A Semi-Abelian Group of Order 768
That Is Not an M-Group

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Abstract

M. Kida conjectured that every finite semi-abelian group is monomial (J. Group Theory, 2025; Kourovka Notebook, Problem 21.68). We show that the conjecture is false. If a finite group W acts on a finite abelian group N, a linear character of N has stabiliser T in W, and T has an irreducible character that is not monomial, then $N \rtimes W$ is not an M-group. We apply this to the augmentation submodule of the permutation module $\mathbb{F}_2[W/T]$, where $W = E \rtimes A_4$ is the index-two subgroup of $C_2 \wr A_4$ and $T \cong \mathrm{SL}(2,3)$ is the binary tetrahedral group acting on the eight quaternion units. The result is a semi-abelian group of order $768 = 2^8 \cdot 3$ with a non-monomial irreducible character of degree 8. The proof uses only Clifford theory. The example was also checked by exact computation, including every subgroup of index 8. This is an unrefereed note; minimality of the order is not claimed.

1 Introduction

All groups are finite and all characters are complex. A character is *monomial* if it is induced from a linear character of some subgroup, and a group is an *M-group* (a *monomial group*) if all its irreducible characters are monomial. The following class is defined in the Kourovka Notebook [KM26].

Definition 1.1 (Kourovka Notebook, Problem 21.68)**.** A finite group G is *semi-abelian* if it has a sequence of subgroups $1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$ such that for every i the subgroup G_{i+1} is isomorphic to a quotient of a semidirect product $A_i \rtimes G_i$ for some abelian group A_i.

Groups of this kind occur in inverse Galois theory [Tho84, Den95]. Semi-abelian groups are solvable, and so are M-groups. M. Kida asked in [KM26, Problem 21.68]:

Conjecture: Semi-abelian finite groups are monomial.

The conjecture first appeared as [Kid25, Conjecture 1.3]. There Kida reports a Magma computation showing that no semi-abelian group of order at most 240 fails to be monomial. He also proves [Kid25, Theorem 1.4] that every member of an isoclinism-invariant class of finite groups that is closed under subgroups and homomorphic images, and consists of semi-abelian groups, is monomial.

Theorem 1.2. *There is a semi-abelian group of order* 768 *that is not an M-group. In particular, the conjecture of Problem 21.68 of the Kourovka Notebook is false.*

Corollary 1.3. *There are infinitely many semi-abelian groups that are not M-groups; for instance* $\widehat{G} \times A$ *for every finite abelian group* A*, where* $\widehat{G}$ *is the group of Theorem 1.2.*

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Proof. $\widehat{G} \times A = A \rtimes \widehat{G}$ with trivial action, which adds one step to a semi-abelian sequence of $\widehat{G}$. Quotients of M-groups are M-groups: if an irreducible character with kernel containing M is induced from a linear character λ of U, then $M \leq \mathrm{core}(\ker \lambda) \leq U$ by (K) in Section 2, and the character is induced from U/M. Since $\widehat{G}$ is a quotient of $\widehat{G} \times A$, the latter is not an M-group. □

The example is a split extension $\widehat{G} = B \rtimes W$, where B is elementary abelian of order 8 and W is itself an M-group (Remark 3.2). Its failure to be an M-group comes from a subgroup $T \cong \mathrm{SL}(2,3)$ of W: an irreducible character of $\widehat{G}$ whose Clifford correspondent involves a non-monomial character of T cannot be monomial. Section 2 proves this in general (Theorem 2.2). Section 3 builds the example of order 768. Section 4 describes the computer checks and the scope of the result.

Dade showed that normal subgroups of M-groups need not be M-groups [Dad73]. The phenomenon here is of a different kind, but the tools are the same standard Clifford theory. We use [Isa06] as the reference for character theory.

2 Clifford theory and monomial characters

For $H \leq G$ and a class function φ of H, φ^G denotes the induced class function. For $\mu \in \mathrm{Irr}(N)$ with $N \trianglelefteq G$, G_μ is the stabiliser of μ under conjugation, and $\mathrm{Irr}(G \mid \mu)$ is the set of irreducible characters of G whose restriction to N contains μ. We use the following standard facts.

(C) (Clifford correspondence [Isa06, Theorem 6.11]) Induction $\theta \mapsto \theta^G$ is a bijection $\mathrm{Irr}(G_\mu \mid \mu) \to \mathrm{Irr}(G \mid \mu)$. For $\theta \in \mathrm{Irr}(G_\mu \mid \mu)$ one has $\theta_N = e\mu$ with $e = [(\theta^G)_N, \mu]$.

(G) (Gallagher [Isa06, Corollary 6.17]) If μ extends to $\tilde{\mu} \in \mathrm{Irr}(G_\mu)$, then $\mathrm{Irr}(G_\mu \mid \mu) = \{\tilde{\mu}\beta : \beta \in \mathrm{Irr}(G_\mu/N)\}$.

(K) $\ker(\varphi^G) = \mathrm{core}_G(\ker \varphi)$ for a character φ of $H \leq G$ [Isa06, Lemma 5.11], and $(\varphi^G)\zeta = (\varphi\, \zeta_H)^G$ for every class function ζ of G [Isa06, Problem 5.3].

Lemma 2.1. *Let $N \trianglelefteq G$ be abelian, let $\chi \in \mathrm{Irr}(G)$, let $\mu \in \mathrm{Irr}(N)$ be a constituent of χ_N, and let $\theta \in \mathrm{Irr}(G_\mu \mid \mu)$ be the Clifford correspondent of χ, so that $\theta^G = \chi$. If χ is monomial, then θ is monomial.*

Proof. Write $\chi = \lambda^G$ with λ a linear character of a subgroup $U \leq G$, and put $\eta := \lambda^{NU}$. Since $\eta^G = \chi$ is irreducible, η is irreducible. As NU consists of a single (N,U)-double coset, Mackey's formula gives $\eta_N = (\lambda_{N\cap U})^N$. Because N is abelian, by Frobenius reciprocity this is the sum of the $[N : N \cap U]$ distinct linear characters of N extending $\lambda_{N\cap U}$, each with multiplicity one.

The constituents of χ_N form one G-orbit and contain those of η_N. Replacing (μ, G_μ, θ) by a G-conjugate $(\mu^g, G_\mu^g, \theta^g)$ does not affect whether θ is monomial, and θ^g is the Clifford correspondent of χ over μ^g. So we may assume that μ is a constituent of η_N, necessarily with multiplicity one.

Since N is abelian, it acts trivially on $\mathrm{Irr}(N)$, so the stabiliser of μ in NU is NU_μ. By (C) applied to $\eta \in \mathrm{Irr}(NU \mid \mu)$, there is $\theta_0 \in \mathrm{Irr}(NU_\mu \mid \mu)$ with ${\theta_0}^{NU} = \eta$ and $(\theta_0)_N = [\eta_N, \mu]\, \mu = \mu$. Thus θ_0 is linear. Since $NU_\mu \leq G_\mu$, the character ${\theta_0}^{G_\mu}$ is defined, and $({\theta_0}^{G_\mu})^G = {\theta_0}^G = \eta^G = \chi$ is irreducible. Hence ${\theta_0}^{G_\mu}$ is irreducible. It lies over μ, because θ_0 does and G_μ fixes μ. By the uniqueness in (C), ${\theta_0}^{G_\mu} = \theta$. So θ is induced from the linear character θ_0 and is monomial. □

Theorem 2.2. *Let a finite group W act on a finite abelian group N, and put $G := N \rtimes W$. Suppose that $\mu \in \mathrm{Irr}(N)$ has stabiliser $W_\mu = T$ in W, and that $\psi \in \mathrm{Irr}(T)$ is not monomial. Then G has an irreducible character of degree $[W : T]\, \psi(1)$ that is not monomial. In particular G is not an M-group.*

Proof. Since N is abelian, $G_\mu = N \rtimes T$. The formula $\tilde\mu(nt) := \mu(n)$ ($n \in N$, $t \in T$) defines a linear character of $N \rtimes T$ extending μ. It is multiplicative because μ is T-invariant. Let $\hat\psi$ be the inflation of ψ to $N \rtimes T \to T$. By (G), $\theta := \tilde\mu\hat\psi \in \mathrm{Irr}(N \rtimes T \mid \mu)$, and by (C), $\chi := \theta^G \in \mathrm{Irr}(G)$ with $\chi(1) = [G : N \rtimes T]\,\psi(1) = [W : T]\,\psi(1)$.

Suppose that χ is monomial. By Lemma 2.1, θ is monomial, say $\theta = \nu^{N\rtimes T}$ with ν linear on some $V \le N \rtimes T$. By the second part of (K), applied with the linear character $\zeta = \overline{\tilde\mu}$ of $N \rtimes T$,

$$\hat\psi = \theta\,\overline{\tilde\mu} = \left(\nu\,\overline{\tilde\mu}_V\right)^{N\rtimes T} = \nu'^{N\rtimes T}, \qquad \nu' := \nu\,\overline{\tilde\mu}_V \in \mathrm{Lin}(V).$$

By the first part of (K), $N \le \ker\hat\psi = \mathrm{core}_{N\rtimes T}(\ker\nu') \le \ker\nu' \le V$. Hence $V = N \rtimes (V \cap T)$ and ν' is the inflation of a linear character ν'' of $V \cap T \cong V/N$. Induction commutes with inflation, so $\hat\psi$ is the inflation of ν''^T. Thus $\psi = \nu''^T$ is monomial, a contradiction. ☐

Corollary 2.3. *Let W be a finite group, $T \le W$ a subgroup that is not an M-group, and p a prime. Let $M = \mathbb{F}_p[W/T]$ be the permutation module on the right cosets of T, with coordinates $m = (m_\kappa)_{\kappa\in W/T}$, and let $N \le M$ be either M itself or, when $[W : T] \ge 3$, the augmentation submodule $\{m : \sum_\kappa m_\kappa = 0\}$. Then $N \rtimes W$ is not an M-group. If W is semi-abelian, then so is $N \rtimes W$.*

Proof. Let $\zeta_p = e^{2\pi i/p}$ and let $\mu_\kappa(m) := \zeta_p^{m_\kappa}$ for $m \in N$. An element $w \in W$ transforms μ_T into $\mu_{Tw^{\pm1}}$ (the sign depends on conventions), so $w \in W_{\mu_T}$ if and only if $\mu_{Tw} = \mu_T$ on N. Distinct cosets give distinct characters of N. For $N = M$ this is clear. For the augmentation submodule and cosets $\kappa \ne \kappa'$, pick a third coset κ''; then $m = e_\kappa - e_{\kappa''} \in N$ has $m_\kappa = 1$ and $m_{\kappa'} = 0$. Hence $W_{\mu_T} = \{w : Tw = T\} = T$. Since T is not an M-group, some $\psi \in \mathrm{Irr}(T)$ is not monomial, and Theorem 2.2 applies. The last statement holds because $1 = G_0 \le \cdots \le G_n = W \le N \rtimes W$ extends a semi-abelian sequence of W by one step with the abelian group $A_n = N$. ☐

3 The example of order 768

Let $\mathcal{Q} = \{\pm1, \pm i, \pm j, \pm k\} \subset \mathbb{H}^\times$ be the quaternion group of order 8, regarded also as an 8-element set. Its elements fall into four pairs $\{\pm1\}, \{\pm i\}, \{\pm j\}, \{\pm k\}$, and $1, i, j, k$ are the chosen representatives. Every permutation of $\mathcal{Q}$ that maps pairs to pairs has the form $\pm u \mapsto \pm\varepsilon_u\,\pi(u)$ with π a permutation of $\{1, i, j, k\}$ and $\varepsilon_u \in \{\pm1\}$. The maps to $\mathrm{sgn}(\pi)$ and to $\prod_u \varepsilon_u$ are homomorphisms on the group $C_2 \wr S_4$ of such permutations.

- Let W be the group of permutations of $\mathcal{Q}$ that preserve the four pairs, induce an *even* permutation π of the pairs, and change an *even* number of signs. Then $W = E \rtimes A_4$, where $E \cong C_2^3$ consists of the even sign changes ($\pi = 1$) and A_4 of the even permutations π without sign changes. So $|W| = 96$, and W is the index-two subgroup of $C_2 \wr A_4$ singled out by the sign character.
- Let α be conjugation by $\omega = \frac{1}{2}(-1 + i + j + k)$, which fixes ±1 and permutes $i \mapsto j \mapsto k \mapsto i$. Let T be the group of permutations $x \mapsto q\,\alpha^m(x)$ of $\mathcal{Q}$ ($q \in \mathcal{Q}$, $m \in \mathbb{Z}/3$).

Lemma 3.1. *1. W is semi-abelian, via $1 \le C_3 \le A_4 \le W$ with $A_4 = V_4 \rtimes C_3$ and $W = E \rtimes A_4$.*

2. $T \le W$, and $T \cong \mathcal{Q} \rtimes C_3 \cong \mathrm{SL}(2,3)$. In particular $|T| = 24$ and $[W : T] = 4$.

3. $\mathrm{SL}(2,3)$ *is not an M-group.*

Proof. (1) Each step is a semidirect product $A \rtimes H$ of an abelian group A by the previous group H, and H sits inside the next group as a complement: $C_3 = C_3 \rtimes 1$, $A_4 = V_4 \rtimes C_3$, $W = E \rtimes A_4$.

(2) Left multiplication L_q by $q \in \mathcal{Q}$ preserves the pairs. The map L_i induces the even permutation $(1\,i)(j\,k)$ of the pairs and changes exactly two signs ($i \mapsto -1$ and $k \mapsto -j$); since $\alpha L_i \alpha^{-1} = L_{\alpha(i)}$, the same holds for L_j and L_k. The map L_{-1} changes all four signs, and $L_{-q} = L_{-1}L_q$. So all L_q lie in W. The permutation α induces the 3-cycle $(i\,j\,k)$ of the pairs and changes no signs, so $\alpha \in W$. The map $\mathcal{Q} \rtimes \langle\alpha\rangle \to \mathrm{Sym}(\mathcal{Q})$, $(q, \alpha^m) \mapsto (x \mapsto q\,\alpha^m(x))$, is a homomorphism because α is an automorphism of $\mathcal{Q}$. It is injective: evaluating at $x = 1$ recovers q, and then $\alpha^m = 1$. Hence $T \cong \mathcal{Q} \rtimes C_3$ with C_3 acting on $\mathcal{Q}$ by the automorphism $i \mapsto j \mapsto k$. This is the binary tetrahedral group, isomorphic to $\mathrm{SL}(2,3)$.

(3) $\mathrm{SL}(2,3)$ has irreducible characters of degree 2, for example the character of $\mathrm{SL}(2,3) \cong \langle\mathcal{Q}, \omega\rangle \subset \mathbb{H}^\times$ acting on $\mathbb{H} \cong \mathbb{C}^2$. A monomial character of degree 2 would be induced from a subgroup of index 2. But the derived subgroup of $\mathrm{SL}(2,3)$ is $\mathcal{Q}$, of index 3, so there is no subgroup of index 2. □

Proof of Theorem 1.2. Let B be the augmentation submodule of $\mathbb{F}_2[W/T]$, the even-weight vectors in $\mathbb{F}_2^{W/T} \cong \mathbb{F}_2^4$, so $B \cong C_2^3$. Let $\widehat{G} := B \rtimes W$, of order $8 \cdot 96 = 768$. By Lemma 3.1(1) and Corollary 2.3, $\widehat{G}$ is semi-abelian, with the explicit sequence

$$1 \le C_3 \le A_4 \le W \le \widehat{G}.$$

Since $[W : T] = 4 \ge 3$ and $T \cong \mathrm{SL}(2,3)$ is not an M-group by Lemma 3.1(3), Corollary 2.3 shows that $\widehat{G}$ is not an M-group. More precisely, with $\psi \in \mathrm{Irr}(T)$ of degree 2, Theorem 2.2 gives a non-monomial $\chi \in \mathrm{Irr}(\widehat{G})$ of degree $[W : T] \cdot 2 = 8$. □

Remark 3.2. The group W itself is an M-group. For a semidirect product $A \rtimes H$ with A abelian, (C) and (G) show that every irreducible character is $(\tilde{\mu}\hat{\beta})^{A\rtimes H}$ with $\mu \in \mathrm{Irr}(A)$ and $\beta \in \mathrm{Irr}(H_\mu)$. It is therefore monomial whenever β is monomial: if $\beta = \lambda^{H_\mu}$ with λ linear on $K \le H_\mu$, then $\tilde{\mu}\hat{\beta} = (\tilde{\mu}_{A\rtimes K}\hat{\lambda})^{A\rtimes H_\mu}$. All subgroups of A_4 are M-groups, so $W = E \rtimes A_4$ is an M-group. Thus $\widehat{G} = B \rtimes W$ is a split extension of an elementary abelian group by an M-group that is not an M-group.

Remark 3.3. The example does not contradict [Kid25, Theorem 1.4], because the class of semi-abelian groups is not closed under subgroups. The subgroup $T \cong \mathrm{SL}(2,3)$ of $\widehat{G}$ is not semi-abelian: a nontrivial semi-abelian group has an abelian normal subgroup not contained in its Frattini subgroup [Den95], whereas the abelian normal subgroups of $\mathrm{SL}(2,3)$ are 1 and the centre, which is the Frattini subgroup. Kida already observes [Kid25, Example 5.5] that a semi-abelian group $C_2^3 \rtimes A_4$ of order 96 contains a subgroup isomorphic to $\mathrm{SL}(2,3)$. The group W above has this shape, and the construction turns that failure of subgroup-closure into a failure of monomiality one level up. By [Kid25, Theorem 1.1] and [Tap76], semi-abelianity and monomiality are both invariant under isoclinism, so every group isoclinic to $\widehat{G}$ is also a semi-abelian group that is not an M-group.

Remark 3.4. The argument uses only that some semi-abelian group W contains a non-M subgroup T with $[W : T] \ge 3$. For every such pair and every prime p, Corollary 2.3 produces semi-abelian groups $\mathbb{F}_p[W/T] \rtimes W$ that are not M-groups. By Kida's computation [Kid25], the smallest order of a semi-abelian group that is not an M-group lies between 241 and 768; we have not determined it.

4 Computer verification and scope

The proof above is complete without computation. As a safeguard, the example was checked by exact computer calculations. The scripts are included in the source archive of this note. Points $0, \ldots, 7$ are the quaternion units, in the order $+1, -1, +i, -i, +j, -j, +k, -k$. Points $8 + 2\kappa + \epsilon$ ($\kappa \in \{0, 1, 2, 3\}$

indexing W/T, with $\kappa = 0$ the coset T, and $\epsilon \in \{0,1\}$) carry the permutation action of $\widehat{G}$ on $W/T \times \mathbb{F}_2$. Here $b \in B$ adds b_κ to ϵ. In this labelling $\widehat{G}$ is generated by

$$
\begin{aligned}
&(8,9)(10,11), \quad (8,9)(12,13), \quad (8,9)(14,15),\\
&(0,1)(2,3)(8,10)(9,11)(12,14)(13,15), \quad (2,3)(4,5)(8,14)(9,15)(10,12)(11,13),\\
&(4,5)(6,7)(8,10)(9,11)(12,14)(13,15), \quad (0,2,4)(1,3,5)(8,12,14)(9,13,15),\\
&(0,2)(1,3)(4,6)(5,7)(8,12)(9,13)(10,14)(11,15).
\end{aligned}
$$

The first three generate B and the remaining five generate W. The subgroup T is generated by $L_i = (0,2,1,3)(4,6,5,7)$, $L_j = (0,4,1,5)(2,7,3,6)$ and $\alpha = (2,4,6)(3,5,7)$ on the first eight points.

The following checks were performed with exact arithmetic.

1. *Quaternion model.* W, T, B and $\widehat{G}$ were built directly from the definitions of Section 3. The computation confirms $|W| = 96$, $T \le W$ with $|T| = 24$, a unique involution, a normal quaternion subgroup and derived subgroup of order 8; also $[W : T] = 4$, $|\widehat{G}| = 768$, and that the stabiliser of μ_T in W is exactly T.

2. *Structure.* A separately written check of the permutation group confirms $|\widehat{G}| = 768$, that B is a normal elementary abelian subgroup of order 8, and that W is a complement. It also confirms the semidirect-product decompositions $W = E \rtimes A_4$ and $A_4 = V_4 \rtimes C_3$, and computes the stabilisers of the seven nontrivial characters of B. Four of these stabilisers are isomorphic to $\mathrm{SL}(2,3)$ and three have order 32.

3. *Direct non-monomiality test, independent of Section 2.* The character $\chi = (\tilde{\mu}\hat{\psi})^{\widehat{G}}$ was computed with the induction formula, with ψ the rational character of degree 2 of T. It satisfies $\chi(1) = 8$ and $\langle \chi, \chi \rangle = 1$. If χ were induced from a linear character of U, then $[\widehat{G} : U] = 8$ and χ_U would have a linear constituent. Fix a Sylow 2-subgroup P of $\widehat{G}$, of order 256. Every subgroup of order 96 is conjugate to one whose Sylow 2-subgroup lies in P. Such a subgroup is generated by one of the 1587 subgroups of order 32 of P together with an element of order 3. All these subgroups were enumerated; there are 24 of them. For each of them, $\sum_{v \in U'} \chi(uv) = 0$ for every $u \in U$. So χ_U has no linear constituent, and χ is not monomial. The same test was run on a second realisation of the construction, with a different subgroup $T \cong \mathrm{SL}(2,3)$ of W, with the same outcome.

4. *Independent audit.* A separately written program by an independent reviewer enumerated all subgroups of order 96 in two ways, the second by classifying complements through 1-cocycles; both give the same 24 subgroups. It also computed the full character table of $\widehat{G}$, with 35 classes and degrees $1^3\, 3^{21}\, 4^6\, 8^3\, 12^2$. Exactly the three irreducible characters of degree 8 are not monomial, namely χ and its twists by the linear characters of $\widehat{G}$; all others are monomial.

Readers with a computer algebra system can confirm the conclusion directly from the generators above.

Scope and priority. The note answers the conjecture of Problem 21.68 in the negative. It uses exactly the definition printed in the notebook. Every step of the semi-abelian sequence is a genuine semidirect product with an abelian kernel, so variants of the definition that do not allow quotients are also covered. Version 46 of the notebook (27 September 2026) lists the problem without a solution. Searches of the arXiv and Crossref databases on the same date found no earlier answer. This negative search is not a proof of priority, and no priority claim is made. The minimal order of a semi-abelian group that is not an M-group (between 241 and 768) is left open.

AI-assisted tools were used to search the problem corpus and the literature, to find the construction, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proof and the sources and is responsible for all claims. The note has not been peer reviewed.

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