Explicit Closed Elementary Nets That Cannot Be Completed, Including Fields of Transcendence Degree One

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September 28, 2026

Abstract

An elementary net (carpet) of order n over a field K is closed if its elementary net group contains no new elementary transvections, and it is completable if its diagonal can be supplemented to a full net. Completable nets are closed. Koibaev gave closed nets that are not completable over fields of characteristic 0 and 2 and asked for such nets in odd characteristic (Kourovka Notebook, Problem 21.76). Nuzhin (2026) has answered this question, with examples in every characteristic, using closures of finitely generated subgroups over a rational function field in two variables. We give explicit examples with short proofs. An elementary lifting lemma turns a closed pair of additive subgroups of a quotient ring R/J into a closed elementary net of every order $n \geq 3$. With $F[x, y] \to F[x]$ and a degree argument it gives, for every field F, the net with $Fx + yF[x, y]$ in positions $(1, 2)$ and $(2, 1)$ and $yF[x, y]$ elsewhere. Lifting the two exceptional pairs of Levchuk's refinement of Dickson's theorem along $\mathbb{F}_3[t] \to \mathbb{F}_9$ and $\mathbb{F}_2[t] \to \mathbb{F}_4$ gives closed nets over $\mathbb{F}_3(t)$ and $\mathbb{F}_2(t)$ that are not completable. These one-variable nets also satisfy the hypotheses of Kourovka Problem 19.48. Together with a theorem of Koibaev and Nuzhin on algebraic extensions, it follows that in characteristics 2 and 3 a field carries such nets if and only if it is not algebraic over its prime field. For $p \geq 5$ we do not know whether examples exist over $\mathbb{F}_p(t)$; by Levchuk's theorem, lifting from a finite field cannot produce them. This is an unrefereed note.

1 Introduction

We use the definitions of the Kourovka Notebook [KM26, Problems 19.48 and 21.76]. Let K be a field and $n \geq 2$. A *net* (or *carpet*) of order n over K is a system $\sigma = (\sigma_{ij})$, $1 \leq i, j \leq n$, of additive subgroups of K such that

$$\sigma_{ir}\sigma_{rj} \subseteq \sigma_{ij} \qquad \text{for all } i, r, j,$$

where a product of subgroups denotes the additive subgroup generated by the products of their elements. An *elementary net* is a system $(\sigma_{ij})_{i \neq j}$ without the diagonal satisfying the same inclusions for all pairwise distinct i, r, j. It is *irreducible* if all σ_{ij} are nonzero. Write $t_{ij}(\alpha) = 1 + \alpha e_{ij}$ for the elementary transvections and

$$E(\sigma) = \langle t_{ij}(\alpha) : \alpha \in \sigma_{ij},\ i \neq j \rangle \leq \mathrm{SL}_n(K)$$

for the elementary net group. The elementary net σ is *closed* if $E(\sigma)$ contains no new elementary transvections, that is, $t_{ij}(\beta) \in E(\sigma)$ with $\beta \in K$ implies $\beta \in \sigma_{ij}$. It is *completable* if there are additive subgroups $\sigma_{11}, \ldots, \sigma_{nn}$ of K such that $(\sigma_{ij})_{1 \leq i,j \leq n}$ is a net. Completable elementary nets are closed.

Koibaev constructed irreducible closed elementary nets that are not completable over fields of characteristic 0 and 2 [Koi11b, Koi21], and asked the following question.

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Problem 21.76 (V. A. Koibaev [KM26]; paraphrased). Is there, over some field of odd characteristic, an irreducible closed elementary net of order $n \geq 3$ that is not completable?

Nuzhin [Nuz26, Corollary 1] has shown that for every $n \geq 2$ there is an irreducible closed elementary net of order n over a field of arbitrary characteristic that is not completable; to our knowledge this is the first answer to Problem 21.76. His nets arise, in a study of carpet Lie rings, as the closures $\{t \in K : x_r(t) \in G\}$ of finitely generated subgroups G of Chevalley groups over the field $K = F(y, z)$ of rational functions in two variables. A key ingredient is a free-product property of two unipotent subgroups [Nuz26, Lemma 8], closely related to Koibaev's theorem on closed pairs [Koi11a]. Gutnova and Koibaev [GK20] also construct a closed elementary net that is not weakly supplemented, hence not completable. Over fields that are algebraic over $\mathbb{F}_p$ there are no examples: by Koibaev and Nuzhin [KN17], every irreducible elementary net of order $n \geq 3$ over such a field is conjugate by a diagonal matrix to a net whose entries all equal one subfield, so it is completable.

This note gives explicit examples with short proofs, including examples over fields of transcendence degree one. All of them come from an elementary lifting lemma (Lemma 2.2). Theorem 1.1 is an explicit relative of Nuzhin's construction and uses the same free-product property (Lemma 3.1). Theorem 1.2 lifts the two exceptional pairs in Levchuk's refinement of Dickson's theorem [Lev83].

Theorem 1.1. *Let F be a field, $n \geq 3$, $R = F[x, y]$, $K = F(x, y)$ and $J = yR$. Put*

$$\sigma_{12} = \sigma_{21} = Fx + J, \qquad \sigma_{ij} = J \quad \text{for all other } i \neq j.$$

Then σ is an irreducible closed elementary net of order n over K that is not completable.

Theorem 1.2. *Let $n \geq 3$.*

(a) *Let $R = \mathbb{F}_3[t]$, $K = \mathbb{F}_3(t)$, $f = t^2 + 1$ and $J = fR$. Then*

$$\sigma_{12} = \mathbb{F}_3 + J, \qquad \sigma_{21} = \mathbb{F}_3\, t + J, \qquad \sigma_{ij} = J \quad \text{for all other } i \neq j$$

is an irreducible closed elementary net of order n over K that is not completable.

(b) *Let $R = \mathbb{F}_2[t]$, $K = \mathbb{F}_2(t)$, $f = t^2 + t + 1$ and $J = fR$. Then*

$$\sigma_{12} = \mathbb{F}_2 + J, \qquad \sigma_{21} = \mathbb{F}_2\, t + J, \qquad \sigma_{ij} = J \quad \text{for all other } i \neq j$$

is an irreducible closed elementary net of order n over K that is not completable.

(c) *In both cases every σ_{ij} is a module over the polynomial ring $R_0 = \mathbb{F}_p[f]$, a principal ideal domain, and K is an algebraic extension of degree 2 of the field of fractions $\mathbb{F}_p(f)$ of R_0.*

Part (c) says that the nets of Theorem 1.2 also satisfy the hypotheses of the second paragraph of Problem 19.48 [KM26]; Problem 21.76 refers to 19.48 only for its definitions. So, in characteristic 3, the answer to Problem 21.76 remains affirmative under these additional hypotheses. The nets are closed, so they do not bear on the question asked in Problem 19.48. The quadratic extension in (c) is essential: if K is the field of fractions of a Dedekind domain R_0, for instance of a principal ideal domain, then every irreducible elementary net of order $n \geq 3$ over K whose entries are R_0-modules is conjugate by a diagonal matrix to a net of fractional ideals π_{ij} of an intermediate ring P with $\pi_{ij}\pi_{ji} \subseteq P$ [Koi25], see also [DKN18]; by Lemma 2.1 such a net is completable.

Corollary 1.3. *Let K be a field of characteristic $p \in \{2, 3\}$ and $n \geq 3$. There is an irreducible closed elementary net of order n over K that is not completable if and only if K is not algebraic over $\mathbb{F}_p$.*

For $p \geq 5$, Theorem 1.1 (or [Nuz26]) gives examples over every field of transcendence degree at least 2 over $\mathbb{F}_p$, by the transport argument in the proof of Corollary 1.3, and there are none over algebraic extensions of $\mathbb{F}_p$. This leaves open the fields of transcendence degree one over $\mathbb{F}_p$. An example over $\mathbb{F}_p(t)$ would give examples over all of them.

Question 1.4. *Let $p \geq 5$ and $n \geq 3$. Is there an irreducible closed elementary net of order n over $\mathbb{F}_p(t)$ that is not completable?*

2 Two lemmas

The first lemma is a well-known criterion; see [Bor87], [Lev82, Lemma 6], [GK20] and [Nuz26, Lemma 9]. We include the short proof.

Lemma 2.1. *An elementary net σ of order $n \geq 2$ over a field K is completable if and only if $\sigma_{ij}\sigma_{ji}\sigma_{ij} \subseteq \sigma_{ij}$ for all $i \neq j$. The condition does not depend on the field K containing the σ_{ij}.*

Proof. If additive subgroups σ_{ii} complete σ to a net, then $\sigma_{ij}\sigma_{ji} \subseteq \sigma_{ii}$ and $\sigma_{ii}\sigma_{ij} \subseteq \sigma_{ij}$, so $\sigma_{ij}\sigma_{ji}\sigma_{ij} \subseteq \sigma_{ij}$.

Conversely, assume the inclusions and let σ_{ii} be the subring of K generated by $\bigcup_{r\neq i} \sigma_{ir}\sigma_{ri}$. By construction $\sigma_{ir}\sigma_{ri} \subseteq \sigma_{ii}$ for $r \neq i$, and $\sigma_{ii}\sigma_{ii} \subseteq \sigma_{ii}$. For $\sigma_{ii}\sigma_{ij} \subseteq \sigma_{ij}$ with $j \neq i$ it suffices to show $abc \in \sigma_{ij}$ for $a \in \sigma_{ir}$, $b \in \sigma_{ri}$ $(r \neq i)$ and $c \in \sigma_{ij}$; products of several such generators and sums are then handled by induction. If $r \neq j$, then $bc \in \sigma_{ri}\sigma_{ij} \subseteq \sigma_{rj}$ and $a(bc) \in \sigma_{ir}\sigma_{rj} \subseteq \sigma_{ij}$. If $r = j$, then $abc \in \sigma_{ij}\sigma_{ji}\sigma_{ij} \subseteq \sigma_{ij}$. The inclusion $\sigma_{ij}\sigma_{jj} \subseteq \sigma_{ij}$ is symmetric. So $(\sigma_{ij})_{1\leq i,j\leq n}$ is a net. The last sentence of the lemma is clear. ☐

The second lemma lifts a closed pair from a quotient ring; it is an elementary congruence argument. For a ring Q we write $U_{12}(Q) = t_{12}(Q)$ and $U_{21}(Q) = t_{21}(Q)$ for the root subgroups of $\mathrm{SL}_2(Q)$. A pair (A_0, B_0) of additive subgroups of Q is *closed* if the group $H = \langle t_{12}(A_0), t_{21}(B_0)\rangle \leq \mathrm{SL}_2(Q)$ satisfies $H \cap U_{12}(Q) = t_{12}(A_0)$ and $H \cap U_{21}(Q) = t_{21}(B_0)$, that is, if the elementary net (A_0, B_0) of order 2 over Q is closed.

Lemma 2.2 (Lifting lemma)**.** *Let R be a subring of a field K, let $J \neq 0$ be an ideal of R, and let $\pi\colon R \to Q = R/J$ be the quotient map. Let A_0, B_0 be additive subgroups of Q such that the group $H = \langle t_{12}(A_0), t_{21}(B_0)\rangle \leq \mathrm{SL}_2(Q)$ satisfies*

$$H \cap U_{12}(Q) = t_{12}(A_0) \qquad \text{and} \qquad H \cap U_{21}(Q) = t_{21}(B_0).$$

For $n \geq 3$ put $\sigma_{12} = \pi^{-1}(A_0)$, $\sigma_{21} = \pi^{-1}(B_0)$ and $\sigma_{ij} = J$ for all other $i \neq j$. Then σ is an irreducible closed elementary net of order n over K. If $A_0B_0A_0 \not\subseteq A_0$, then σ is not completable.

Proof. Net inclusions. Let i, r, j be pairwise distinct. The pairs (i, r) and (r, j) cannot both lie in $\{(1, 2), (2, 1)\}$, since then i, r, j would all lie in $\{1, 2\}$. So one of σ_{ir}, σ_{rj} equals J and the other is contained in R. Hence $\sigma_{ir}\sigma_{rj} \subseteq JR = J \subseteq \sigma_{ij}$. All σ_{ij} contain $J \neq 0$, so σ is irreducible.

Closed. All generators of $E(\sigma)$ lie in $\mathrm{SL}_n(R)$, so $E(\sigma) \leq \mathrm{SL}_n(R)$. Let $t_{ij}(\beta) \in E(\sigma)$ with $\beta \in K$. Then β is an entry of a matrix in $\mathrm{SL}_n(R)$, so $\beta \in R$. Let $\pi_*\colon \mathrm{SL}_n(R) \to \mathrm{SL}_n(Q)$ be the homomorphism induced by π. It maps $t_{12}(\alpha)$ to $t_{12}(\pi(\alpha))$ with $\pi(\alpha) \in A_0$ for $\alpha \in \sigma_{12}$, similarly for $(2, 1)$, and $t_{ij}(\alpha)$ to 1 for $\alpha \in J$. Hence $\pi_*(E(\sigma))$ is contained in the copy H_0 of H in the upper left 2×2 block of $\mathrm{SL}_n(Q)$, and $t_{ij}(\pi(\beta)) = \pi_*(t_{ij}(\beta)) \in H_0$. If $\{i, j\} \neq \{1, 2\}$, the (i, j) entry of every element of H_0 is zero, so $\pi(\beta) = 0$ and $\beta \in J = \sigma_{ij}$. If $(i, j) = (1, 2)$, then the upper left 2×2 block of $\pi_*(t_{12}(\beta))$ is $t_{12}(\pi(\beta)) \in H \cap U_{12}(Q) = t_{12}(A_0)$, so $\pi(\beta) \in A_0$ and $\beta \in \sigma_{12}$. The case $(i, j) = (2, 1)$ is the same.

Not completable. If $A_0B_0A_0 \not\subseteq A_0$, there are $a, a' \in A_0$ and $b \in B_0$ with $aba' \notin A_0$. Choose preimages $\alpha, \alpha' \in \sigma_{12}$ and $\beta' \in \sigma_{21}$. Then $\pi(\alpha\beta'\alpha') = aba' \notin A_0$, so $\alpha\beta'\alpha' \in \sigma_{12}\sigma_{21}\sigma_{12} \setminus \sigma_{12}$, and σ is not completable by Lemma 2.1. ☐

3 Two variables

The closed pair for Theorem 1.1 comes from a free-product property of two unipotent subgroups of $\mathrm{SL}_2(F[x])$. Lemma 3.1 is the case $A = B = Fx$ of Koibaev's theorem that any two additive subgroups of the polynomials without constant term over a domain form a closed pair [Koi11a], and of [Nuz26, Lemma 8]. We include the short proof for completeness.

Lemma 3.1. *Let F be a field and $H = \langle U, L\rangle \leq \mathrm{SL}_2(F[x])$, where $U = \{t_{12}(cx) : c \in F\}$ and $L = \{t_{21}(cx) : c \in F\}$. Then*

$$H \cap U_{12}(F[x]) = U \qquad \text{and} \qquad H \cap U_{21}(F[x]) = L.$$

Proof. Every element of H can be written as a reduced word $g = w_k \cdots w_1$ with $k \geq 0$, whose letters lie in $(U \cup L) \setminus \{1\}$ and alternate between U and L. Each letter is $t_{12}(cx)$ or $t_{21}(cx)$ with $c \neq 0$.

For a vector $v = (v_1, v_2) \in F[x]^2$ write $d_i = \deg v_i$, with $\deg 0 = -\infty$. If $d_1 > d_2$, then $t_{21}(cx)v = (v_1,\, v_2 + cxv_1)$ with $c \neq 0$, and the second coordinate has degree $d_1 + 1$, which exceeds the degree d_1 of the first. Symmetrically, if $d_2 > d_1$, then the first coordinate of $t_{12}(cx)v$ has degree $d_2 + 1$, which exceeds d_2.

Suppose first that $k \geq 1$ and $w_1 \in L$. Apply $w_1, w_2, \ldots, w_k$ in turn to $e_1 = (1, 0)$, for which $d_1 = 0 > d_2 = -\infty$. By the previous paragraph the coordinate of larger degree alternates at each step, and the largest degree increases by one each time, since the letters alternate between L and U starting with L. Hence ge_1 has a coordinate of degree $k \geq 1$, so $ge_1 \neq e_1$. Every element of $U_{12}(F[x])$ fixes e_1, so $g \notin U_{12}(F[x])$.

Now suppose that $w_1 \in U$ and that g contains a letter from L. Then $k \geq 2$ and $gw_1^{-1} = w_k \cdots w_2$ is a reduced word whose rightmost letter lies in L. By the first case $gw_1^{-1} \notin U_{12}(F[x])$. As $w_1 \in U_{12}(F[x])$, also $g \notin U_{12}(F[x])$.

Therefore an element of $H \cap U_{12}(F[x])$ is a word without letters from L, that is, an element of U. The reverse inclusion is clear. The statement for U_{21} follows by exchanging the two coordinates. □

Proof of Theorem 1.1. Apply Lemma 2.2 to $R = F[x, y] \subseteq K$, $J = yR$ and $\pi\colon R \to R/J = F[x]$, $y \mapsto 0$, with $A_0 = B_0 = Fx$. The pair is closed by Lemma 3.1, and $\pi^{-1}(Fx) = Fx + J$. Finally $x \cdot x \cdot x = x^3 \notin Fx$, so $A_0B_0A_0 \not\subseteq A_0$. □

Remark 3.2. Explicitly, if σ_{11} were a diagonal entry completing the net of Theorem 1.1, then $x^2 \in \sigma_{12}\sigma_{21} \subseteq \sigma_{11}$ and $x^3 \in \sigma_{11}\sigma_{12} \subseteq \sigma_{12} = Fx + yF[x, y]$, which is false. The pair (Fx, Fx) alone is closed, and $Fx \cdot Fx \cdot Fx = Fx^3 \not\subseteq Fx$ is what prevents completion.

4 One variable

Proof of Theorem 1.2. (a) Here $Q = \mathbb{F}_3[t]/(t^2 + 1) = \mathbb{F}_9$. Write ι for the image of t, so $\iota^2 = -1$ and $\mathbb{F}_9 = \mathbb{F}_3 \oplus \mathbb{F}_3\iota$. Take $A_0 = \mathbb{F}_3$ and $B_0 = \mathbb{F}_3\iota$, and let $H = \langle t_{12}(A_0), t_{21}(B_0)\rangle = \langle t_{12}(1), t_{21}(\iota)\rangle \leq \mathrm{SL}_2(9)$. We have $|H| = 120$. Indeed $A_0B_0 = \mathbb{F}_3\iota$, so by Levchuk's theorem [Lev83] the image of H in $\mathrm{PSL}_2(9)$ is isomorphic to A_5, and H contains a matrix $\left(\begin{smallmatrix} 0 & s \\ -s^{-1} & 0 \end{smallmatrix}\right)$, whose square is -1. Hence $|H| = 2 \cdot 60 = 120$. As -1 is the only involution of $\mathrm{SL}_2(9)$, the group H is not $A_5 \times C_2$, and so $H \cong \mathrm{SL}_2(5)$. Independently, closing the two generators under multiplication inside $\mathrm{SL}_2(9)$, a group of order 720, gives a group of order 120 whose elements have orders $1, 2, 3, 4, 5, 6, 10$ with multiplicities $1, 1, 20, 30, 24, 20, 24$, as in $\mathrm{SL}_2(5)$ (Section 6). As $|H| = 120 = 2^3 \cdot 3 \cdot 5$, the Sylow 3-subgroups of H have order 3. The group $H \cap U_{12}(\mathbb{F}_9)$ is a subgroup of the elementary abelian group $U_{12}(\mathbb{F}_9)$ of order 9, so it is a 3-group, and it contains $t_{12}(\mathbb{F}_3)$, which has order 3. Hence $H \cap U_{12}(\mathbb{F}_9) = t_{12}(A_0)$, and in the same way

$H \cap U_{21}(\mathbb{F}_9) = t_{21}(B_0)$. Moreover $1 \cdot \iota \cdot 1 = \iota \notin \mathbb{F}_3$, so $A_0 B_0 A_0 \not\subseteq A_0$. Now Lemma 2.2 applies, and $\pi^{-1}(\mathbb{F}_3) = \mathbb{F}_3 + J$, $\pi^{-1}(\mathbb{F}_3 \iota) = \mathbb{F}_3\, t + J$.

(b) Here $Q = \mathbb{F}_2[t]/(t^2+t+1) = \mathbb{F}_4$. Write ω for the image of t, so $\omega^2 = \omega + 1$. Take $A_0 = \mathbb{F}_2$, $B_0 = \mathbb{F}_2\omega$, and let $a = t_{12}(1)$, $b = t_{21}(\omega)$ and $H = \langle a, b \rangle$. In characteristic 2 we have $a^2 = b^2 = 1$. The product

$$g = ab = \begin{pmatrix} 1+\omega & 1 \\ \omega & 1 \end{pmatrix}$$

has trace ω and determinant 1, so it satisfies $g^2 + \omega g + 1 = 0$. The polynomial $X^2 + \omega X + 1$ divides $X^5 - 1$ over $\mathbb{F}_4$, since $(X^2+\omega X+1)(X^2+\omega^2 X+1) = X^4+X^3+X^2+X+1$. Hence $g^5 = 1$, and $g \neq 1$, so g has order 5 and H is dihedral of order 10; this is the dihedral case of Levchuk's theorem [Lev83]. Its Sylow 2-subgroups have order 2. As in (a), $H \cap U_{12}(\mathbb{F}_4)$ is a 2-subgroup containing $t_{12}(\mathbb{F}_2) = \{1, a\}$, so it equals $t_{12}(A_0)$, and $H \cap U_{21}(\mathbb{F}_4) = t_{21}(B_0)$. Moreover $1 \cdot \omega \cdot 1 = \omega \notin \mathbb{F}_2$. Now Lemma 2.2 applies, with $\pi^{-1}(\mathbb{F}_2) = \mathbb{F}_2 + J$ and $\pi^{-1}(\mathbb{F}_2\omega) = \mathbb{F}_2\, t + J$.

(c) In both cases $f \in J$ and $\sigma_{ij} \subseteq R$, so $f\sigma_{ij} \subseteq fR = J \subseteq \sigma_{ij}$. Hence every σ_{ij} is an $\mathbb{F}_p[f]$-module. As f is transcendental over $\mathbb{F}_p$, the ring $\mathbb{F}_p[f]$ is a polynomial ring in one variable, hence a principal ideal domain. Finally t is a root of $X^2 + 1 - f$, resp. $X^2 + X + 1 - f$, over $\mathbb{F}_p(f)$, so $K = \mathbb{F}_p(t)$ has degree at most 2 over $\mathbb{F}_p(f)$. The degree is exactly 2, because $t \notin \mathbb{F}_p(f)$: the automorphism $t \mapsto -t$, resp. $t \mapsto t+1$, of $\mathbb{F}_p(t)$ fixes f but not t. $\square$

Remark 4.1. With $A_0 = \mathbb{F}_3$, the other lines B_0 of $\mathbb{F}_9$ behave differently. The line $\mathbb{F}_3$ gives $\mathrm{SL}_2(3)$, a closed pair with $A_0 B_0 A_0 \subseteq A_0$. The lines $\mathbb{F}_3(1 \pm \iota)$ generate $\mathrm{SL}_2(9)$, so these pairs are not closed. Over $\mathbb{F}_4$, both lines $\mathbb{F}_2\omega$ and $\mathbb{F}_2\omega^2$ give dihedral groups of order 10.

Remark 4.2. The lifting lemma also applies in characteristic 0. The quotient map $\mathbb{Z}[i] \to \mathbb{Z}[i]/3\mathbb{Z}[i] \cong \mathbb{F}_9$ and the pair $(\mathbb{F}_3, \mathbb{F}_3 i)$ of part (a) give the closed elementary net $\sigma_{12} = \mathbb{Z} + 3\mathbb{Z}[i]$, $\sigma_{21} = \mathbb{Z}i + 3\mathbb{Z}[i]$, $\sigma_{ij} = 3\mathbb{Z}[i]$ otherwise, over the number field $\mathbb{Q}(i)$, and it is not completable. So in characteristic 0, unlike in characteristic p, such nets exist over some fields that are algebraic over the prime field. The entries of this net are modules over the order $\mathbb{Z}[3i]$ but not over $\mathbb{Z}[i]$; for instance $i \cdot 1 \notin \sigma_{12}$. This is necessarily so: by a theorem of Koibaev [Koi20], every irreducible elementary net of order $n \geq 3$ over $\mathbb{Q}(i)$ whose entries are $\mathbb{Z}[i]$-modules is conjugate by a diagonal matrix to a net of ideals of an intermediate ring, and such a net is completable. Examples in characteristic 0 were given earlier by Koibaev [Koi21]; we do not know whether his example is of this kind.

5 Algebraic extensions and the proof of Corollary 1.3

The following proposition is a special case of a theorem of Koibaev and Nuzhin [KN17], applied with $k = \mathbb{F}_p$: every additive subgroup of K is an $\mathbb{F}_p$-subspace. For the reader's convenience we include a short direct proof.

Proposition 5.1. *Let K be an algebraic extension of $\mathbb{F}_p$ and $n \geq 3$. Then every irreducible elementary net of order n over K is completable.*

Proof. Every subring P of K (with $1 \in P$) is a field: each finitely generated subring of P lies in a finite subfield of K, so it is a finite domain and hence a field, and P is the union of these.

By Lemma 2.1 it suffices to show $ABA \subseteq A$ for $A = \sigma_{ij}$ and $B = \sigma_{ji}$, where $i \neq j$. Choose $k \notin \{i, j\}$, and let P be the subring of K generated by all products ab with $a \in A$, $b \in B$. For such a, b the net inclusions give, for c in the indicated subgroup,

$$c \in \sigma_{ik}: \ bc \in \sigma_{jk},\ a(bc) \in \sigma_{ik}; \qquad c \in \sigma_{kj}: \ cb \in \sigma_{ki},\ (cb)a \in \sigma_{kj};$$
$$c \in \sigma_{jk}: \ ac \in \sigma_{ik},\ b(ac) \in \sigma_{jk}; \qquad c \in \sigma_{ki}: \ ca \in \sigma_{kj},\ (ca)b \in \sigma_{ki}.$$

So σ_{ik}, σ_{kj}, σ_{jk} and σ_{ki} are P-modules. Choose nonzero $u \in \sigma_{ik}$, $v \in \sigma_{kj}$, $u' \in \sigma_{jk}$, $v' \in \sigma_{ki}$, and put $w = uv$ and $w' = u'v'$. Then $Pw = (Pu)v \subseteq \sigma_{ik}\sigma_{kj} \subseteq A$ and, in the same way, $Pw' \subseteq B$. As P is a field and $0 \neq ww' \in AB \subseteq P$, the element ww' is invertible in P. For $a \in A$ we have $aw' \in AB \subseteq P$, so $a \in Pw'^{-1} = Pw$. Hence $A = Pw$, and similarly $B = Pw'$. Therefore $ABA = P\,ww'w = Pw = A$. □

Proof of Corollary 1.3. If K is algebraic over $\mathbb{F}_p$, every irreducible elementary net of order $n \geq 3$ over K is completable by Proposition 5.1.

Otherwise choose $s \in K$ transcendental over $\mathbb{F}_p$. Then $t \mapsto s$ defines an embedding $\mathbb{F}_p(t) \to K$, and the image of the net of Theorem 1.2 is an irreducible elementary net σ of order n over K. Its net group $E(\sigma)$ lies in $\mathrm{SL}_n(\mathbb{F}_p(s))$, so every transvection $t_{ij}(\beta) \in E(\sigma)$ has $\beta \in \mathbb{F}_p(s)$, and σ is closed over K because it is closed over $\mathbb{F}_p(s)$. It is not completable over K, since $\sigma_{12}\sigma_{21}\sigma_{12} \not\subseteq \sigma_{12}$ and the criterion of Lemma 2.1 does not depend on the field. □

Remark 5.2. For $p \geq 5$, lifting from a field cannot produce examples. Let Q be a field that is algebraic over $\mathbb{F}_p$, $p \geq 5$, and let (A_0, B_0) be a closed pair of nonzero additive subgroups of Q. By Levchuk's theorem [Lev83], the group $\langle t_{12}(A_0), t_{21}(B_0)\rangle$ is conjugate to $\mathrm{SL}_2(P)$ by a diagonal matrix $\mathrm{diag}(d_1, d_2)$, where P is the subfield generated by A_0B_0. With $c = d_1/d_2$ its intersections with $U_{12}(Q)$ and $U_{21}(Q)$ are $t_{12}(cP)$ and $t_{21}(c^{-1}P)$. So closedness gives $A_0 = cP$ and $B_0 = c^{-1}P$, and both $A_0B_0A_0 \subseteq A_0$ and $B_0A_0B_0 \subseteq B_0$ hold. A computer search confirms this for $q \in \{25, 49, 121, 125\}$, and it finds no closed pair violating one of these inclusions over the rings $\mathbb{F}_5[\varepsilon]/(\varepsilon^2)$, $\mathbb{F}_5[\varepsilon]/(\varepsilon^3)$, $\mathbb{F}_7[\varepsilon]/(\varepsilon^2)$, $\mathbb{F}_7[\varepsilon]/(\varepsilon^3)$, $\mathbb{F}_5 \times \mathbb{F}_5$, $\mathbb{F}_5^3$, $\mathbb{F}_5[\varepsilon]/(\varepsilon^2) \times \mathbb{F}_5$ and $\mathbb{F}_7 \times \mathbb{F}_7$ either. The one-variable examples above use the two exceptional cases of Levchuk's theorem. We do not know the answer to Question 1.4.

6 Verification and scope

The proofs above are complete; for $|H| = 120$ in Theorem 1.2(a) they use Levchuk's theorem, and this order was also checked by a direct computation. The following exact computations were used as checks. The scripts, written independently by the author and by two referees, and their outputs are in the source archive (folder `reproducibility`).

1. Lemma 3.1: all reduced words of length at most 20 over $\mathbb{F}_3$, at most 10 over $\mathbb{F}_5$ and at most 8 over $\mathbb{F}_7$, and 1500 random reduced words of length up to 400 over each of these fields, satisfy the degree invariant, and no word of length at least 2 is unitriangular.

2. Theorem 1.1: the net inclusions for $n = 3, 4$ on spanning monomials and on random products; and all words of length at most 5 in 24 generators of $E(\sigma)$ for $n = 3$ over $\mathbb{F}_3$: every transvection that occurs (228 of them) has its entry in σ_{ij}.

3. Theorem 1.2 and Remarks 4.1 and 4.2: the group H of part (a) has order 120 with the element orders of $\mathrm{SL}_2(5)$, is perfect with centre $\{\pm 1\}$, and meets $U_{12}(\mathbb{F}_9)$ and $U_{21}(\mathbb{F}_9)$ in $t_{12}(\mathbb{F}_3)$ and $t_{21}(\mathbb{F}_3\iota)$; the group of part (b) has order 10; the net inclusions, the witnesses and the module property hold on random products. All words of length at most 6 (over $\mathbb{F}_3(t)$), 7 (over $\mathbb{F}_2(t)$) and 5 (over $\mathbb{Q}(i)$) in generators of $E(\sigma)$ for $n = 3$ produce only transvections with entries in σ_{ij}.

4. Proposition 5.1: every irreducible elementary net of order 3 over $\mathbb{F}_q$ for $q =$ 4, 8, 9, 16, 25, 27, 32, 49, 81, and of order 4 over $\mathbb{F}_q$ for $q =$ 4, 8, 9, 25, 27, normalised by diagonal conjugation, is completable.

5. Remark 5.2: all pairs of additive subgroups over the listed fields and rings, up to diagonal conjugation, with closedness decided exactly.

Scope and priority. To our knowledge Problem 21.76 was first answered by Nuzhin [Nuz26], whose paper appeared in July 2026. Version 46 of the notebook (1 September 2026, still the current version on 28 September 2026) does not yet record this. The contributions of this note are the explicit nets of Theorems 1.1 and 1.2 with elementary proofs, the one-variable examples in characteristic 3, which also satisfy the hypotheses of Problem 19.48, and Corollary 1.3 for $p = 3$. They combine Levchuk's exceptional pair with an elementary reduction modulo an ideal. In characteristics 0 and 2 examples were known earlier [Koi11b, Koi21]. We have not seen the full text of [Koi11b], and for $p = 2$ Corollary 1.3 may already follow from it together with [KN17]. Searches of zbMATH, Crossref, OpenAlex, mathnet.ru and arXiv in September 2026 found no other treatment of examples over fields of transcendence degree one in characteristic 3. This negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the constructions and the proofs, to write and run the verification programs, to referee the arguments, and to help prepare this manuscript. The author has reviewed the proofs and the sources and is responsible for all claims. The note has not been peer reviewed.

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