

Chordal Enhanced Power Graphs of Finite Simple Groups and Induced 4-Cycles: On Kourovka Notebook Problem 21.23

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Abstract

For a finite group G , the enhanced power graph $\mathcal{E}(G)$ has vertex set G , two distinct elements being adjacent when they generate a cyclic subgroup. Bubboloni, Fumagalli and Praeger asked (Kourovka Notebook, Problem 21.23) for which finite nonabelian simple groups $\mathcal{E}(G)$ has no induced cycle of a given length $n \geq 4$, and for which $\mathcal{E}(G)$ is chordal. By their results, $\mathcal{E}(G)$ is chordal for $\text{PSL}_2(q)$ ($q \geq 4$), the Suzuki groups, $\text{PSL}_3(4)$ and A_7 . We show that $\mathcal{E}(G)$ has an induced cycle of length at least 4 for M_{11} , J_1 , the Tits group, $\text{PSL}_3(q)$ with q odd, $\text{PSU}_3(q)$ with $q \geq 3$, and the Ree groups ${}^2G_2(q)$ with $q \geq 27$; for the infinite families the cycles are given by direct constructions. Combined with the inclusion, proved by Bubboloni, Fumagalli and Praeger in an unrefereed preprint by a case analysis over the classification of finite simple groups, of all simple groups without induced 4-cycles in an explicit list, this gives: $\mathcal{E}(G)$ is chordal if and only if G is one of the groups of the first list. For $n = 4$ we show that $\mathcal{E}(G)$ has no induced 4-cycle for the Tits group and, using the published lists of maximal subgroups of Bray, Holt and Roney-Dougal, for $\text{PSL}_3(p)$ and $\text{PSU}_3(p)$ when p is a prime with $p - 1$, respectively $p + 1$, equal to 2^a or $3 \cdot 2^a$, while it has one for all other $\text{PSU}_3(q)$ and for ${}^2G_2(q)$ with $q \geq 27$; under the same two assumptions this determines the set for $n = 4$. For $n \geq 5$ the question remains open; we determine the least length of an induced cycle of length at least 4 for some small groups. Several steps are computer-assisted, and all certificates are included. This is an unrefereed note.

1 Introduction

1.1 The problem

All groups are finite. For a group G , the *enhanced power graph* $\mathcal{E}(G)$ [AAC⁺17] has vertex set G , and distinct $x, y \in G$ are adjacent, written $x \sim y$, if and only if $\langle x, y \rangle$ is cyclic. For $n \geq 3$, C_n is the cycle on n vertices. A graph is *C_n -free* if it has no induced subgraph isomorphic to C_n ; an induced cycle of length $n \geq 4$ is called a *hole*; a graph is *chordal* if it has no hole, and a *cograph* if it has no induced path on four vertices [BFP25, Definition 2.1].

The following question was proposed by D. Bubboloni, F. Fumagalli and C. E. Praeger as Problem 21.23 of the Kourovka Notebook [KM26]; it is Open Problem 1 of their preprint [BFP25]. Simple groups are taken up to isomorphism.

Problem ([KM26, Problem 21.23]). (a) *For a given integer $n \geq 4$, determine the set S_n of all finite nonabelian simple groups G such that $\mathcal{E}(G)$ is C_n -free.* (b) *Determine the set of all finite nonabelian simple groups G such that $\mathcal{E}(G)$ is chordal, that is, $\bigcap_{n \geq 4} S_n$.*

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The problem is record KOU-21.23 of the UnsolvedMath collection [ula26]. We use two results of [BFP25]. *Theorem A* of [BFP25]: if $\mathcal{E}(G)$ is a cograph, then $\mathcal{E}(G)$ is chordal. *Theorem B* of [BFP25]: for a finite nonabelian simple group G , $\mathcal{E}(G)$ is a cograph if and only if G lies in

$$\mathcal{L}_1 = \{\mathrm{PSL}_2(q) \ (q \geq 4), \mathrm{PSL}_3(4), \mathrm{Sz}(q) \ (q = 2^{2m+1} \geq 8)\},$$

and

$$\mathcal{L}_1 \cup \{A_7, M_{11}, J_1\} \subseteq S_4 \subseteq \mathcal{L}_1 \cup \{A_7, M_{11}, J_1, \mathrm{PSL}_3(q), \mathrm{PSU}_3(q), {}^2G_2(q), {}^2F_4(2)'\}, \quad (1)$$

where q is an odd prime with $(q-1)/\gcd(3, q-1)$ a prime power for $\mathrm{PSL}_3(q)$, $q > 2$ for $\mathrm{PSU}_3(q)$, and $q = 3^{2m+1} > 3$ for ${}^2G_2(q)$ [BFP25, p. 3]. Moreover $\mathcal{E}(A_7)$ is chordal and not a cograph [BFP25, Proposition 6.1]. The second inclusion in (1), which we call *the upper bound* (1), is proved in [BFP25] by a case analysis over the classification of finite simple groups; Section 5 lists the facts it rests on. We use the usual isomorphisms $\mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5) \cong A_5$, $\mathrm{PSL}_2(9) \cong A_6$, $\mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$; ${}^2G_2(3) \cong \mathrm{P}\Gamma\mathrm{L}(2, 8)$ and ${}^2F_4(2)$ are not simple, and ${}^2F_4(2)'$ is the Tits group. Put

$$\mathcal{P}_L = \{p \text{ odd prime} : p-1 \in \{2^a, 3 \cdot 2^a\}\}, \quad \mathcal{P}_U = \{p \text{ odd prime} : p+1 \in \{2^a, 3 \cdot 2^a\}\};$$

so $\mathcal{P}_L = \{3, 5, 7, 13, 17, 97, 193, 257, 769, \dots\}$ and $\mathcal{P}_U = \{3, 5, 7, 11, 23, 31, 47, 127, 191, 383, \dots\}$. For an odd prime q , $(q-1)/\gcd(3, q-1)$ is a prime power if and only if $q \in \mathcal{P}_L$.

1.2 Results

The first statement is known; we include short proofs in Section 3.1.

Theorem 1.1 ([BFP25]). *If G is $\mathrm{PSL}_2(q)$ ($q \geq 4$), $\mathrm{Sz}(q)$ ($q = 2^{2m+1} \geq 8$), $\mathrm{PSL}_3(4)$ or A_7 , then $\mathcal{E}(G)$ is chordal.*

Theorem 1.2. *$\mathcal{E}(G)$ has a hole in each of the following cases: $G = M_{11}$ (a hole of length 10), $G = J_1$ or $G = {}^2F_4(2)'$ (a hole of length 6), $G = \mathrm{PSL}_3(q)$ or $G = \mathrm{PSU}_3(q)$ with q odd (a hole of length 6), $G = \mathrm{PSU}_3(q)$ with $q \geq 4$ even, and $G = {}^2G_2(q)$ with $q = 3^{2m+1} \geq 27$ (an induced 4-cycle). For M_{11} and J_1 the proof is a computer certificate.*

Corollary 1.3. *Assume the upper bound (1). Then for a finite nonabelian simple group G , the graph $\mathcal{E}(G)$ is chordal if and only if G is isomorphic to $\mathrm{PSL}_2(q)$ with $q \geq 4$, to $\mathrm{Sz}(q)$ with $q = 2^{2m+1} \geq 8$, to $\mathrm{PSL}_3(4)$ or to A_7 .*

For $n = 4$ we prove the following. We write “ G has a C_4 ” if $\mathcal{E}(G)$ has an induced 4-cycle.

Theorem 1.4. (a) $\mathcal{E}({}^2F_4(2)')$ is C_4 -free (computer-assisted).

(b) If $p \in \mathcal{P}_L$, then $\mathcal{E}(\mathrm{PSL}_3(p))$ is C_4 -free; if $p \in \mathcal{P}_U$, then $\mathcal{E}(\mathrm{PSU}_3(p))$ is C_4 -free. For $\mathrm{PSL}_3(p)$ with $p \geq 5$ and $\mathrm{PSU}_3(p)$ with $p \geq 7$ the proof uses the lists of maximal subgroups of these groups in [BHRD13]; the remaining cases $\mathrm{PSL}_3(3)$, $\mathrm{PSU}_3(3)$, $\mathrm{PSU}_3(5)$ are computer-assisted. Independently of [BHRD13], (b) is verified by an exhaustive computation for $\mathrm{PSL}_3(p)$, $p \in \{3, 5, 7, 13, 17\}$, and for $\mathrm{PSU}_3(p)$, $p \in \{3, 5, 7, 11, 23\}$.

(c) G has a C_4 if $G = \mathrm{PSU}_3(q)$ with $q \geq 4$ even, or with q odd and $q+1 \notin \{2^a, 3 \cdot 2^a\}$; if $G = {}^2G_2(q)$ with $q \geq 27$; and ([BFP25, Lemma 6.5(b)]) if $G = \mathrm{PSL}_3(q)$ with $q \notin \{2, 4\} \cup \mathcal{P}_L$.

Corollary 1.5. *Assume the upper bound (1) and the lists of maximal subgroups of $\mathrm{PSL}_3(p)$ and $\mathrm{PSU}_3(p)$ in [BHRD13]. Then*

$$S_4 = \mathcal{L}_1 \cup \{A_7, M_{11}, J_1, {}^2F_4(2)'\} \cup \{\mathrm{PSL}_3(p) : p \in \mathcal{P}_L\} \cup \{\mathrm{PSU}_3(p) : p \in \mathcal{P}_U\}.$$

The inclusion “ $\supseteq$ ” does not use (1), and it uses [BHRD13] only for the groups $\mathrm{PSL}_3(p)$ and $\mathrm{PSU}_3(p)$ with p not among the primes listed in Theorem 1.4(b).

For $n \geq 5$ we prove only the following (Section 6).

Proposition 1.6. *The least length of a hole of $\mathcal{E}(G)$ is 10 for $G = M_{11}$, and 6 for J_1 , ${}^2F_4(2)'$, $\text{PSL}_3(q)$ and $\text{PSU}_3(q)$ with $q \in \{3, 5, 7\}$. Hence $M_{11} \in S_n$ for $4 \leq n \leq 9$ and $M_{11} \notin S_{10}$, and the other groups lie in $S_4 \cap S_5$ and not in S_6 . The values for M_{11} and J_1 rest on an exhaustive computer search; for the other groups the proof uses Theorem 1.4 (computer-assisted for ${}^2F_4(2)'$, $\text{PSL}_3(3)$, $\text{PSU}_3(3)$ and $\text{PSU}_3(5)$) and their element orders, computed from the conjugacy classes. Whether these groups lie in S_n for larger n is not decided here.*

1.3 What is new, what is used, and what stays open

Theorem 1.1 and Lemmas 2.2 to 2.4 are due to Bubboloni, Fumagalli and Praeger [BFP25], who also proved that $\mathcal{E}(M_{11})$ and $\mathcal{E}(J_1)$ are C_4 -free and not cographs, and that $\text{PSL}_3(q)$ has a C_4 for $q \notin \{2, 4\} \cup \mathcal{P}_L$ [BFP25, Lemma 6.5(b)]; for $\text{PSL}_2(q)$, Theorem 1.1 combines Theorem A with Cameron's result that $\mathcal{E}(\text{PSL}_2(q))$ is a cograph [Cam22, Proposition 8.7], as stated in [BFP25, p. 3]. To our knowledge (see “Scope and priority” in Section 7) the rest of Theorems 1.2 and 1.4, the two corollaries, Proposition 1.6, and the completion of one step of the proof of (1) in Section 5 are not in print.

External inputs. (i) Corollaries 1.3 and 1.5 assume the upper bound (1), proved in the unrefered preprint [BFP25]. After the re-check of Section 5, its proof rests on the classification of finite simple groups, on [LSS92, Table 5.1] and [Cra23] (subgroups of exceptional groups of Lie type), on [KL90, Lemma 4.1.1, pp. 185–187] (subgroups of orthogonal groups), and on ATLAS containments [CCN⁺85] for some classical groups over $\mathbb{F}_2$ and for ten sporadic groups, each confirmed with the maximal-subgroup data of CTblLib [Bre25] (those over $\mathbb{F}_2$ by the second verification run of Section 7); the other steps were re-derived or recomputed here. (ii) Theorem 1.4(b) for $p \geq 5$ (resp. $p \geq 7$) and Corollary 1.5 use the maximal subgroups of $\text{PSL}_3(p)$ and $\text{PSU}_3(p)$ as classified in [BHRD13, Propositions 3.2.1–3.2.6, Theorem 4.10.2, Tables 8.3–8.6], going back to Mitchell [Mit11] for odd q and Hartley [Har25] for even q (see also Bloom [Blo67]). We did not consult these texts or the printed tables of [BHRD13]; we used the lists computed by the GAP package ClassicalMaximals [HHP⁺26], which implements [BHRD13], and for $p \leq 7$ they agree with GAP's general algorithm and with the ATLAS (Section 4.2). (iii) Standard structure: the partitions of $\text{PSL}_2(q)$ [Hup67, II.8.5] and $\text{Sz}(q)$ [HB82, XI.3.10], and the centraliser of an involution of ${}^2G_2(q)$ [War66].

Open. Part (a) for $n \geq 5$, apart from Proposition 1.6 and $\mathcal{L}_1 \cup \{A_7\} \subseteq S_n$; and the dependence of the two corollaries on (1) and of Corollary 1.5 on [BHRD13].

Computations. The statements marked computer-assisted, the small cases of Theorem 1.4(b) and parts of Section 5 use GAP 4.16.1 [The26] with PrimGrp [HKL⁺26], CTblLib [Bre25], TomLib [MNP24], ClassicalMaximals [HHP⁺26] and AtlasRep [WPN⁺26] (data shipped with GAP; nothing was downloaded). All certificates are explicit lists of group elements, re-checked by independent, AI-assisted verification runs with their own programs (Section 7).

2 Tools

We write $|x|$ for the order of x ; a p -element is an element of order p^k with $k \geq 1$, and an element of prime-power order is a p -element for some prime p .

Lemma 2.1. (a) *If $\gcd(|x|, |y|) = 1$, then $\langle x, y \rangle$ is cyclic if and only if $xy = yx$.* (b) *If x, y are p -elements, then $\langle x, y \rangle$ is cyclic if and only if $x \in \langle y \rangle$ or $y \in \langle x \rangle$.* (c) *If $x_1, \dots, x_k$ pairwise generate cyclic subgroups, then $\langle x_1, \dots, x_k \rangle$ is cyclic.*

Proof. (a) If $xy = yx$, then $\langle x, y \rangle = \langle x \rangle \times \langle y \rangle$ is cyclic. (b) The subgroups of a cyclic p -group form a chain. (c) The x_i commute, so $H = \langle x_1, \dots, x_k \rangle$ is abelian. For a prime p , the p -parts of the x_i are

powers of the x_i , so they pairwise generate cyclic p -groups and, by (b), the subgroups they generate form a chain. The Sylow p -subgroup of H is generated by them, so it is the largest of them and is cyclic. An abelian group with cyclic Sylow subgroups is cyclic. $\square$

For $k = 3$, (c) is [AAC⁺17, Lemma 32], quoted as [BFP25, Lemma 4.12]; [BFP25] apply it repeatedly in the proof of their Lemma 4.13.

Lemma 2.2 ([BFP25, Lemmas 4.13 and 4.9(ii)]). *If $\mathcal{E}(G)$ has a hole and n is the least length of a hole, then $\mathcal{E}(G)$ has a hole $(a_1, \dots, a_n)$ of length n whose vertices have prime-power order, with $\gcd(|a_i|, |a_{i+1}|) = 1$ for all i (indices modulo n). Moreover $\langle a_i \rangle \neq \langle a_j \rangle$ for $i \neq j$.*

Proof. Let $C = (a_1, \dots, a_n)$ be a hole of length n . (i) 1 is not on C , since it is adjacent to every vertex. (ii) For no i is $a_i \in \langle a_{i+1} \rangle$: otherwise $\langle a_i, a_{i+2} \rangle \leq \langle a_{i+1}, a_{i+2} \rangle$ would be cyclic, a chord; symmetrically $a_{i+1} \notin \langle a_i \rangle$. Suppose $|a_1|$ is not a prime power, and for a prime p dividing $|a_1|$ let $a_1(p)$ be the p -part of a_1 . Each $a_1(p)$ lies in $\langle a_1 \rangle$, so $\langle a_1(p), a_2 \rangle$ and $\langle a_1(p), a_n \rangle$ are cyclic. If $\langle a_1(p), a_3 \rangle$ were cyclic for every p , then Lemma 2.1(c), applied to a_3 and the $a_1(p)$, would make $\langle a_1, a_3 \rangle$ cyclic. So there is p with $b := a_1(p) \not\sim a_3$ and $b \neq a_3$. Then b is not on C : $b = a_2$ or $b = a_n$ contradicts (ii), and $b = a_j$ with $4 \leq j \leq n-1$ would give the chord $a_j \sim a_2$, as $\langle a_j, a_2 \rangle \leq \langle a_1, a_2 \rangle$. Let $k \in \{4, \dots, n\}$ be least with $b \sim a_k$ (it exists, as $b \sim a_n$). Then $(b, a_2, \dots, a_k)$ is a hole of length k , so $k = n$. Replacing a_1 by b lowers the number of vertices of non-prime-power order; repeat. If $|a_i|$ and $|a_{i+1}|$ were powers of one prime, $\langle a_i, a_{i+1} \rangle$ would be a cyclic p -group, against (ii) and Lemma 2.1(b). Finally, $\langle a_i \rangle = \langle a_j \rangle$ with $i \neq j$ would make $a_i \sim a_j$, so $j = i \pm 1$, against (ii). $\square$

A C_4 -configuration in G is a quadruple (a_1, a_2, b_1, b_2) of elements of prime-power order with $\gcd(|a_i|, |b_j|) = 1$ and $a_i b_j = b_j a_i$ for all i, j , such that $\langle a_1, a_2 \rangle$ and $\langle b_1, b_2 \rangle$ are not cyclic.

Lemma 2.3 ([BFP25, Proposition 4.14]). (a) *If $a_1, a_2, b_1, b_2 \in G$ satisfy $\gcd(|a_i|, |b_j|) = 1$, $a_i b_j = b_j a_i$, and $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle$ are not cyclic, then (a_1, b_1, a_2, b_2) is an induced 4-cycle of $\mathcal{E}(G)$; prime-power orders are not needed here.* (b) *$\mathcal{E}(G)$ has an induced 4-cycle if and only if G contains a C_4 -configuration, if and only if there are elements a_1, a_2 of prime-power order with $\langle a_1, a_2 \rangle$ not cyclic such that the elements of prime-power order of $C_G(\langle a_1, a_2 \rangle)$ of order prime to $|a_1||a_2|$ generate a non-cyclic group.*

Proof. (a) By Lemma 2.1(a), $a_i \sim b_j$; the other two pairs are not adjacent. The four elements are distinct and non-trivial, since the pairs generate non-cyclic groups and the orders are coprime. (b) By (a), a C_4 -configuration gives an induced 4-cycle. Conversely, 4 is the least possible length of a hole, so Lemma 2.2 gives an induced 4-cycle (a_1, b_1, a_2, b_2) with vertices of prime-power order and coprime orders of consecutive vertices; the edges are commutations by Lemma 2.1(a). For the last form, a set of elements generating a non-cyclic group contains two that generate a non-cyclic group (Lemma 2.1(c)). $\square$

Lemma 2.4. (a) ([BFP25, Lemma 3.2]) *If $H \leq G$, then $\mathcal{E}(H)$ is the subgraph of $\mathcal{E}(G)$ induced on H ; holes of $\mathcal{E}(H)$ are holes of $\mathcal{E}(G)$.* (b) ([BFP25, Corollary 4.15]) *If G is nonabelian simple, then $\mathcal{E}(G)$ is C_4 -free if and only if $\mathcal{E}(M)$ is C_4 -free for one representative M of each conjugacy class of maximal subgroups.*

Proof. (a) $\langle x, y \rangle$ does not depend on the ambient group. (b) Let (a_1, a_2, b_1, b_2) be a C_4 -configuration, $A = \langle a_1, a_2 \rangle$, $B = \langle b_1, b_2 \rangle$ and $H = \langle A, B \rangle$. Then A is normal in H , $A \neq 1$, and $A \neq G$ since $b_1 \neq 1$ centralises A and $Z(G) = 1$. As G is simple, $H \neq G$; so H lies in a maximal subgroup, and (a) applies; conjugation preserves $\mathcal{E}$. $\square$

Lemma 2.5. *Let $S \leq \mathrm{GL}_n(F)$ with $\mathrm{char} F = p$, let $Z = S \cap F^\times I_n$, and write $\bar{x}$ for the image of x in S/Z . Let $u, v \in S$ be unipotent. (a) If $g \in S$ and $[g, u] \in Z$, then $[g, u] = 1$. (b) If $\bar{u} \in \langle \bar{v} \rangle$, then $u \in \langle v \rangle$. (c) $|\bar{u}| = |u|$.*

Proof. (a) $gug^{-1} = zu$ with z scalar; both sides are unipotent, so $z = 1$. (b) $u = zv^k$ with z scalar, so $z = 1$. (c) A scalar power of u is a unipotent scalar, hence 1. $\square$

3 Chordal groups and holes

3.1 The chordal groups

For $G \in \mathcal{L}_1$, Theorem 1.1 follows from [BFP25, Theorems A and B], and for A_7 it is [BFP25, Proposition 6.1]. The following lemma, a variant of the partition arguments of [BFP25, Theorem 5.3 and Lemmas 6.3, 6.4], gives a short direct proof; a *partition* of G is a set of subgroups such that every non-identity element lies in exactly one of them.

Lemma 3.1. *If G has a partition into subgroups each of which is cyclic or of prime-power order, then $\mathcal{E}(G)$ is chordal.*

Proof. Suppose not, and take a hole as in Lemma 2.2, with consecutive vertices $a \sim b$ of coprime prime-power orders. Then $\langle a, b \rangle = \langle ab \rangle$, and ab lies in a member K , which contains a and b ; so K is not of prime-power order and is cyclic. If $h \sim a$, then $\langle a, h \rangle = \langle g \rangle$ for some g , which lies in a member K' ; as $1 \neq a \in K \cap K'$, $K' = K$ and $h \in K$. So the closed neighbourhood of a is the cyclic group K , a clique, and the two neighbours of a on the hole are adjacent: a contradiction. $\square$

Proof of Theorem 1.1. $\mathrm{PSL}_2(q)$, $q \geq 4$, is partitioned by its Sylow p -subgroups and its cyclic subgroups of orders $(q \pm 1)/\gcd(2, q - 1)$ [Hup67, II.8.5]; $\mathrm{Sz}(q)$ by the conjugates of a Sylow 2-subgroup and of three cyclic subgroups [HB82, XI.3.10] (as used in [BFP25, Lemma 6.4]). Lemma 3.1 applies. Every element of $\mathrm{PSL}_3(4)$ has prime-power order [CCN⁺85, p. 23], while two adjacent vertices of coprime prime-power orders $r^i, s^j > 1$ commute and have a product of order $r^i s^j$; so Lemma 2.2 excludes holes. For A_7 see [BFP25, Proposition 6.1]: the elements of orders 4, 5, 6, 7 and the products of two disjoint 3-cycles generate maximal cyclic subgroups, so their closed neighbourhoods are cliques and they lie on no hole; hence a hole as in Lemma 2.2 alternates between double transpositions and 3-cycles, and after conjugation the two 3-cycles next to $(12)(34)$ commute with it, so both lie in $\langle (567) \rangle$, against the last assertion of Lemma 2.2. $\square$

3.2 $\mathrm{PSL}_3(q)$, q odd: a hole of length 6

Let $q = p^f$ be odd and $K = \mathbb{F}_q$. Row vectors $x \in K^2$ are the points of the affine plane, and the affine map $x \mapsto xA + v$ ($A \in \mathrm{SL}_2(q)$, $v \in K^2$) is the matrix $M(A, v) = \begin{pmatrix} A & 0 \\ v & 1 \end{pmatrix} \in \mathrm{SL}_3(q)$, since $(x, 1)M(A, v) = (xA + v, 1)$. These matrices form a group $\cong \mathrm{ASL}(2, q)$ meeting the scalars trivially, so $\mathrm{ASL}(2, q) \leq \mathrm{PSL}_3(q)$, and by Lemma 2.4(a) a hole of $\mathcal{E}(\mathrm{ASL}(2, q))$ is a hole of $\mathcal{E}(\mathrm{PSL}_3(q))$. For $c \in K^2$ let $t_c(x) = 2c - x$, an involution, with $t_c = t_{c'}$ only if $c = c'$. For a line $l = m + Kd$ ($d \neq 0$) fix a linear form $f \neq 0$ with $f(d) = 0$ and let $s_l(x) = x + f(x - m)d$. Then $s_l^k(x) = x + kf(x - m)d$, so $|s_l| = p$ and s_l^k fixes exactly the points of l when $p \nmid k$. Moreover

$$t_c s_l t_c(x) = x + f(x - (2c - m))d,$$

which equals $s_l(x)$ for all x if and only if $f(2c - 2m) = 0$, that is, $c \in l$. By Lemma 2.1: $t_c \sim s_l$ if and only if $c \in l$; $t_c \not\sim t_{c'}$ for $c \neq c'$ (a cyclic group has one involution); and $s_l \not\sim s_{l'}$ for $l \neq l'$ (if $s_l \in \langle s_{l'} \rangle$, the fixed point sets agree). Take non-collinear points c_1, c_2, c_3 and let l_{ij} be the line $c_i c_j$. Then

$$(t_{c_1}, s_{l_{12}}, t_{c_2}, s_{l_{23}}, t_{c_3}, s_{l_{31}})$$

is an induced 6-cycle: consecutive elements commute, $c_k \notin l_{ij}$ for $k \notin \{i, j\}$, and the reflections, respectively the shears, are pairwise non-adjacent. For $c_1 = (0, 0)$, $c_2 = (1, 0)$, $c_3 = (0, 1)$ the six matrices have integer entries (for instance $s_{l_{23}} = M\left(\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, (1, -1)\right)$), and the programs of this note check them in $\text{PSL}_3(q)$ for 16 odd values of q up to 81 (Section 7).

3.3 $\text{PSU}_3(q)$, q odd: a hole of length 6

Let $q = p^f$ be odd, $V = \mathbb{F}_{q^2}^3$ with a non-degenerate Hermitian form B (linear in the first variable), $\text{SU} = \text{SU}_3(q)$, Z its group of scalars, and $\text{PSU}_3(q) = \text{SU}/Z$. For a non-isotropic vector u let $t_u(v) = -v + 2B(v, u)B(u, u)^{-1}u$; it is $+1$ on $\langle u \rangle$ and -1 on $u^\perp$, an isometry of determinant 1 depending only on the point $P = \langle u \rangle$, written t_P . For an isotropic $w \neq 0$ and $a \in \mathbb{F}_{q^2}^\times$ with $a^q = -a$ let $\tau_w(v) = v + aB(v, w)w$. Since $B(w, w) = 0$ and $a + a^q = 0$, τ_w is an isometry; it is unipotent, $\tau_w^k = 1 + k(\tau_w - 1)$, so $|\tau_w| = p$, and the image of $\tau_w^k - 1$ is $\langle w \rangle$ for $p \nmid k$. For $g \in \text{SU}$, $g\tau_w g^{-1} = \tau_{gw}$, and $\tau_{w'} = \tau_w$ if and only if $w' = \mu w$ with $\mu^{q+1} = 1$. Hence t_P commutes with τ_w if and only if $t_P w = \pm w$, that is, $w \in P^\perp$ ($w \notin P$, as P is non-isotropic). By Lemma 2.5(a),(c) the same holds for the images in $\text{PSU}_3(q)$, where $|\tau_w|$ is still p ; the image of t_P has order 2. So, by Lemma 2.1, $t_P \sim \tau_w$ if and only if $w \perp P$. For $P \neq P'$, $t_P \not\sim t_{P'}$: otherwise $t_P = z t_{P'}$ with z scalar, and comparing the eigenvalue of multiplicity one gives $z = 1$ and $P = P'$. For $\langle w \rangle \neq \langle w' \rangle$, $\tau_w \not\sim \tau_{w'}$, by Lemma 2.1(b) and Lemma 2.5(b).

Let w_1, w_2, w_3 be linearly independent isotropic vectors and $P_{ij} = \langle w_i, w_j \rangle^\perp$. The Witt index of V is 1, so $\langle w_i, w_j \rangle$ is a non-degenerate plane and P_{ij} is a non-isotropic point; P_{ij} is orthogonal to w_i, w_j and not to w_k , $k \notin \{i, j\}$; the P_{ij} are distinct. Hence

$$(t_{P_{31}}, \tau_{w_1}, t_{P_{12}}, \tau_{w_2}, t_{P_{23}}, \tau_{w_3})$$

is an induced 6-cycle of $\mathcal{E}(\text{PSU}_3(q))$. Explicitly, for the Gram matrix with 1's on the antidiagonal (the form of GAP), one can take $w_1 = e_1$, $w_2 = e_3$, $w_3 = (-\frac{1}{2}, 1, 1)$, $P_{12} = \langle e_2 \rangle$, $P_{23} = \langle (0, 1, 2) \rangle$, $P_{31} = \langle (1, -1, 0) \rangle$ (each spanned by a vector u with $B(u, u) = 1$) and $a = z^{(q+1)/2}$ for a primitive element z of $\mathbb{F}_{q^2}$; this is checked for 11 odd values of q up to 27.

3.4 $\text{PSU}_3(q)$, $q \geq 4$ even: an induced 4-cycle

Take the Gram matrix J with 1's on the antidiagonal, so $B(u, v) = u_1 v_3^q + u_2 v_2^q + u_3 v_1^q$, and let $q \geq 4$ be even. Put $x(\beta) = I + \beta E_{13}$ ($\beta \in \mathbb{F}_q$), $h(\nu) = \text{diag}(\nu, \nu^{-2}, \nu)$ ($\nu^{q+1} = 1$) and $u = \begin{pmatrix} 1 & \alpha & \beta' \\ 0 & 1 & \alpha^q \\ 0 & 0 & 1 \end{pmatrix}$ with $\alpha \neq 0$ and $\beta' + \beta'^q = \alpha^{q+1}$ (such β' exists, as the trace map $\mathbb{F}_{q^2} \rightarrow \mathbb{F}_q$ is onto). Multiplying out $gJ\bar{g}^\top$ (where $\bar{g}$ is g with q -th powers of the entries) shows that these matrices lie in $\text{SU}_3(q)$; for $x(\beta)$ and u this uses characteristic 2. The matrix $x(\beta)$ commutes with every upper unitriangular matrix and with every $h(\nu)$. Let $e = (q+1)/\gcd(3, q+1) > 1$ and let r be a prime divisor of e (r is odd). For λ of order $q+1$, $h(\lambda)^k$ is scalar if and only if $\lambda^{3k} = 1$, so the image of $h(\lambda)$ in $\text{PSU}_3(q)$ has order e ; put $b_1 = h(\lambda)^{e/r} = h(\nu)$, of image order r , and $b_2 = u^{-1}b_1u$. Then: $t_1 = x(1)$ and $t_2 = x(\beta)$ with $\beta \in \mathbb{F}_q \setminus \mathbb{F}_2$ have distinct images of order 2 (Lemma 2.5(b),(c)), so $t_1 \not\sim t_2$; b_1, b_2 commute with t_1, t_2 ; and $b_2 \notin \langle b_1 \rangle Z$, since otherwise $u^{-1}b_1u = D$ is diagonal, and the entries (1, 2) and (2, 2) of $b_1u = uD$ give $\nu\alpha = \alpha\nu^{-2}$, so $\nu^3 = 1$ and b_1 would be scalar. Hence (t_1, b_1, t_2, b_2) is an induced 4-cycle of $\mathcal{E}(\text{PSU}_3(q))$ (Lemma 2.3(a)). This is checked for $q = 4, 8, 16, 32$ ($r = 5, 3, 17, 11$). The elements $x(\beta)$, $h(\lambda)$ and the conjugation by an element of the unipotent radical are those of the proof of [BFP25, Lemma 6.6], where they show that $\mathcal{E}(\text{PSU}_3(q))$ is not a cograph.

3.5 The Ree groups: an induced 4-cycle

We use three facts about $G = {}^2G_2(q)$, $q = 3^{2m+1}$: (R1) $|G| = q^3(q^3 + 1)(q - 1)$; (R2) for an involution J of G , $C_G(J) = \langle J \rangle \times L$ with $L \cong \text{PSL}_2(q)$ [War66] (this is also fact (3) in the proof of [BFP25, Lemma 6.7], where it is taken from [LN85]); (R3) a Sylow 3-subgroup E of L is elementary abelian of order q and $N_L(E) = E \rtimes C_{(q-1)/2}$, so the 3-elements of $N_L(E)$ lie in E [Hup67, II.8].

Proposition 3.2. *For $q = 3^{2m+1} \geq 27$, $\mathcal{E}({}^2G_2(q))$ has an induced 4-cycle.*

Proof. Let E be a Sylow 3-subgroup of $L \leq C_G(J)$. As $|E| = q < q^3 = |G|_3$, E lies properly in a Sylow 3-subgroup U of G , and $N_U(E) > E$. Take $s \in N_U(E) \setminus E$ and $J' = s^{-1}Js$, an involution centralising $s^{-1}Es = E$. If $J' = J$, then s is a 3-element of $C_G(J) = \langle J \rangle \times L$, so $s \in L$, and $s \in N_L(E)$, so $s \in E$ by (R3): a contradiction. So $J' \neq J$ and $\langle J, J' \rangle$ is not cyclic. Since E is elementary abelian of rank $2m + 1 \geq 3$, there are $e_1, e_2 \in E$ with $\langle e_1, e_2 \rangle \cong C_3^2$. By Lemma 2.3(a), (e_1, J, e_2, J') is an induced 4-cycle. $\square$

For $q = 3$ the argument fails exactly where expected ($E \cong C_3$), and $\mathcal{E}({}^2G_2(3))$ is indeed C_4 -free. For $q = 27$ the programs of this note repeat the proof in ${}^2G_2(27)$, acting on 19684 lines; four 7×7 matrices over $\mathbb{F}_{27}$ forming such a 4-cycle are stored. A second proof that $\mathcal{E}({}^2G_2(q))$ is not chordal: ${}^2G_2(3) \cong \text{P}\Gamma\text{L}(2, 8)$ lies in ${}^2G_2(q)$ (the fixed points of the exceptional endomorphism σ of $G_2(\mathbb{F}_3)$, with σ^2 the Frobenius map, are fixed by σ^{2m+1} ; checked with GAP's matrices for $q = 27$), and $\mathcal{E}(\text{P}\Gamma\text{L}(2, 8))$ has a hole of length 10 (and none shorter, by computer search).

3.6 The finite cases, and the proofs of Theorem 1.2 and Corollary 1.3

For the three remaining groups we give certificates: lists of permutations in a fixed representation for which $\langle x, y \rangle$ was computed for all pairs (Section 7).

- M_{11} , as GAP's `MathieuGroup(11)` on 11 points: the hole of length 10, with orders 2, 3, 2, 3, $\dots$,

$$\begin{aligned} & (1, 3)(5, 8)(6, 11)(7, 9), (2, 4, 10)(5, 7, 11)(6, 8, 9), (1, 3)(2, 5)(4, 7)(10, 11), \\ & (1, 4, 5)(2, 3, 7)(6, 8, 9), (1, 6)(4, 8)(5, 9)(10, 11), (1, 9, 10)(2, 3, 7)(5, 11, 6), \\ & (2, 6)(3, 5)(4, 8)(7, 11), (1, 9, 10)(3, 8, 11)(4, 7, 5), (1, 4)(2, 6)(5, 10)(7, 9), \\ & (1, 5, 6)(2, 4, 10)(3, 8, 11). \end{aligned}$$

- J_1 , as the primitive group `PrimitiveGroup(266, 1)` of order 175560 (simple, hence J_1): a hole of length 6 whose vertices have orders 3, 5, 2, 5, 3, 2.
- ${}^2F_4(2)'$, as the group of the table of marks "`2F4(2)'`" of TomLib (order 17971200, simple, hence the Tits group): a hole of length 6 with orders 2, 3, 2, 3, 2, 3, lying in a subgroup $\text{PSL}_3(3)$. Here no computer is needed: ${}^2F_4(2)'$ has a maximal subgroup $\text{PSL}_3(3):2$ [CCN⁺85, p. 74] (also listed in CTblLib), and Section 3.2 with $q = 3$ applies.

Proof of Theorem 1.2. Combine Sections 3.2 to 3.4, Proposition 3.2 and the certificates above. $\square$

Proof of Corollary 1.3. A chordal $\mathcal{E}(G)$ is C_4 -free, so G lies in the right-hand side of (1). The groups there that are not in $\mathcal{L}_1 \cup \{A_7\}$ are M_{11} , J_1 , ${}^2F_4(2)'$, $\text{PSL}_3(q)$ with q an odd prime, $\text{PSU}_3(q)$ with $q > 2$ and ${}^2G_2(q)$ with $q > 3$, all of which have a hole by Theorem 1.2. The converse is Theorem 1.1. $\square$

4 Induced 4-cycles

4.1 Tori

Lemma 4.1. *Let q be odd. If $q + 1$ has an odd prime divisor $r \neq 3$, or $9 \mid q + 1$, then $\text{PSU}_3(q)$ has a C_4 . If $q - 1$ has an odd prime divisor $r \neq 3$, or $9 \mid q - 1$, then $\text{PSL}_3(q)$ has a C_4 (this is the second half of [BFP25, Lemma 6.5(b)]).*

Proof. In an orthonormal basis, $T = \{\text{diag}(a, b, (ab)^{-1}) : a^{q+1} = b^{q+1} = 1\} \cong C_{q+1}^2$ lies in $\text{SU}_3(q)$ and meets the scalars in a group of order $d = \gcd(3, q + 1)$; the automorphism $(a, b) \mapsto (ab^{-1}, b)$ of T shows $T/(T \cap Z) \cong C_{q+1} \times C_{(q+1)/d}$. This group contains C_2^2 and, under the hypothesis, C_r^2 (with $r = 3$ if $9 \mid q + 1$). So $\text{PSU}_3(q)$ contains $C_2^2 \times C_r^2$, and Lemma 2.3(a) applies. For $\text{PSL}_3(q)$ use the diagonal torus of $\text{SL}_3(q)$. $\square$

If $q = p^f$ is odd with $f \geq 2$, then $q + 1 \notin \{2^a, 3 \cdot 2^a\}$: for f even $q + 1 \equiv 2 \pmod{4}$ and $q + 1 > 6$, and for f odd $q + 1 = (p + 1)(p^{f-1} - p^{f-2} + \dots + 1)$ with an odd second factor at least 7. Hence, by Lemma 4.1, if q is odd and $\text{PSU}_3(q)$ is C_4 -free, then $q \in \mathcal{P}_U$. For $\text{PSL}_3(q)$ the analogue fails ($q = 9, 25, 49$ have $q - 1 = 8, 24, 48$), and the non-prime q are excluded by the first half of [BFP25, Lemma 6.5(b)]: for $q = p^f \notin \{2, 4\}$ with $f \geq 2$ and w a generator of $\mathbb{F}_q^\times$, the images of $a_1 = I + E_{13}$, $a_2 = I + wE_{13}$, $b_1 = \text{diag}(w, w^{-2}, w)$ and $b_2 = c^{-1}b_1c$, $c = I + E_{23}$, satisfy the hypotheses of Lemma 2.3(a). (Here $|b_j| = (q - 1)/\gcd(3, q - 1)$ need not be a prime power, which Lemma 2.3(a) allows; $\langle b_1, b_2 \rangle$ is not cyclic because b_2 is not diagonal while $|b_1| = |b_2|$.) The programs of this note check these 4-cycles for $q = 9$ and $q = 25$.

4.2 The maximal subgroups of $\text{PSL}_3(p)$ and $\text{PSU}_3(p)$

$\text{SL}_3(p)$, $p \in \mathcal{P}_L$, $p \geq 5$	$\text{SU}_3(p)$, $p \in \mathcal{P}_U$, $p \geq 7$	image C_4 -free by
$E_p^2 : \text{GL}_2(p)$ (two classes)	$E_p^{1+2} : C_{p^2-1}$ (Borel)	Lemma 4.6 / Lemma 4.3
	$\text{GU}_2(p)$ (non-isotropic point)	Lemma 4.4(a)
$C_{p-1}^2 : S_3$ (monomial)	$C_{p+1}^2 : S_3$ (monomial)	Lemma 4.5
$C_{p^2+p+1} : 3$	$C_{p^2-p+1} : 3$	Lemma 4.2
$3^{1+2} : Q_8$ if $3 \mid p - 1$	$3^{1+2} : Q_8$ if $3 \mid p + 1$	Lemma 3.1
$\text{SO}_3(p) \times \gcd(3, p - 1)$	$\text{SO}_3(p) \times \gcd(3, p + 1)$	Lemma 3.1
$\text{PSL}_2(7)$, $3.A_6$ (if p satisfies congruence conditions)	$\text{PSL}_2(7)$, $3.A_6$ (if p satisfies congruence conditions)	Lemma 3.1

Table 1: Classes of maximal subgroups of $\text{SL}_3(p)$ and $\text{SU}_3(p)$ for the primes considered, as given by ClassicalMaximals [HHP⁺26] following [BHRD13], and the lemma used for their images in $\text{PSL}_3(p)$, $\text{PSU}_3(p)$.

Up to conjugacy, the maximal subgroups of $\text{PSL}_3(p)$, $p \in \mathcal{P}_L$, $p \geq 5$, and of $\text{PSU}_3(p)$, $p \in \mathcal{P}_U$, $p \geq 7$, are the images of the groups in Table 1 [BHRD13]. For these primes $9 \nmid p \mp 1$, which is why the third-to-last row has Q_8 ; the image of $3^{1+2} : Q_8$ is a Frobenius group $3^2 : Q_8$ (the centre of 3^{1+2} is scalar, and $Q_8 \leq \text{SL}_2(3)$ acts fixed-point-freely on $\mathbb{F}_3^2$), partitioned by its kernel and its complements; the image of $\text{SO}_3(p)$ is $\text{PGL}_2(p)$, partitioned by its Sylow p -subgroups and its cyclic subgroups of orders $p \pm 1$ (it acts sharply 3-transitively on the projective line); and the last row gives $\text{PSL}_2(7)$ and $A_6 \cong \text{PSL}_2(9)$. So Lemma 3.1 applies to these three rows. The lists computed by ClassicalMaximals have the predicted class orders for $p = 5, 7, 11, 13, 17, 23$ (for instance 53240, 5280, 288, 111, 3×1320 , 72, 3×360 for $\text{PSU}_3(11)$, the last row being A_6); for $p \leq 7$ they agree with the lists computed by

GAP's general algorithm `MaximalSubgroupClassReps` and with the ATLAS [CCN⁺85] (as compared in the first computation and by verification run A of Section 7).

Lemma 4.2. *If M has a cyclic normal subgroup K of prime index r , then $\mathcal{E}(M)$ is C_4 -free.*

Proof. In a C_4 -configuration, $\langle a_1, a_2 \rangle \not\leq K$, so some a_i maps onto a generator of M/K and is an r -element; likewise some b_j . This contradicts $\gcd(|a_i|, |b_j|) = 1$. $\square$

Lemma 4.3. *Let $M = Q:H$ with Q a p -group, $|Z(Q)| = p$, H cyclic of order prime to p , such that no $h \neq 1$ in H fixes a non-trivial element of $Q/Z(Q)$. Then $\mathcal{E}(M)$ is C_4 -free.*

Proof. Put $Z = Z(Q)$. For $1 \neq h \in H$, $C_Q(h) \leq Z$ by coprime action, so $C_Q(h) \in \{1, Z\}$, and $C_M(h) = C_Q(h)H$ as H is abelian. The p -elements of M lie in Q , and the p' -elements are Q -conjugate into H . Take a C_4 -configuration; by symmetry b_1, b_2 are p' -elements. (i) If a_1 is a p -element, conjugate so that $b_1 \in H$; then $a_1 \in C_Q(b_1) \leq Z$, so $Z = \langle a_1 \rangle$. If a_2 were a p -element, then $a_2 \in Z$ and $\langle a_1, a_2 \rangle$ would be cyclic; so a_2 is a p' -element, which we conjugate into H by an element of Q (fixing a_1). As $\langle a_1, a_2 \rangle$ is not cyclic, a_2 does not centralise a_1 , so $C_Q(a_2) = 1$, $C_M(a_2) = H$, and $b_1, b_2 \in H$ generate a cyclic group: a contradiction. (ii) If all four are p' -elements, conjugate a_1 into H . If $C_Q(a_1) = 1$, then $b_1, b_2 \in H$, a contradiction. So $b_1, b_2 \in C_M(a_1) = ZH$, and $b_j = h_j^{z_j}$ with $h_j \in H$, $z_j \in Z$. If some h_j acts non-trivially on Z , then $C_M(b_j) = H^{z_j}$ contains a_1, a_2 , a contradiction; otherwise $b_j = h_j \in H$ for both j , again a contradiction. $\square$

For the Borel subgroup of $\text{PSU}_3(p)$ (Gram matrix J as in Section 3.4), $Q = \{u(\alpha, \beta)\}$ with $u(\alpha, \beta) = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & -\alpha^p \\ 0 & 0 & 1 \end{pmatrix}$, $\beta + \beta^p + \alpha^{p+1} = 0$, has centre $Z(Q) = \{u(0, \beta)\}$ of order p , and α identifies $Q/Z(Q)$ with $(\mathbb{F}_{p^2}, +)$. The torus element $y = \text{diag}(w^{-p}, w^{p-1}, w)$, w of order $p^2 - 1$, satisfies $y^{-1}u(\alpha, \beta)y = u(w^{2p-1}\alpha, *)$, so y acts on $Q/Z(Q)$ as multiplication by w^{2p-1} . The kernel of this action on $\langle y \rangle$ has order $\gcd(p^2 - 1, 2p - 1) = \gcd(3, p + 1)$ (as $4(p^2 - 1) = (2p - 1)(2p + 1) - 3$), and it is the group of scalars in $\langle y \rangle$. So the image H of $\langle y \rangle$ acts faithfully by field multiplications, hence fixed-point-freely, and Lemma 4.3 applies for every prime p .

Lemma 4.4. *Let p be odd, $Y = \text{GU}_2(p)$ or $Y = \text{GL}_2(p)$, $X \leq Y$, and $C \leq X$ a group of scalars of order 1 or 3. A split torus of Y is the stabiliser of an orthogonal decomposition into two non-degenerate lines ($\cong C_{p+1}^2$) if $Y = \text{GU}_2(p)$, and of two distinct lines ($\cong C_{p-1}^2$) if $Y = \text{GL}_2(p)$. If for every split torus T the group $(X \cap T)/C$ contains no subgroup $C_r^2 \times C_s^2$ with primes $r \neq s$, then $\mathcal{E}(X/C)$ is C_4 -free. In particular: (a) $\mathcal{E}(\text{GU}_2(p)/C)$ is C_4 -free for $p \in \mathcal{P}_U$ and $|C| = \gcd(3, p + 1)$; (b) $\mathcal{E}(X)$ is C_4 -free for $p \in \mathcal{P}_L$ and $X = \{h \in \text{GL}_2(p) : \det h \text{ is a cube}\}$.*

Proof. (1) For a non-scalar element $x \in Y$ of prime-power order, $C_Y(x)$ is abelian, and cyclic unless it is a split torus. If x is unipotent, $C_Y(x)$ is the product of the scalars and a root group, cyclic. If x is semisimple, it has two distinct eigenvalues and $C_Y(x)$ is the stabiliser of the two eigenlines. For $\text{GL}_2(p)$ this is a split torus if the eigenvalues lie in $\mathbb{F}_p$, and $\mathbb{F}_{p^2}^\times$ otherwise. For $\text{GU}_2(p)$: if one eigenline is non-degenerate, unitarity forces the two eigenlines to be orthogonal, and $C_Y(x)$ is a split torus; if both are isotropic, $C_Y(x) \cong \mathbb{F}_{p^2}^\times$; and if the eigenvalues lie outside $\mathbb{F}_{p^2}$, $C_Y(x)$ lies in $\mathbb{F}_{p^4}^\times$. (2) Given a C_4 -configuration in X/C , choose preimages of prime-power order (the preimages of a 3-element are 3-elements, and the r -part of a preimage of an r -element, $r \neq 3$, is again a preimage). In each pair (a_i, b_j) one preimage y has order prime to 3; if $xyx^{-1} = zy$ with $z \in C$, then $z^{|y|} = 1$, so $z = 1$. So the preimages form a C_4 -configuration $(\tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \tilde{b}_2)$ in X . (3) The scalars of X form a cyclic group, so some $x = \tilde{a}_i$ and some $y = \tilde{b}_j$ are not scalar. Then $\tilde{B} = \langle \tilde{b}_1, \tilde{b}_2 \rangle \leq C_Y(x)$ and $\tilde{A} = \langle \tilde{a}_1, \tilde{a}_2 \rangle \leq C_Y(y)$ are abelian, so by Lemma 2.1(a) $\tilde{A}$ is an s -group and $\tilde{B}$ an r -group, $r \neq s$, both non-cyclic, and $\tilde{A}\tilde{B} \leq C_Y(x)$. By (1), $C_Y(x) = T$ is a split torus, and the images of $\tilde{A}$ and $\tilde{B}$ give

$C_s^2 \times C_r^2 \leq (X \cap T)/C$. For (a), $T/C \cong C_{p+1} \times C_{(p+1)/|C|}$ (as in Lemma 4.1) contains no such subgroup when $p+1 \in \{2^a, 3 \cdot 2^a\}$. For (b), $X \cap T$ is a 2-group if $p-1 = 2^a$; if $p-1 = 3 \cdot 2^a$, its 3-elements are the $\text{diag}(\alpha, \alpha^{-1})$ with $\alpha^3 = 1$, since $9 \nmid p-1$ and no element of order 3 of $\mathbb{F}_p^\times$ is a cube. $\square$

The stabiliser in $\text{SU}_3(p)$ of a non-isotropic point P acts faithfully as $\text{GU}_2(p)$ on $P^\perp$, and the scalars of $\text{SU}_3(p)$ become the scalars of order $\gcd(3, p+1)$ of $\text{GU}_2(p)$; so Lemma 4.4(a) covers the second row of Table 1.

Lemma 4.5. *Let $q-1$ (for $\text{PSL}_3(q)$) or $q+1$ (for $\text{PSU}_3(q)$) be 2^a or $3 \cdot 2^a$, and let N be the image of the group of monomial matrices of determinant 1 with respect to a basis (an orthonormal basis for $\text{PSU}_3(q)$). Then $\mathcal{E}(N)$ is C_4 -free.*

Proof. $N = \bar{T}:S_3$, where $\bar{T}$, the image of the diagonal group, is $C_{q \mp 1} \times C_{(q \mp 1)/d}$ with $d \in \{1, 3\}$, and $|N|$ is a $\{2, 3\}$ -number. So in a C_4 -configuration one side consists of 2-elements and the other of 3-elements. The Sylow 3-subgroup of $\bar{T}$ has order at most 3, so some element $b = \sigma t$ of the 3-side maps onto a 3-cycle of S_3 (σ a permutation matrix, $t \in \bar{T}$). A 2-element a of the other side commutes with b , so its image in S_3 is trivial; thus $a \in \bar{T}$ commutes with σ . If $\sigma \text{diag}(x, y, z) \sigma^{-1} = \zeta \text{diag}(x, y, z)$ with ζ scalar, then $\zeta^3 = 1$ and $\text{diag}(x, y, z) = x \text{diag}(1, \zeta^{\pm 1}, \zeta^{\pm 2})$; so $C_{\bar{T}}(\sigma)$ has order dividing 3 and $a = 1$, a contradiction. $\square$

Lemma 4.6. *Let $p \in \mathcal{P}_L$, $X = \{h \in \text{GL}_2(p) : \det h \text{ a cube}\}$, and $P = V:X$ the group of affine maps $x \mapsto xh + v$ of $V = \mathbb{F}_p^2$. Both classes of maximal parabolic subgroups of $\text{PSL}_3(p)$ are isomorphic to P , and $\mathcal{E}(P)$ is C_4 -free.*

Proof. Isomorphism. The stabiliser of $\{x_3 = 0\}$ in $\text{SL}_3(p)$ is $\left\{ \begin{pmatrix} A & 0 \\ v & \det(A)^{-1} \end{pmatrix} \right\}$, and conjugating the translation $\begin{pmatrix} I & 0 \\ v & 1 \end{pmatrix}$ by $\text{diag}(A, \det(A)^{-1})$ gives the translation by $\det(A)vA$. So this group is $V:\text{GL}_2(p)$ with A acting by $\rho(A) = \det(A)A$; $\ker \rho = \{\lambda I : \lambda^3 = 1\}$ is its intersection with the scalars, and $\rho(\text{GL}_2(p)) = X$ (as $\det \rho(A) = \det(A)^3$, and $h = \rho(c^{-1}h)$ if $\det h = c^3$). The point stabilisers are the images under the inverse-transpose automorphism.

Facts. (F1) A p' -element $x \neq 1$ of P fixes a point (it is conjugate into X), its linear part g is semisimple of the same order, and $\text{Fix}(x)$ is a point or a line; in the second case $g = \text{diag}(1, \lambda)$, $\lambda \neq 1$, in a basis (e, f) with e along the line (a *homology*). (F2) If x, y commute, y preserves $\text{Fix}(x)$ and the linear parts commute. (F3) If the linear part of a p' -element $x \neq 1$ is λI (a *homothety*) or $\text{diag}(1, \lambda)$, then x is a 2-element: λ^2 resp. λ is a cube, and the cubes form the Sylow 2-subgroup of $\mathbb{F}_p^\times$ (for $p-1 = 2^a$ trivially). So no two homologies or homotheties have coprime orders.

Take a C_4 -configuration; the roles of the a 's and b 's are symmetric. *All four are p' -elements.* If some element, say a_1 , fixes only one point c , then b_1, b_2 fix c . If the linear parts of b_1, b_2 have no common fixed vector, every element commuting with b_1, b_2 fixes c , so the configuration lies in the point stabiliser $\cong X$, against Lemma 4.4(b). Otherwise b_1, b_2 are homologies with the same axis $c + \langle e \rangle$, and their eigenlines $\langle f_1 \rangle, \langle f_2 \rangle$ for $\lambda \neq 1$ differ (else $\langle b_1, b_2 \rangle$ is cyclic); the linear part of a_1 preserves $\langle e \rangle, \langle f_1 \rangle, \langle f_2 \rangle$, so it is scalar, and (F3) for a_1, b_1 gives a contradiction. If every element fixes more than one point, all four are homologies, against (F3).

Some element, say a_1 , is a p -element; then b_1, b_2 are p' -elements. (i) a_1 is the translation by $w \neq 0$. The linear parts of b_j fix w , so b_j are homologies with axes in direction w and λ -eigenlines $\langle f_j \rangle$. The Sylow p -subgroup of $C_P(b_1)$ consists of the translations along w ; so if a_2 were a p -element, $\langle a_1, a_2 \rangle$ would be cyclic. Thus a_2 is a p' -element whose linear part g preserves $\langle w \rangle, \langle f_1 \rangle, \langle f_2 \rangle$ and satisfies $wg = \alpha w$ with $\alpha \neq 1$ (a_2 does not commute with a_1). If $\langle f_1 \rangle \neq \langle f_2 \rangle$, g is scalar and (F3) applies to a_2, b_1 . Otherwise the axes $l_1 \neq l_2$ of b_1, b_2 are parallel, a_2 preserves both and acts on each with ratio $\alpha \neq 1$, so it fixes a point on each; thus g has the eigenvalue 1, $g = \text{diag}(\alpha, 1)$ in the basis (w, f_1) , and (F3) applies. (ii) The linear part u of a_1 is a transvection. The p' -elements of $C_{\text{GL}_2(p)}(u)$ are scalars,

so b_1, b_2 are homotheties, with distinct centres c_1, c_2 (equal centres would make $\langle b_1, b_2 \rangle$ cyclic), and a_1 fixes c_1, c_2 , hence the line $m = c_1 c_2$ pointwise. So does a_2 . If a_2 is a p -element, a_1 and a_2 lie in the group of order p of maps fixing m pointwise with unipotent linear part, so $\langle a_1, a_2 \rangle$ is cyclic; otherwise a_2 is a homology and (F3) applies to a_2, b_1 . $\square$

Theorem 4.7. *Assume that the maximal subgroups of $\mathrm{PSL}_3(p)$, $p \in \mathcal{P}_L$, $p \geq 5$, and of $\mathrm{PSU}_3(p)$, $p \in \mathcal{P}_U$, $p \geq 7$, are the images of the groups in Table 1. Then these groups have C_4 -free enhanced power graphs.*

Proof. Each row of Table 1 is covered by the indicated lemma, so Lemma 2.4(b) applies. $\square$

4.3 Proofs of Theorem 1.4 and Corollary 1.5

(a) The eight classes of maximal subgroups of ${}^2F_4(2)'$ (orders 11232 twice, 10240, 7800, 6144, 1440 twice, 1200), taken from the table of marks in TomLib and the same as in the ATLAS [CCN⁺85, p. 74], all have C_4 -free enhanced power graphs, and an exhaustive search on the whole group (by Lemma 2.3(b), without maximal subgroups) finds no C_4 -configuration (Section 7). (b) For $p \geq 5$ (resp. $p \geq 7$) this is Theorem 4.7 with [BHRD13]; $\mathrm{PSL}_3(3)$, $\mathrm{PSU}_3(3)$ and $\mathrm{PSU}_3(5)$ are decided by computation, and the exhaustive search of Lemma 2.3(b) on the whole group finds no C_4 -configuration in $\mathrm{PSL}_3(p)$, $p \in \{3, 5, 7, 13, 17\}$, and $\mathrm{PSU}_3(p)$, $p \in \{3, 5, 7, 11, 23\}$. (c) For q even this is Section 3.4; for q odd, Lemma 4.1 and the remark after it; for ${}^2G_2(q)$, Proposition 3.2; for $\mathrm{PSL}_3(q)$, [BFP25, Lemma 6.5(b)].

Proof of Corollary 1.5. By (1), S_4 is contained in the right-hand side of (1). There, $\mathcal{L}_1 \cup \{A_7\} \subseteq S_4$ by Theorem 1.1, $M_{11}, J_1 \in S_4$ by [BFP25], ${}^2F_4(2)' \in S_4$ and $\mathrm{PSL}_3(p), \mathrm{PSU}_3(p) \in S_4$ for $p \in \mathcal{P}_L$, resp. $p \in \mathcal{P}_U$, by Theorem 1.4(a),(b), and all other groups have a C_4 by Theorem 1.4(c). $\square$

5 The upper bound of Theorem B of Bubboloni, Fumagalli and Praeger

The upper bound (1) says that every finite nonabelian simple group outside the right-hand side of (1) has a C_4 . It is proved in [BFP25, pp. 29–33] family by family; Table 2 records the argument and what we checked. “Re-derived” means that the argument was written out again from the cited facts; “computed” means an explicit 4-cycle in an explicitly constructed group (Section 7).

The explicit 4-cycles are in $M_{12}, M_{22}, M_{23}, M_{24}, J_2, J_3, HS, McL, Co_3, Co_2, Suz, He, Ru, Fi_{22}$. The other ten sporadic groups (besides M_{11}, J_1) are covered by the containments $Co_1 > Co_2$, $Fi_{23} > S_{12} > A_8$, $Fi'_{24} > Fi_{23}$, $B > Fi_{23}$, $M > (A_5 \times A_{12}):2$, $HN > A_{12}$, $Th > {}^3D_4(2)$, $Ly > G_2(5)$ (which contains $\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)$ [LSS92]), $O'N > (3^2:4 \times A_6).2 > C_3^2 \times C_2^2$ and $J_4 > M_{22}:2$ of the ATLAS [CCN⁺85]; each is confirmed by the maximal-subgroup data of CTblLib [Bre25].

Remarks on [BFP25]. We note four points; none affects the statement of [BFP25, Theorem B]. (1) For $G = {}^3D_4(q)$ or $G_2(q)$ the proof uses the subgroup $\mathrm{SL}_2(q) \times \mathrm{SL}_2(q)$ (q even) or $\mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)$ (q odd), “and so by Corollary 4.16, $E(G)$ is not C_4 -free” [BFP25, p. 31]; part (e) of that corollary needs $q \geq 4$, and $\mathcal{E}(S_3 \times S_3)$ is C_4 -free (in a C_4 -configuration one side would consist of 3-elements generating the Sylow 3-subgroup C_3^2 , which is self-centralising). As $G_2(2)$ is not simple and $G_2(2)' \cong \mathrm{PSU}_3(3)$, only ${}^3D_4(2)$ is affected directly. But the proof then passes from ${}^3D_4(q) < F_4(q)$ to $E_6(q)$, ${}^2E_6(q)$ and on to $E_7(q)$, $E_8(q)$ [BFP25, p. 31], so for $q = 2$ the steps for $F_4(2)$, $E_6(2)$, ${}^2E_6(2)$, $E_7(2)$ and $E_8(2)$ rest on ${}^3D_4(2)$ as well, and so do the rows of Th and M in [BFP25, Table 1] (B also has the row A_8). The gap, and with it all these steps, is closed by the maximal subgroup $S_3 \times \mathrm{PSL}_2(8)$ of ${}^3D_4(2)$ [CCN⁺85, p. 89]: two involutions generating S_3 and two elements of order 7 generating $\mathrm{PSL}_2(8)$ form

family	subgroup with a C_4 used in [BFP25]	status here
$A_n, n \geq 8$	$((12)(34), (567), (13)(24), (568))$	re-derived, computed
$\text{PSL}_n(q), n \geq 4$	$A_8 \cong \text{SL}_4(2)$ (q even); $\text{SL}_2(p)^2$ or $\text{SL}_2(p) \circ \text{SL}_2(p)$ (q odd), [BFP25, Cor. 4.16(c),(d)]	re-derived
$\text{PSL}_3(q)$	[BFP25, Lemma 6.5(b)]	re-derived; computed for $q = 9, 25$
$\text{PSU}_n(q), n \geq 4$	$\text{SU}_2(p)^2$ (q odd); $\text{SL}_2(q)^2$ ($q > 2$ even); $A_8 < \text{Sp}_6(2) < \text{PSU}_6(2)$ [CCN ⁺ 85, pp. 46, 115] and matrices in $\text{SU}_4(2)$ ($q = 2$)	re-derived; the $\text{SU}_4(2)$ matrices computed
$\text{PSp}_n(q), \text{P}\Omega_n^\epsilon(q)$	$\text{Sp}_2(q)^2$; $A_8 \leq \Omega_6^+(2)$; $A_8 \leq \Omega_7(p)$ [KL90, Lemma 4.1.1, pp. 185–187]	re-derived
$G_2(q), {}^3D_4(q)$	$\text{SL}_2(q)^2$ or $\text{SL}_2(q) \circ \text{SL}_2(q)$ [LSS92, Table 5.1]	from [LSS92]; gap at ${}^3D_4(2)$, closed below
other exceptional	$\text{Sp}_4(q) \leq {}^2F_4(q)'$ ($q > 2$), ${}^3D_4(q) \leq F_4(q) \leq E_6(q), {}^2E_6(q), E_6(q)$ or ${}^2E_6(q) \leq E_7(q), E_8(q)$ [LSS92, Cra23]	from [LSS92, Cra23], for $q = 2$ via ${}^3D_4(2)$ (see below)
sporadic	[BFP25, Table 1] and the text	C_4 's computed for 14 groups; containments for ten

Table 2: The proof of the upper bound (1) in [BFP25], and what was checked here.

a C_4 -configuration; this 4-cycle is also computed in the primitive permutation group of degree 819 and order 211341312. (2) The first row of [BFP25, Table 1] lists the subgroup $A_4 \times C_2^2$ for J_1 ; this group has order 48, while the Sylow 2-subgroups of J_1 have order 8, and the proof shows that $\mathcal{E}(J_1)$ is C_4 -free. The group J_3 does not appear elsewhere in the proof; we found $A_4 \times C_2^2 \leq J_3$, with an explicit 4-cycle in the ATLAS representation of degree 6156, so the upper bound is not affected. (3) For ${}^2F_4(2)'$ the proof uses $\text{PSL}_3(3) < {}^2F_4(2)'$ and states “In either case we showed above that $\mathcal{E}(H)$ is not C_4 -free” [BFP25, p. 31]; by Theorem 1.4, $\mathcal{E}(\text{PSL}_3(3))$ and even $\mathcal{E}({}^2F_4(2)')$ are C_4 -free, and ${}^2F_4(2)'$ is kept in the right-hand side of (1). (4) The sentence “we will show that $\mathcal{E}(G)$ is not C_4 -free for $G = \text{M11}$ and $G = \text{J1}$ ” [BFP25, p. 29] should read “ C_4 -free”, which is what the proofs on pp. 32–33 show.

6 Cycles of length at least 5

Lemma 2.2 controls only holes of minimal length, so our methods decide membership in S_n only below and at the least hole length.

Lemma 6.1. *Suppose that every pair $\{r, s\}$ of distinct primes such that G has an element of order divisible by rs (an edge of the prime graph of G) contains 2. Then every hole of $\mathcal{E}(G)$ of minimal length has even length.*

Proof. Take a hole as in Lemma 2.2. Consecutive vertices are r - and s -elements with $r \neq s$ that commute, so their product has order divisible by rs ; hence $2 \in \{r, s\}$, and 2-elements alternate with elements of odd prime-power order. $\square$

Proof of Proposition 1.6. For $G = {}^2F_4(2)'$ the element orders are 1, 2, 3, 4, 5, 6, 8, 10, 12, 13, 16 (computed from the conjugacy classes), so the edges of the prime graph are $\{2, 3\}$ and $\{2, 5\}$; for $\text{PSL}_3(q)$ and $\text{PSU}_3(q)$ with $q \in \{3, 5, 7\}$ every edge contains 2 as well (the edges are $\{2, 3\}$, and $\{2, p\}$ for $p = 5, 7$; computed from the conjugacy classes). These groups are C_4 -free (Theorem 1.4) and have holes of length 6 (Sections 3.2, 3.3 and 3.6), so by Lemma 6.1 the least hole length is 6. For J_1 the prime graph has the triangle $\{2, 3, 5\}$; $\mathcal{E}(J_1)$ is C_4 -free [BFP25], has the hole of length 6 of Section 3.6,

and has no hole of length 5 by an exhaustive search. For M_{11} the only edge is $\{2, 3\}$, and an exhaustive search shows that there is no hole of length 6 or 8; the hole of length 10 is in Section 3.6. $\square$

The searches use the graph $\Gamma(G)$ on the non-trivial cyclic subgroups of prime-power order, $X \sim Y$ if $\langle X, Y \rangle$ is cyclic: generators of the vertices of a hole of $\Gamma(G)$ form a hole of $\mathcal{E}(G)$, and by Lemma 2.2 a shortest hole of $\mathcal{E}(G)$ gives one of $\Gamma(G)$ of the same length; G acts on $\Gamma(G)$, so one starting vertex per class suffices ($\Gamma(M_{11})$ has 1915 vertices, $\Gamma(J_1)$ has 14631). For the non-simple $\text{P}\Gamma\text{L}(2, 8)$ the least hole length is 10. Beyond Proposition 1.6 we know for $n \geq 5$ only that $\mathcal{L}_1 \cup \{A_7\} \subseteq S_n$ and that $\text{PSL}_3(q), \text{PSU}_3(q) \notin S_6$ for odd q and ${}^2G_2(q) \notin S_{10}$ (Section 3.5).

7 Computations and reproducibility

The theorems on infinite families do not depend on computations; the computations prove the statements marked computer-assisted, check the constructions for small q , and re-check parts of [BFP25]. They use GAP 4.16.1 with the packages cited in Section 1 and Python 3 with networkx; the package contains the programs, their recorded outputs, the certificates, and a README saying which program checks which statement. Ambient groups: $M_{11}, M_{22}, M_{23}, M_{24}$ are GAP's `MathieuGroup`; J_1 is `PrimitiveGroup(266, 1)`; J_3 is generated by the two permutations of degree 6156 of the ATLAS of Group Representations shipped with AtlasRep; ${}^2F_4(2)'$ is the group of the TomLib table of marks; $J_2, HS, McL, Co_3, Co_2, Suz, He, Ru, Fi_{22}$ are the primitive groups of degrees 100, 100, 275, 276, 2300, 1782, 2058, 4060, 3510 of the right orders; ${}^2G_2(27)$ is GAP's 7-dimensional matrix group acting on 19684 lines. Each group is identified by its order and simplicity.

Programs of the first computation. `epg_build.g` and `epg_analyze.py` decide for $\Gamma(G)$ (Section 6) the existence of a C_4 , the least hole length and chordality; every hole found is re-checked in GAP pair by pair. The programs `cert_*.g` apply them, or explicit constructions, to the groups of Sections 3 to 5, with maximal subgroups computed by GAP ($p \leq 7$), ClassicalMaximals ($\text{PSU}_3(11), \text{PSL}_3(13), \text{PSU}_3(23)$) and TomLib (${}^2F_4(2)'$). As a sanity check, `cert_sanity.g` treats all 56 nonabelian simple groups of order less than 10^6 : exactly the 39 groups $\text{PSL}_2(q)$ of this range, $A_7, \text{PSL}_3(4)$ and $\text{Sz}(8)$ have chordal enhanced power graphs, as Theorem 1.1 and Corollary 1.3 predict; $\text{PSL}_3(3), \text{PSU}_3(3), M_{11}, \text{PSU}_3(5), J_1, \text{PSL}_3(5)$ are C_4 -free and not chordal; the other eight have a C_4 . An independent program of verification run B below agrees for all 56 groups.

Programs written with this note. `check_note.g` re-checks in about 15 seconds every stored hole and 4-cycle pair by pair, the constructions of Sections 3 and 4 for small q (including Lemma 4.1 for $\text{PSL}_3(11), \text{PSU}_3(9), \text{PSU}_3(13)$ and [BFP25, Lemma 6.5(b)] for $q = 9, 25$), Proposition 3.2 for $q = 27$ with the membership of the stored matrices in ${}^2G_2(27)$, the 4-cycles of ${}^3D_4(2)$ and M_{12} , and the prime graphs of Section 6. `check_c4_direct.g` runs the exhaustive search of Lemma 2.3(b) on the whole group (b_1 up to conjugacy, $a_1 \in C_G(b_1)$, a_2 up to $C_G(\langle a_1, b_1 \rangle)$, then $C_G(\langle a_1, a_2 \rangle)$) for ${}^2F_4(2)', M_{11}, J_1, \text{PSL}_3(p)$ ($p = 3, 5, 7, 13, 17$) and $\text{PSU}_3(p)$ ($p = 3, 5, 7, 11, 23$), finding no C_4 ; it finds one in each of the controls $A_8, M_{12}, \text{PSU}_3(4), \text{PSU}_3(9), \text{PSL}_3(11)$, and none in A_7 .

Verification. The results were first obtained, with proofs and with the programs of the first computation. A first verification followed in two independent, AI-assisted runs, A and B, on that first write-up; each re-derived the proofs and wrote its own programs. Run A re-derived every proof of Sections 2 to 4 that does not rest on a computation, and the closing of the gap at ${}^3D_4(2)$, and compared all statements taken from [BFP25] with its text. Its own exhaustive whole-group test found no C_4 in ${}^2F_4(2)', A_7, \text{PSL}_3(q)$ and $\text{PSU}_3(q)$ ($q = 3, 5, 7$), $M_{11}, J_1, \text{P}\Gamma\text{L}(2, 8)$, and one in $A_8, M_{12}, \text{PSU}_3(4)$; it repeated Proposition 3.2 for $q = 27$ for all 26 non-trivial cosets of E in a Sylow 3-subgroup of $N_G(E)$, re-checked the certificates, and compared the lists of maximal subgroups with the source

of ClassicalMaximals. It required six corrections: stating the external inputs exactly; Lemma 2.3(a) without prime powers, and the exclusion of $\mathrm{PSL}_3(9)$, $\mathrm{PSL}_3(25)$ by [BFP25, Lemma 6.5(b)]; the exponent $2p - 1$ in the Borel action; precise references for the maximal subgroups and for the centraliser in ${}^2G_2(q)$; and the least hole length 6 of ${}^2F_4(2)'$. Run B built $\Gamma(G)$ with its own programs, recomputed the 56 groups of order below 10^6 , found the least hole length 6 for $\mathrm{PSL}_3(7)$, $\mathrm{PSU}_3(7)$ and (by the argument of Lemma 6.1) ${}^2F_4(2)'$, found no C_4 by a whole-group test in $\mathrm{PSL}_3(p)$, $p = 3, 5, 7, 13, 17$, and $\mathrm{PSU}_3(p)$, $p = 3, 5, 7, 11$, and found a C_4 in thirteen groups $\mathrm{PSL}_3(q)$, $\mathrm{PSU}_3(q)$ ($q \leq 29$) covered by Theorem 1.4(c). It rebuilt the constructions of Section 3 in other coordinates, J_3 with its own reader of the ATLAS files, and the 4-cycles of ${}^3D_4(2)$ and ${}^2G_2(27)$, tested the upper bound (1) on thirteen further groups, and repeated the literature search. It required five corrections: the orders of the stored hole of J_1 ; the least hole length of ${}^2F_4(2)'$; the labelling of the dependence on [BHRD13] in Theorem 1.4; quotations of the passages of [BFP25] behind the remarks of Section 5; and the ambient representations of the certificates. Neither run found a mathematical error, and all corrections were made. When this note was written, the whole-group search was repeated with `check_c4_direct.g` and extended to $\mathrm{PSU}_3(23)$ (12654 triples, no C_4 , about 15 minutes), which both earlier computations had treated only through the lists of ClassicalMaximals.

A second verification run, AI-assisted and independent of runs A and B, then examined the text of this note; its programs and outputs are in `independent_run_2/`. It compared every theorem, hypothesis and list of groups and primes with the first write-up and every statement taken from [BFP25] with the text of [BFP25], re-derived all proofs, and found no mathematical error. Its own programs computed, without GAP, the graph $\Gamma(G)$ and its least hole length for A_7 (chordal), S_6 (6), M_{11} (10), $\mathrm{PTL}(2, 8)$ (10), $\mathrm{PSL}_3(3)$, $\mathrm{PSU}_3(3)$, $\mathrm{PSL}_3(5)$ and J_1 (6; J_1 from Janko's matrices over $\mathbb{F}_{11}$), with the numbers of vertices and edges of the first computation, and checked the hole of M_{11} printed in Section 3.6. An exhaustive whole-group search of its own, organised by the elements whose order has exactly two prime divisors, found no C_4 -configuration in M_{11} , J_1 , ${}^2F_4(2)'$ (in its primitive action on 1755 points), $\mathrm{PSL}_3(p)$ for $p = 3, 5, 7, 13, 17$ and $\mathrm{PSU}_3(p)$ for $p = 3, 5, 7, 11, 23$, none in A_7 , S_6 , $\mathrm{PTL}(2, 8)$, and one in each of A_8 , M_{12} , $\mathrm{PSU}_3(4)$, $\mathrm{PSL}_3(9)$, $\mathrm{PSU}_3(9)$, $\mathrm{PSL}_3(11)$. It rebuilt from the text the constructions of Sections 3.2 to 3.4 and the 4-cycle of [BFP25, Lemma 6.5(b)] for 21 values of q ; recomputed the prime graphs of Section 6, a 4-cycle in $S_3 \times \mathrm{PSL}_2(8) < {}^3D_4(2)$ and the centraliser $\langle J \rangle \times \mathrm{PSL}_2(27)$ of an involution of ${}^2G_2(27)$; re-checked all stored certificates (among them the subgroup $A_4 \times C_2^2$ of J_3) in ambient groups rebuilt by itself, and the containments of Section 5 with CTblLib; re-ran the programs of the package from the archive, with the recorded summary results; and repeated the literature search. It required seven corrections, all made: the consequences of the gap at ${}^3D_4(2)$ in remark (1) of Section 5; the CTblLib confirmation of the containments over $\mathbb{F}_2$ and the subgroup $M_{22}:2$ of J_4 ; the computational input of Proposition 1.6; the word “explicit” in the abstract; the credits to Cameron, to [BFP25, Lemmas 4.9, 4.12 and 6.6] and to [AAC⁺17]; this description of the verification; and short additions to the proofs of Lemma 2.4(b), Theorem 1.1 and Section 3.4 and to the end of Section 6.

Scope and priority. The note answers part (b) of Problem 21.23 assuming the upper bound (1) of [BFP25] (Corollary 1.3), and part (a) for $n = 4$ assuming (1) and the lists of [BHRD13] (Corollary 1.5); for $n \geq 5$ it gives only Proposition 1.6. The problem, Theorems A and B, Proposition 6.1, Lemmas 3.2, 4.9(ii), 4.13 and 6.5, Proposition 4.14, Corollaries 4.15 and 4.16, and the elements used in Section 3.4 (from the proof of their Lemma 6.6) are due to Bubboloni, Fumagalli and Praeger [BFP25]; Lemma 2.1(c) for three elements is due to Aalipour et al. [AAC⁺17], and the cograph property of $\mathcal{E}(\mathrm{PSL}_2(q))$ to Cameron [Cam22]; Section 5 re-checks much of the proof of (1) but does not replace it. We read [BFP25] (version 1, the only version on arXiv on 27 October 2026) in full and the entry of [KM26]. We did not read [BHRD13], [Mit11], [Har25], [Blo67], [LSS92], [Cra23],

[KL90], [War66], [LN85] or the ATLAS; what we use from them is quoted in [BFP25], implemented in ClassicalMaximals or CTblLib, or (for [War66]) checked for $q = 27$. The second verification run could not read [War66], [Mit11] or [Blo67] either (the publisher’s archive refused automated access). The searches were made on 27 October 2026: the arXiv API (enhanced power graphs with chordal, cograph, simple, perfect, hole, induced cycle, Ree, unitary, Tits, sporadic; the authors of [BFP25]; Kourovka Problem 21.23), the works citing [BFP25] in OpenAlex (none), zbMATH Open, Crossref (no journal version of [BFP25]), and one web search; Semantic Scholar refused automated access. The second verification run repeated the queries to the arXiv API, Crossref, OpenAlex and zbMATH Open and one web search on the same day, with the same outcome. No printed answer to the question was found in these sources; a search that finds nothing is not a proof of novelty. The chordality of the *power* graph of simple groups is a different question [BK23], and [vDJMM26] proves, according to its abstract, that a power graph which is a cograph is chordal (Problem 21.24 of [KM26]). No priority is claimed.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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