ all classes were checked. For $n = 8$ all classes of H were checked, which suffices by Lemma 6.5(a), and the fourth verification run checked all classes, which needs no lemma. For $n = 10$ all classes of A^* and of B^* were checked, by two programs, which suffices by Lemma 6.5, and the fourth verification run also checked all classes of H , which suffices by Lemma 6.5(a) alone. $\square$

Proof of Theorem A for (23, 8, 3) and (29, 10, 3). We have $23 = 3 \cdot 8 - 1$ and $29 = 3 \cdot 10 - 1$, so Theorem A' applies, and its hypothesis holds by Proposition 6.6. The same argument, with (L1₄) over $\mathbb{F}_{11}$ and (L1₆) over $\mathbb{F}_{17}$, proves the cases (11, 4, 3) and (17, 6, 3) again, independently of the finite check of Section 5. The rest follows from Lemma 2.1. $\square$

The recorded outputs also show that (L1₈) holds over $\mathbb{F}_{11}$ (its hyperplane H was checked), which is not needed here.

7 Consequences, conjectures and the open region

Proof of Corollaries 1 to 3. Corollary 1, first sentence: for $3n-1 < p$ regular by (K1); for $3n-1 = p$, i.e. $(p, n) = (11, 4), (17, 6), (23, 8)$ or $(29, 10)$, regular by Theorem A; for $3n-1 > p$, i.e. $n \geq (p+4)/3$, irregular by Theorem B. Second sentence, prime by prime $(P_1(\mathbb{Z}/p^3))$ is abelian, and Lemma 2.1 is used throughout):

(n, p)	set	lemma	classes	failures	programs
(4, 11)	all		12	0	<code>L1check.py</code> , <code>check_note.py</code> ; run 4 (3 programs)
(6, 17)	all		5220	0	<code>L1check.py</code> , <code>check_note.py</code> ; run 4 (3 programs)
(8, 23)	H	(a)	292561	0	<code>L1fast.py</code> , <code>check_note.py</code> ; run 4
(8, 23)	A^*, B^*	(a), (b)	2×12720	0	<code>L1fast2.py</code> ; run 4
(8, 23)	all		6728904	0	run 4
(10, 29)	A^*, B^*	(a), (b)	2×21243690	0	<code>L1fast2.py</code> (2576 s); run 4 (2×42 s)
(10, 29)	H	(a)	616067011	0	run 4 (2×630 s)
(5, 11)	all		133	1	the palindromic class (Remark 6.3)
(5, 19)	all		381	1	the palindromic class
(7, 13)	all		30941	14	the 14 palindromic classes
(10, 7)	all		960800	8	$p < n$; run 4 only

Table 3: The computer check of $(L1_n)$ (Proposition 6.6) and controls. The column “lemma” names the parts of Lemma 6.5 that make the set sufficient. “Run 4” is the fourth independent verification run, with its own programs (Section 8). The times printed by `L1fast2.py` are cumulative: 1305 s after A^* , 2576 s after A^* and B^* . The controls for $n = 5, 7$ were run with `L1check.py` and, in run 4, with three programs, and `check_note.py` repeats (5, 11) and (7, 13); the control (10, 7) was run in run 4 only.

- $p = 2$: $3n - 1 \leq 2$ means $n = 1$. For $n \geq 2$, $P_n(\mathbb{Z}/8)$ is irregular, because $P_2(\mathbb{Z}/4)$ is irregular by (K3) ($2n - 1 = 3 > p$), or because a regular 2-group is abelian [Hup67, Chapter III] and $P_2(\mathbb{Z}/8)$ is not.
- $p = 3$: $3n - 1 \leq 3$ means $n = 1$. For $n \geq 2$, $P_n(\mathbb{Z}/27)$ is irregular, because $P_2(\mathbb{Z}/9)$ is irregular by (K3) ($2n - 1 = 3 = p$).
- $p = 5$: $3n - 1 \leq 5$ means $n \leq 2$. $P_2(\mathbb{Z}/125)$ is regular by (K4) ($n^2 = 4 < 5$), and also by Theorem A' with $n = 2$; for $n \geq 3$, $P_n(\mathbb{Z}/125)$ is irregular, because $P_3(\mathbb{Z}/25)$ is irregular by (K3) ($2n - 1 = 5 = p$).
- $p = 7, 13, 19$ ($p \equiv 1 \pmod{3}$): regular for $3n - 1 < p$, i.e. $n \leq (p - 1)/3$, by (K1), and irregular for $n \geq (p + 2)/3$ by (K5) [KL22, Theorem 2]; these ranges cover all n , and $3n - 1 = p$ is impossible.
- $p = 11, 17, 23, 29$ ($p \equiv 2 \pmod{3}$): the first sentence.

The rows $p = 11, 17, 23, 29$ rest on Theorems A and B, which are computer-assisted and unrefereed; the other rows rest on the cited literature (for $p = 2, 3$ the fourth verification run also found witnesses of irregularity in $P_2(\mathbb{Z}/4)$, $P_2(\mathbb{Z}/8)$, $P_2(\mathbb{Z}/9)$ and $P_2(\mathbb{Z}/27)$ by a direct search). Corollary 2: $P_1(\mathbb{Z}/11^m)$ is abelian; for $m = 1$ use (K2), for $m = 2$ (K3) ($2n - 1 < 11$ if and only if $n \leq 5$), for $m = 3$ Corollary 1; for $m \geq 4$, $n \leq 3$ gives $n^2 < 11$ (K4), and $n \geq 5$ is irregular by Theorem B. Corollary 3: by Theorem B; for $p \in \{17, 23, 29\}$ these n satisfy $n < (p + 1)/2$ and $n^2 > p$, and $p \equiv 2 \pmod{3}$, so none of (K1)–(K5) applies. $\square$

7.1 Conjectures

The following are conjectures, not results.

Conjecture 1 ($m = 3$). *For every prime p , $P_n(\mathbb{Z}/p^3)$ is regular if and only if $3n - 1 \leq p$.*

Status: the statement holds for $p \equiv 1 \pmod{3}$, where $3n - 1 \leq p$ means $3n - 1 < p$ (by (K1) and (K5)), and it is proved here for every prime $p \leq 29$ (Corollary 1; computer-assisted for $p = 11, 17, 23, 29$). Its remaining content concerns the primes $p \equiv 2 \pmod{3}$ with $p \geq 41$: the regularity of $P_{(p+1)/3}(\mathbb{Z}/p^3)$, which by Theorem A' follows from (L1_n) over $\mathbb{F}_p$ for $n = (p+1)/3$, and the irregularity of $P_{(p+4)/3}(\mathbb{Z}/p^3)$, for which the pair (2) is the natural candidate witness.

Conjecture 2 (the boundary for $m \geq 3$). *If $nm - 1 = p$ and $m \geq 3$, then $P_n(\mathbb{Z}/p^m)$ is regular.*

Evidence: all seven boundary cases with $n^2 > p$ and $m \geq 3$ that were computed are regular (Theorem A), and those with $n^2 < p$ are regular by (K4). For $m = 3$ the conjecture is the “if” half of Conjecture 1 for $p \equiv 2 \pmod{3}$, and by Theorem A' it follows from Conjecture 3. For $m = 1$ and $m = 2$ the boundary groups are irregular, by (K2) and (K3); Remark 5.7 and the remark after the proof of Theorem A' describe how $m = 2$ and $m = 3$ differ.

Conjecture 3 ((L1_n) for even n). *For every even $n \geq 2$ and every prime $p > 2n$, (L1_n) holds over $\mathbb{F}_p$.*

Evidence: Proposition 6.6, that is $(n, p) = (4, 11), (6, 17), (8, 23), (10, 29)$; and, by the program `check_note.py` of this note (Part D, with the bracket formulas of Lemma 6.4), (L1₄) over $\mathbb{F}_p$ for all primes $11 \leq p \leq 31$, (L1₆) for all primes $13 \leq p \leq 37$ (all classes), and (L1₈) for $p = 17, 19$ (the hyperplane H , which suffices by Lemma 6.5(a)). Random elements of H for $n = 10, 12, 14$ and several primes $p > 2n$ (20000 for each pair) gave no failure, which is weak evidence, since the failures found elsewhere are special classes. By Remark 6.3, the restriction to even n is necessary, and some restriction on p is needed as well: (L1₁₀) fails over $\mathbb{F}_7$ (Table 3), and (L1₈) fails over $\mathbb{F}_5$ for 6 classes. By Theorem A', Conjecture 3 for the even n with $3n - 1$ prime implies the “if” half of Conjecture 1 and Conjecture 2 for $m = 3$.

7.2 Evidence only

The following computations are not proofs. (The pair (2) also satisfies (1) in $P_8(\mathbb{Z}/23^3)$ and $P_{10}(\mathbb{Z}/29^3)$, by the original program and in the second verification run; these two groups are no longer evidence only, since Theorem A proves them regular.) For $P_4(\mathbb{Z}/11^4)$ and $P_4(\mathbb{Z}/11^5)$, the open cases of Corollary 2, the pair (2) and 340 further pairs (products of weight-one elements with disjoint supports and random sparse elements) satisfy (1), by the exact test of Section 4 with the derived series; the second verification run tested 600 further pairs (400 and 200) with its own exact method and found no witness either. For $m \geq 4$, the pair (2) is not a witness in $P_6(\mathbb{Z}/19^4)$, $P_5(\mathbb{Z}/19^5)$ and $P_4(\mathbb{Z}/13^4)$ (the original exact program), which suggests that the mechanism of Theorem B is specific to $m = 3$. The exact test as implemented needs $2p \geq mn$ (for Lemmas 3.2 and 3.3 with $w_0 = 2$); it does not apply to $P_4(\mathbb{Z}/11^m)$ for $m \geq 6$.

7.3 What stays open

By (K1)–(K4), the undecided triples with $n \geq 2$ satisfy $m \geq 3$, $n^2 > p$, $nm - 1 \geq p$ and $n < (p+1)/2$. After this note, the triples of this kind that are decided by (K5), by Theorems A and B or by these with Lemma 2.1 are known; all others are open. For example:

- $p \equiv 1 \pmod{3}$ and $m \geq 4$ below the bound of (K5): $(p, n) = (13, 4)$ for $m \geq 4$; $(19, 5)$ for $m \geq 5$; $(19, 6)$ for $m \geq 4$;
- $p \equiv 2 \pmod{3}$ above the boundary, $m \geq 4$: $(11, 4)$, $(17, 5)$, $(17, 6)$ for $m \geq 4$, and similarly for $p = 23, 29$ (for instance $(23, 7)$ and $(23, 8)$ for $m \geq 4$, $(23, 5)$ and $(23, 6)$ for $m \geq 5$, $(29, 8)$, $(29, 9)$ and $(29, 10)$ for $m \geq 4$, $(29, 6)$ for $m \geq 6$);

- $m = 3$: for $p \equiv 2 \pmod{3}$ and $p \geq 41$, the boundary case $n = (p+1)/3$ (which by Theorem A' follows from (L1_n) over $\mathbb{F}_p$) and the range $(p+4)/3 \leq n < (p+1)/2$.

In particular, for every prime $p \geq 11$ there is an n (for instance $n = \lceil \sqrt{p} \rceil$) for which the regularity of $P_n(\mathbb{Z}/p^m)$ is open for all large m .

8 Computations and reproducibility

The proofs of Theorems A and B and of Proposition 6.6 use computations; everything else, including Theorem A' and Lemma 6.5, is proved in the text. The reproducibility package (Python 3 with the standard library; the programs of the second verification run, two programs of the third and one of the fourth use numpy) contains three parts. First, the programs with which the results were first obtained, with their outputs: `preg.py` (the group computations, with $\mathcal{U}_1$ along the derived series by Lemma 3.2), `liebound.py`, `allV.py` and `allV_gen.py` (the graded check; about 8 minutes for (17, 6, 3), 46 minutes for (23, 6, 4) and three hours for (29, 6, 5)), `family.py`, `sanity.py`, `mech.py`, `targeted.py`, `rand_pairs.py`, and, for Proposition 6.6, `L1check.py` (all classes), `L1fast.py` (the hyperplane H) and `L1fast2.py` (the sets A^* , B^* ; for (10, 29) about 40 minutes). Second, the program of this note, `check_note.py`, a separate implementation of the computations: Part A, Theorem B with end-of-run certificates, the cross-check along the derived series and the controls (7 s); Part B, Theorem A for (11, 4, 3) over all subspaces and over the normal forms and for (19, 5, 4), with checks of Lemma 5.1 and of the torus symmetry on random samples (20 s); Part C, the validation (1 s); Part D, for Section 6: Lemma 6.2 and the two cases of the proof of Theorem A' in the graded algebra for $(p, n) = (5, 2)$, (11, 4), (17, 6), (23, 8), (29, 10), the conclusion $\bar{t} \in \mathfrak{o}(V)$ for sample subspaces of (23, 8, 3) and (29, 10, 3), Lemmas 6.4 and 6.5 against matrix arithmetic, (L1₄) over $\mathbb{F}_{11}$ and (L1₆) over $\mathbb{F}_{17}$ on all classes and (L1₈) over $\mathbb{F}_{23}$ on H , with a second method (matrices) on samples, the controls of Remark 6.3 and the evidence for Conjecture 3 (10 s); Part L, the normal forms for (17, 6, 3), (23, 6, 4) and (29, 6, 5) (1, 5 and 17 minutes). It represents elements of A_w as vectors indexed by rows and tests σ by Lemma 5.4(iii) and (ii) (for (11, 4, 3) also by the criterion of the original program, all p^4 pairs); it reproduces Tables 1 and 2 and the rows of Table 3 that it treats, and its last line is **TOTAL failures: 0**. The case (10, 29) of Proposition 6.6 is not run by `check_note.py`; it takes about 40 minutes with `L1fast2.py` and is documented by the saved outputs of that program and of the fourth verification run. Third, the programs and outputs of the four independent verification runs. All programs with recorded outputs were run again from the package folder, with the same results up to running times, except the two longest runs of `allV_gen.py`, for (23, 6, 4) and (29, 6, 5), which are covered by Part L and by the faster programs of the second and third runs, and `L1fast2.py` for (10, 29), which is covered by the programs of the fourth run (all of them were run again, including the two pieces of its hyperplane check for (10, 29)); the third verification run repeated the re-runs of the first version of the package from a copy extracted from the archive, with the same results. Command lines and running times are in `README.md` and `RERUN_LOG.txt`.

Verification. The results were first obtained with written proofs and the programs listed above. Four independent verification runs, all AI-assisted, followed; each re-derived the proofs and wrote its own programs, and none found a mathematical error. The first three examined Theorem B and the cases of Theorem A in Table 2; the fourth examined Theorem A', Proposition 6.6 and the consequences drawn from them, which were added afterwards.

The first run checked the statement against the Notebook and re-derived Lemmas 2.2, 2.3, 3.1 and 3.2, the exactness of sifting, Propositions 5.2 and 5.3, the torus symmetry and the normal forms. Its exact program, with end-of-run certificates and validated against an enumeration of the subgroups in five small groups, reproduced the four certificates of Table 1 (for $p = 17, 23, 29$ it supplied

the numbers; the first program had printed only the verdict), and its graded program reproduced Table 2 for $(11, 4, 3)$ over all subspaces and for $(17, 6, 3)$. Exact tests on 489 pairs with random terms of higher weight, in five groups, found no exception to Propositions 5.2 and 5.3. It required nine corrections of the first write-up, none mathematical, which were made; among them are the congruence $g_{ii} \equiv 1 \pmod{p}$ in the definition of G , the notation $\mathcal{U}_1$, the range $m \geq 2$ in (K3), the remark after Proposition 5.5, the full data of Table 1, and an end-of-run certificate in the package.

The second run used a different method for the exact part, the Lazard correspondence on G_2 with Howell forms of modules over $\mathbb{Z}/p^3$ and closure certificates, and confirmed Theorem B in all four cases (the orders of Table 1, and for $(11, 5, 3)$ the residue modulo $\text{gr}_{11}(\mathcal{U}_1(H'))$). Its own program for the graded check reproduced Table 2 for $(11, 4, 3)$ (all subspaces, and two families of normal forms) and for $(17, 6, 3)$ (two families); it completed the validation case $(13, 7, 2)$, made the test with the wrong rule, tested Propositions 5.2 and 5.3 exactly on 3200 pairs, and computed the Hall–Petresco words directly on 1410 pairs, in which $c_p(a, b) = t^{-\sigma(x, y)}$ held every time (wrong exponents or a wrong sign are detected). It read [KL22] and [KP25] in full and version 49 of [KM26], checked the bibliography with Crossref, and computed $(19, 5, 4)$, $(23, 6, 4)$ and $(29, 6, 5)$ with its program, which suggested Conjecture 2. It required seven corrections, none mathematical, which were made; among them are the hypotheses of [Kol13], the disclosure about [EKL23], the description of the open region and the DOIs.

Besides, the boundary cases $(19, 5, 4)$, $(23, 6, 4)$ and $(29, 6, 5)$ were computed with the original program (the first while the verification runs were in progress, the others after them), with the same counts as in the second run, and `mech.py` was added (Remark 5.7); the two runs had examined the first write-up, which contained the cases $m = 3$ of Theorem A that are in Table 2. When this note was written, `check_note.py` was added, with the lemma on filtered generating sets (Lemma 3.3), which it uses besides Lemma 3.2.

The third run examined the text of this note, beginning with the parts written after the first two runs (Lemma 3.3 and the cases $m = 4, 5$ of Theorem A). It compared the statement again with entry 8.3 in versions 48 and 49 of [KM26] and with the corpus record, read the introduction, Theorems 1 and 2, Corollary 1 and Section 4 of [KL22], the introduction, Theorem 1 and Section 3 of [Kol13] and the zbMATH reviews of [Yag94, Kol06, EKL23, KP25], and re-derived all lemmas and propositions line by line. Its own programs, written without the code of the package, reproduced Table 1 exactly (with another pivot convention for the sifting, end-of-run certificates, and a check against an enumeration of the subgroups for 40 pairs in six small groups) and every count of Table 2 and of the validation, including the wrong rule. It computed the closures both from their definitions and in the simplified form in which the components of weight w of $\mathfrak{h}(V)$ and $\mathfrak{d}(V)$ are $[\mathfrak{h}(V)_{w-1}, V] + \pi(\mathfrak{h}(V)_{w-n})$ and $[\mathfrak{h}(V)_{w-1}, V]$ and $\mathfrak{o}(V) = \pi(\mathfrak{d}(V))$, which follows from the definitions by the Jacobi identity, since π is multiplication by a central element; both gave the same subspaces. Over all subspaces for $(5, 3, 2)$, $(7, 4, 2)$ and $(11, 4, 3)$ it found that each torus orbit contains exactly one normal form and that the verdict is constant on orbits. In exact computations in the group on 4100 random pairs with prescribed leading terms and random terms of higher weight, in the five groups of Table 2 and in $P_3(\mathbb{Z}/5^2)$, $P_4(\mathbb{Z}/7^2)$ and $P_6(\mathbb{Z}/11^2)$, the Hall–Petresco words computed by Hall’s recursion gave $c_p(a, b) = t^{-\sigma(x, y)}$ every time, the bound of Proposition 5.3 held, and $Z(a, b) \in \mathcal{U}_1(\langle a, b \rangle')$ held exactly when $\sigma(x, y) = 0$ or $t \in \mathcal{U}_1(\langle a, b \rangle')$; in the five groups of Table 2 it held for all 3100 pairs. The run also executed the package again from its archive, with the same results up to running times, and repeated the literature search. It required six corrections, none mathematical, which were made; among them are the sentence on the condition $nm - 1 < p$ in the abstract and in Section 1 (that this condition is not necessary for $m \geq 3$ follows already from (K4)), the caption of Table 2 and one sentence of Remark 5.7.

After the third run, Theorem A’ and its proof were found and added, with `L1check.py`, `L1fast.py`

and `L1fast2.py`, and with them the cases $(23, 8, 3)$ and $(29, 10, 3)$ of Theorem A, the second sentence of Corollary 1 and Conjecture 3. The fourth run examined these additions. It re-derived the proof of Theorem A' (Steps 1 and 2, both cases with the content of Lemmas 6.1 and 6.2, the side conditions n even and $n \leq p - 2$, and the case $n = 2$), the points of Section 5 that the proof uses (the top layer, Lemma 2.3 and Propositions 5.2 and 5.3 in outline, and the torus symmetry), the two reductions of Lemma 6.5 and the description of the sets H , A^* , B^* , and Remark 6.3. It wrote three programs of its own before reading those of the package: one computes $\mathfrak{g}(S, DS)$ with $n \times n$ matrices, one evaluates all the brackets of degree $n - 1$ for many classes at once (numpy), and one computes the spaces L_d by elimination. With them it verified (L1₄) over $\mathbb{F}_{11}$ and (L1₆) over $\mathbb{F}_{17}$ on all classes (three programs), (L1₈) over $\mathbb{F}_{23}$ on all 6728904 classes, without any lemma, and (L1₁₀) over $\mathbb{F}_{29}$ on A^* and B^* and on the whole hyperplane H (616067011 classes, 0 failures); the three programs agree wherever all were run, including the failing sets for $n = 5$ and $n = 7$. It also checked both cases of the proof of Theorem A' inside the matrix ring $M_n(\mathbb{Z}/p^3)$ for $(p, n) = (5, 2), (11, 4), (17, 6), (23, 8), (29, 10)$, re-ran `L1check.py`, `L1fast.py` and `L1fast2.py` on the smaller cases from a copy, with identical results, checked the second sentence of Corollary 1 prime by prime, and found witnesses of irregularity for $P_2(\mathbb{Z}/4)$, $P_2(\mathbb{Z}/8)$, $P_2(\mathbb{Z}/9)$ and $P_2(\mathbb{Z}/27)$ by a direct search. It made no external request. It required seven corrections, none mathematical, which were made: Theorem A' stated for $p \geq 5$ (for $p = 2$ one has $n = 1$, and the group is abelian); “odd $n \geq 5$ ” in Remark 6.3 ((L1₃) holds); the list of the primes $p \leq 29$ with their sources and the words computer-assisted and unrefereed for the second sentence of Corollary 1; the consistency of the whole note with the new results; and three presentational points (the cumulative times of `L1fast2.py`, the two reductions named as lemmas proved by hand, and the reasons for the memberships in $\mathfrak{d}(V)$ in the two cases of the proof). Part D of `check_note.py`, Lemmas 6.1, 6.2 and 6.4 as separate statements and the evidence for Conjecture 3 were added after the fourth run.

Conclusion, scope and priority. In the terms of the question, our result is a partial answer: $P_n(\mathbb{Z}/p^m)$ is regular in the boundary cases $nm - 1 = p$ for $(p, n, m) = (11, 4, 3), (17, 6, 3), (23, 8, 3), (29, 10, 3), (19, 5, 4), (23, 6, 4), (29, 6, 5)$, and $P_n(\mathbb{Z}/p^3)$ is irregular for $n = (p + 4)/3$ and $p \in \{11, 17, 23, 29\}$; hence for every prime $p \leq 29$, $P_n(\mathbb{Z}/p^3)$ is regular iff $3n - 1 \leq p$, and for $p = 11$ the answer is complete except for $n = 4, m \geq 4$; the question stays open in general. The note thus decides new cases of Problem 8.3 of [KM26], those of Theorems A and B and Corollaries 1 to 3, with computer-assisted proofs; Theorem A' reduces the boundary for $m = 3$ to the condition (L1 _{n}); the question stays open in general (Section 7.3), and Conjectures 1 to 3 are conjectures. The question is due to B. A. F. Wehrfritz [KM26]. The known results we use are due to Merzlyakov [Mer64] and Levchuk [Lev92] (the class), Yagzhev [Yag94] ($m = 1$), Kolesnikov [Kol01] and Egorychev, Kolesnikov and Leontiev [EKL23] ($m = 2$), Kolesnikov [Kol06] ($n^2 < p$) and Kolesnikov and Leontiev [KL22] ($p \equiv 1 \pmod{3}$); the witness pair of Theorem B is that of [KL22]. Hall's theory of regular p -groups [Hal34], the Hall–Petresco identity [Hal34, Pet54], the Magnus embedding [Mag35] and Jacobson's formula [Jac37] are classical.

What was read: entry 8.3 in versions 48 and 49 of [KM26]; [KL22] in full (by the second verification run) and, for this note, its introduction, statements, Section 4 and references; [Kol13] in the journal's file (introduction, Theorem 1, Section 3 and references); [KP25] in full (by the second verification run) and, for this note, its introduction; the zbMATH reviews of [Yag94, Kol06, EKL23, KP25]; the zbMATH records of [Mer64, KM11, Kol13]; the Crossref records of all cited DOIs. Of [EKL23] only the zbMATH review (Zbl 1516.20037) and the Crossref record were seen: the paper was not read in full, and (K3) relies on the review for its Theorem 1. [Kol01] was not accessible and was not read; we cite it as the Notebook does. [Mer64] and [Lev92] were not read; the bound for the class used here is proved in Section 2. Of [Yag94] only the review was read, and the letter to the editor was not seen.

The searches were made on 22 October 2026: zbMATH Open (the 32 works of S. G. Kolesnikov, the works of V. M. Leontiev and of G. P. Egorychev, the queries “Wehrfritz problem” and “regular Sylow subgroup”); arXiv (queries on regular p -groups, Sylow subgroups, Wehrfritz, the Hall–Petresco identity and the authors); the lists of citing works in OpenAlex and Semantic Scholar for [Kol06, KLE20, KL22, EKL23, KP25]; Crossref; two web searches; the publishers’ pages of [KL22, Kol13, KP25]. No printed decision of the cases of Theorems A and B was found. [KL22] states that its condition allows a partial study of the case $n \geq (p+1)/3$, and its Theorem 2 assumes that $(p+2)/3$ is an integer; [Kol13] proves necessary conditions only; [KP25] concerns the types F_4 and E_6 ; OpenAlex and Semantic Scholar list no work citing [KL22], [EKL23] or [KP25]. The third verification run repeated the search on the same day (arXiv, zbMATH Open, Crossref, OpenAlex and one web search) and found nothing new. The cases $(23, 8, 3)$ and $(29, 10, 3)$, added later, lie in the region covered by these searches, which found no decision of any boundary case $n = (p+1)/3$ with $p \equiv 2 \pmod{3}$ and $p \geq 11$; an arXiv query for the newest abstracts with “regular” and “Sylow”, made when Theorem A’ was added, found nothing new. The literature on Lie algebras of triangular matrices was not searched for $(L1_n)$, which may be known in another form. Conference abstracts were not searched. A search that finds nothing is not a proof of novelty, and we claim no priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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