

On Part b) of Kourovka Notebook Problem 19.93: Finite p -Groups with Unbounded Lower p -Central Factors and No $C_p \wr C_p$ Quotient for $p = 2, 3, 5$

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October 21, 2026

Abstract

Problem 19.93 of the Kourovka Notebook, posed by P. Spiga in 2018, conjectures that there is a function $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that every factor of the lower p -central series of a finite p -group with d generators and without an epimorphic image isomorphic to $C_p \wr C_p$ has order at most $f(p, d)$; part a) concerns $p \geq 7$, part b) $p = 2, 3, 5$. M. Kinyon and J. Urban refuted part a), in a preprint that declares AI assistance, with a filtered Golod–Shafarevich inequality and a counting bound for truncated group algebras; the method of this note is theirs. We show that part b) is false: for each of $p = 2, 3, 5$ no bound $f(p, d)$ exists (for $p = 5$ with any $d \geq 3$, for $p = 3$ with any $d \geq 5$, for $p = 2$ with any $d \geq 7$). What is new is the choice of relators: x_i^p and $(x_i x_j)^p$ on three generators for $p = 5$ and on five for $p = 3$, where the argument of Kinyon and Urban in $C_p \wr C_p$, applied to each pair of generators, excludes this quotient; and twelve products of squares and commutators on seven generators for $p = 2$, where the exclusion of $C_2 \wr C_2$ becomes a finite statement about twelve bilinear forms over $\mathbb{F}_2$, checked exactly in two ways. We also observe why the Golod–Shafarevich argument with the relations used for $p \geq 7$ fails for $p = 2, 3, 5$. Every step is proved in the note, including the Golod–Shafarevich inequality in the form used. With part a), the conjecture fails for every prime. The cases $(p, d) = (5, 2), (3, 2), (3, 3), (3, 4)$ and $(2, 2), \dots, (2, 6)$ are not decided. No printed answer to part b) was found in the sources searched. This is an unrefereed note.

1 Introduction

1.1 The problem

Let p be a prime. The *lower p -central series* of a group G is given by $P_1(G) = G$ and $P_{i+1}(G) = P_i(G)^p [P_i(G), G]$, the subgroup generated by the p -th powers of the elements of $P_i(G)$ and the commutators $[h, g] = h^{-1}g^{-1}hg$ with $h \in P_i(G)$, $g \in G$. Its factors are elementary abelian, and for a finite p -group it reaches 1. We write $d(G)$ for the smallest number of generators of G , and $W_p = C_p \wr C_p$ for the regular wreath product (Section 3), a group of order p^{p+1} ; W_2 is dihedral of order 8.

Problem 19.93 of the Kourovka Notebook [KM26b], posed by P. Spiga in the 19th issue (2018), is a conjecture, which we paraphrase: there is a function $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that, whenever G is a finite p -group with d generators that has no epimorphic image isomorphic to W_p , every factor of the lower p -central series of G (the Notebook says “ p -lower central series”) has order at most $f(p, d)$; part a) is this assertion for $p \geq 7$, part b) for $p = 2, 3, 5$. The proposer suggests a comparison with Shalev’s characterization of finitely generated p -adic analytic pro- p groups [Sha92].

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We read the problem in arXiv v48 (6 October 2026, page 152) and v49 (20 October 2026, printed page 153) of the 21st edition; the text of 19.93 and its comment are the same in both. Part a) carries the star that marks a confirmed solution, and the comment says that the conjecture is refuted for $p \geq 7$ by an AI solution confirmed by P. Spiga, citing the preprint [KU26] of M. Kinyon and J. Urban. Part b) carries no mark. The update file [KM26a] of the Notebook’s web site, read on 21 October 2026, whose header reads “November 2026 update” (sic), lists 19.93 on its page 24 in the same form. The Notebook marks with a sharp sign the problems with an AI solution not yet confirmed by mathematicians and regards them as unsolved [KM26b, KM26a]. In the UnsolvedMath corpus [ula26] the problem is the record KOU-19.93, with status “partially solved”.

Reading. The hypothesis concerns epimorphic images of G itself, not sections of G or quotients of its subgroups. Our groups have $d(G) = d$, which covers both readings, $d(G) \leq d$ and $d(G) = d$, of “ d generators”. As the conjecture asserts one function on all pairs (p, d) , part b) is refuted for a prime p by one value of d and, for every $K \geq 0$, a group as in the hypothesis with a lower p -central factor of order greater than p^K . The Jennings–Zassenhaus reading of the series gives the same conclusion (Remark 7.1).

1.2 The result

Let $x_1, \dots, x_d$ be free generators. We use the following sets of relators.

(R5) $(p, d) = (5, 3)$: x_i^5 for $1 \leq i \leq 3$ and $(x_i x_j)^5$ for $1 \leq i < j \leq 3$; six relators.

(R3) $(p, d) = (3, 5)$: x_i^3 for $1 \leq i \leq 5$ and $(x_i x_j)^3$ for $1 \leq i < j \leq 5$; fifteen relators.

(R2) $(p, d) = (2, 7)$: the twelve relators

$$\rho_k = \prod_{i=1}^7 x_i^{2c_i^{(k)}} \prod_{1 \leq i < j \leq 7} [x_i, x_j]^{c_{ij}^{(k)}} \quad (1 \leq k \leq 12),$$

with the exponents $c_i^{(k)}, c_{ij}^{(k)} \in \{0, 1\}$ of Table 1 in Section 3; the squares come first, then the commutators in the lexicographic order of (i, j) .

Theorem 1. *Let (p, d) be $(5, 3)$, $(3, 5)$ or $(2, 7)$. For every $K \geq 0$ there is a finite p -group Q with $d(Q) = d$ and no epimorphic image isomorphic to $C_p \wr C_p$, such that some factor $P_i(Q)/P_{i+1}(Q)$ of its lower p -central series has order greater than p^K . One can take for Q a group Q_N of Section 5, defined by the relators (R5), (R3) or (R2) and a truncation level N ; the level N and the index i of the large factor depend on K .*

Consequently no function $f(p, d)$ as in Problem 19.93 exists for $p = 2, 3, 5$. By Remark 7.2, the group $Q \times C_p^m$ has $d + m$ generators, no epimorphic image isomorphic to $C_p \wr C_p$, and again a lower p -central factor of order greater than p^K . Hence part b) is false: for each of $p = 2, 3, 5$ no bound $f(p, d)$ exists (for $p = 5$ with any $d \geq 3$, for $p = 3$ with any $d \geq 5$, for $p = 2$ with any $d \geq 7$). Together with the refutation of part a) in [KU26, §3.33], the conjecture fails for every prime. The open cases are listed in Remark 7.3.

1.3 Method, credits and what is new

The method is that of Kinyon and Urban [KU26, §3.33], who refuted part a) for every fixed prime $p \geq 7$ with $d = 2$: they take the groups generated by $1 + X$ and $1 + Y$ in truncations of $\mathbb{F}_p\langle\langle X, Y \rangle\rangle$ modulo the relation elements of $x^p = y^p = (xy)^p = 1$, relations that exclude a quotient $C_p \wr C_p$ by an argument in $\mathbb{F}_p^2$ (parts (i), (ii) and the case $d = 2$ of (iii) of Lemma 3), and use the complete filtered

Golod–Shafarevich inequality [Ers12, Theorem 2.6], due to Vinberg [Vin65], and a counting bound for truncated group algebras. Their counterexamples, like ours, have no quotient $C_p \wr C_p$ of the group itself: the construction “excludes wreath quotients of Q_N itself, exactly as asked” [KU26, §3.33]. The index of results of [KU26] lists 19.93 with the parts “all”, while its Section 3.33 proves the statement for every fixed prime $p \geq 7$. The preprint states that its arguments originated in a 48-hour experiment using GPT-6 Astra through Codex and were later reviewed by an AI model [KU26, §1].

For $p = 2, 3, 5$ the argument with these relations fails: the polynomial $1 - 2t + 3t^p$ is positive on $(0, 1)$ for $p = 3, 5$ (Remark 7.6), and for $p = 2$ the relations give $xy = (xy)^{-1} = y^{-1}x^{-1} = yx$. More generators help the Golod–Shafarevich condition but call for more relators to exclude $C_p \wr C_p$. The new content of this note is the choice of relators: (R5) and (R3), to which the argument of [KU26, §3.33] that excludes $C_p \wr C_p$ applies pair by pair (Lemma 3), and (R2), with the bilinear-form criterion of Lemma 4 and certificate C; and the observation of Remark 7.6. To keep the note independent of [KU26], every step is proved here, including the filtered inequality in the cumulative form used (Lemmas 1 and 2) and the counting bound (Lemma 5).

This note was also prepared with AI assistance (see the last paragraph before the references) and has not been refereed. Under the Notebook’s rule stated above, part b) should therefore be regarded as open until the argument has been checked by experts.

2 The filtered Golod–Shafarevich inequality

Let p be a prime, $d \geq 1$, and let $\mathcal{F} = \mathbb{F}_p\langle\langle X_1, \dots, X_d \rangle\rangle$ be the algebra of formal power series in non-commuting variables. For $n \geq 1$ let $\mathcal{F}_n$ be the set of series all of whose monomials have length at least n , and let $\mathcal{F}_n = \mathcal{F}$ for $n \leq 0$. Each $\mathcal{F}_n$ is a two-sided ideal, $\mathcal{F}_n \mathcal{F}_m \subseteq \mathcal{F}_{n+m}$, and $\mathcal{F}/\mathcal{F}_n$ is finite-dimensional. Every element $1 + m$ with $m \in \mathcal{F}_1$ is a unit, $(1 + m)^{-1} = \sum_{k \geq 0} (-m)^k$, the sum converging coefficientwise. A group word w in $x_1, \dots, x_d$ (an element of the free group) has a value $w(1 + X)$ in the group $1 + \mathcal{F}_1$, obtained by substituting $1 + X_i$ for x_i ; its *relation element* is $f_w = w(1 + X) - 1 \in \mathcal{F}_1$.

Let $w_1, \dots, w_r$ be words with relation elements $f_k = f_{w_k}$, and let $\delta_1, \dots, \delta_r \geq 1$ be integers with $f_k \in \mathcal{F}_{\delta_k}$. Let I be the two-sided ideal of $\mathcal{F}$ generated by $f_1, \dots, f_r$, that is, the set of finite sums $\sum u f_k v$ with $u, v \in \mathcal{F}$, and put

$$B_N = \dim_{\mathbb{F}_p} \mathcal{F}/(\mathcal{F}_N + I) \quad (N \in \mathbb{Z}).$$

Then $B_N = 0$ for $N \leq 0$, and $B_1 = 1$ because $I \subseteq \mathcal{F}_1$. Since $\mathcal{F}_N + I$ contains the open ideal $\mathcal{F}_N$, it is closed in the degree topology; so B_N does not change if I is replaced by its closure, as in [Ers12, §2.3].

Every $h \in \mathcal{F}_1$ has a unique expression $h = \sum_{i=1}^d h_i X_i$ with $h_i \in \mathcal{F}$: the series h_i collects the monomials of h that end with X_i , with this last letter removed. If $h \in \mathcal{F}_n$, then $h_i \in \mathcal{F}_{n-1}$. In particular $f_k = \sum_i g_{k,i} X_i$ with $g_{k,i} \in \mathcal{F}_{\delta_k-1}$.

Lemma 1 (Vinberg; [Ers12, Theorem 2.6]). *For every $N \geq 0$,*

$$B_{N+1} \geq 1 + d B_N - \sum_{k=1}^r B_{N+1-\delta_k}.$$

Proof. Consider the linear map

$$\mu: (\mathcal{F}/(\mathcal{F}_N + I))^d \rightarrow \mathcal{F}_1/(\mathcal{F}_{N+1} + I), \quad (a_i)_i \mapsto \sum_i a_i X_i.$$

It is well defined, because $(\mathcal{F}_N + I)X_i \subseteq \mathcal{F}_{N+1} + I$, and it is onto by the decomposition above. Since $\mathcal{F}_{N+1} + I \subseteq \mathcal{F}_1$ and $\mathcal{F}_1$ has codimension 1 in $\mathcal{F}$, the target has dimension $B_{N+1} - 1$. Hence $B_{N+1} - 1 = dB_N - \dim \ker \mu$, and it suffices to show $\dim \ker \mu \leq \sum_k B_{N+1-\delta_k}$.

Every $h \in I$ is a finite sum $h = \sum_m u_m f_{k_m} v_m$. Write $v_m = c_m + \sum_i v_{m,i} X_i$ with $c_m \in \mathbb{F}_p$. Then $u_m f_{k_m} v_m = \sum_i (c_m u_m g_{k_m,i} + u_m f_{k_m} v_{m,i}) X_i$, and therefore

$$h = \sum_i \left(\sum_k U_k g_{k,i} + J_i \right) X_i \quad \text{with } U_k = \sum_{m: k_m=k} c_m u_m \in \mathcal{F}, \quad J_i = \sum_m u_m f_{k_m} v_{m,i} \in I.$$

Now let $(a_i) \in \ker \mu$, with representatives $\tilde{a}_i \in \mathcal{F}$, so that $\sum_i \tilde{a}_i X_i = \varphi + h$ with $\varphi \in \mathcal{F}_{N+1}$ and $h \in I$. Write $\varphi = \sum_i \varphi_i X_i$ with $\varphi_i \in \mathcal{F}_N$. By the uniqueness of the decomposition, $\tilde{a}_i = \varphi_i + \sum_k U_k g_{k,i} + J_i$, so that $(a_i) = \nu(U_1, \dots, U_r)$ for the map

$$\nu: \bigoplus_{k=1}^r \mathcal{F}/(\mathcal{F}_{N+1-\delta_k} + I) \rightarrow (\mathcal{F}/(\mathcal{F}_N + I))^d, \quad (U_k)_k \mapsto \left(\sum_k U_k g_{k,i} \right)_i.$$

The map ν is well defined. If $U \in \mathcal{F}_{N+1-\delta_k}$, then $U g_{k,i}$ lies in $\mathcal{F}_n$ with $n = \max(0, N+1-\delta_k) + \delta_k - 1 \geq N$, hence in $\mathcal{F}_N$; and if $U \in I$, then $U g_{k,i} \in I$. So $\ker \mu$ is contained in the image of ν , whose dimension is at most $\sum_k B_{N+1-\delta_k}$. $\square$

Multiplied by t^N and summed over $N \geq 0$, Lemma 1 says that $(1 - dt + \sum_k t^{\delta_k}) \sum_{n \geq 0} B_{n+1} t^n \geq 1/(1-t)$ coefficientwise. For the exact degrees δ_k and $A = \mathcal{F}/\bar{I}$ we have $\sum_n B_{n+1} t^n = \text{Hilb}_A(t)/(1-t)$ in the notation of [Ers12, §2.3], and this is the inequality (2.4) of [Ers12, Theorem 2.6]. Ershov attributes (2.4) to Vinberg [Vin65]; the original inequality is that of Golod and Shafarevich [GS64]. The proof above is self-contained, and it allows any $\delta_k \leq \deg f_k$.

Lemma 2 (divergence; cf. [Ers12, Proposition 2.7(a)]). *Let $\tau \in (0, 1)$ with $P(\tau) := 1 - d\tau + \sum_k \tau^{\delta_k} < 0$. Then $\sum_{N \geq 1} B_N \tau^N = +\infty$.*

Proof. Suppose that $S := \sum_{N \geq 0} B_N \tau^N$ is finite. Multiply the inequality of Lemma 1 by τ^{N+1} and sum over $0 \leq N \leq M$. All series that occur have non-negative terms and are bounded by S or by $\tau/(1-\tau)$; as $B_m = 0$ for $m \leq 0$, the limit $M \rightarrow \infty$ gives

$$S \geq \frac{\tau}{1-\tau} + d\tau S - \sum_k \tau^{\delta_k} S, \quad \text{that is,} \quad P(\tau) S \geq \frac{\tau}{1-\tau} > 0.$$

This is impossible, since $P(\tau) < 0$ and $S \geq 0$. $\square$

3 No quotient isomorphic to $C_p \wr C_p$

We index the coordinates of $\mathbb{F}_p^p$ by $\mathbb{Z}/p$ and write W_p for the set of pairs (v, a) with $v \in \mathbb{F}_p^p$ and $a \in \mathbb{Z}/p$, with the product

$$(v, a)(w, b) = (v + t^a \cdot w, a + b), \quad (t^a \cdot w)_k = w_{k-a}.$$

This is the regular wreath product $C_p \wr C_p$: the base group is $\mathbb{F}_p^p$, and $t = (0, 1)$ acts on it by the cyclic shift. Let $s(v) = \sum_k v_k \in \mathbb{F}_p$, let $\pi(v, a) = (s(v), a) \in \mathbb{F}_p^2$, and call $A_1 = \mathbb{F}_p \times \{0\}$ and $A_2 = \{0\} \times \mathbb{F}_p$ the two axes of $\mathbb{F}_p^2$.

Lemma 3 (cf. [KU26, §3.33]). (i) *The map $\pi: W_p \rightarrow \mathbb{F}_p^2$ is a surjective homomorphism.*

(ii) *For $w = (v, a) \in W_p$ we have $w^p = 1$ if and only if $s(v)a = 0$, that is, if and only if $\pi(w) \in A_1 \cup A_2$.*

(iii) *Let $d \geq 2$, and let G be a group generated by elements $y_1, \dots, y_d$ with $y_i^p = 1$ for all i and $(y_i y_j)^p = 1$ for all $i < j$. Then there is no epimorphism from G onto W_p .*

Proof. (i) We have $s(v+t^a \cdot w) = s(v)+s(w)$, since t^a permutes the coordinates, and $\pi((c, 0, \dots, 0), a) = (c, a)$.

(ii) By induction, $(v, a)^n = (\sum_{m=0}^{n-1} t^{ma} \cdot v, na)$. If $a \neq 0$, the elements ma with $0 \leq m < p$ run through $\mathbb{Z}/p$, so the first component of w^p is $\sum_{c \in \mathbb{Z}/p} t^c \cdot v$, the constant vector all of whose entries equal $s(v)$; hence $w^p = 1$ if and only if $s(v) = 0$. If $a = 0$, then $w^p = (pv, 0) = 1$.

(iii) Let $\varphi: G \rightarrow W_p$ be onto, and $u_i = \varphi(y_i)$. The u_i generate W_p , so by (i) the $\pi(u_i)$ generate $\mathbb{F}_p^2$, and by (ii) each $\pi(u_i)$ lies in $A_1 \cup A_2$. If all $\pi(u_i)$ lay in A_1 , they would generate at most A_1 ; so some $\pi(u_i)$ lies in $A_2 \setminus \{0\}$, and likewise some $\pi(u_j)$ lies in $A_1 \setminus \{0\}$, and $i \neq j$. Then $\pi(u_i u_j) = \pi(u_i) + \pi(u_j)$ has two non-zero coordinates, so $(u_i u_j)^p \neq 1$ by (ii). If $i < j$, this contradicts $(y_i y_j)^p = 1$. If $i > j$, then $u_j u_i = u_i^{-1}(u_i u_j)u_i$ is conjugate to $u_i u_j$, so $(u_j u_i)^p \neq 1$, which contradicts $(y_j y_i)^p = 1$. $\square$

Parts (i) and (ii), and (iii) for $d = 2$, are the argument of [KU26, §3.33] for the relations $x^p = y^p = (xy)^p = 1$; for $d > 2$ the proof applies it to a suitable pair u_i, u_j . The relators (R5) and (R3) have the form of Lemma 3(iii). The proof needs no computation; the checks made by programs are listed in Section 9. For $p = 2$ the relators x_i^2 and $(x_i x_j)^2$ would give an elementary abelian group, and we use (R2) instead. Write $D = W_2$, $z = ((1, 1), 0)$, and $\ell_1(v, a) = s(v)$, $\ell_2(v, a) = a$, so that $\pi = (\ell_1, \ell_2)$.

Lemma 4. (a) *The group D is dihedral of order 8, and $Z(D) = \langle z \rangle = \ker \pi = [D, D] = \Phi(D)$.*

(b) *For all $g, h \in D$ we have $g^2 = z^{\ell_1(g)\ell_2(g)}$ and $[g, h] = z^{\ell_1(g)\ell_2(h) + \ell_2(g)\ell_1(h)}$.*

(c) *Let ψ be a homomorphism from the free group on $x_1, \dots, x_7$ to D , let $g_i = \psi(x_i)$, and let $\alpha, \beta \in \mathbb{F}_2^7$ have the coordinates $\alpha_i = \ell_1(g_i)$ and $\beta_i = \ell_2(g_i)$. Then $\psi(\rho_k) = z^{\alpha^\top S_k \beta}$, where S_k is the symmetric 7×7 matrix over $\mathbb{F}_2$ with diagonal entries $c_i^{(k)}$ and off-diagonal entries $(S_k)_{ij} = (S_k)_{ji} = c_{ij}^{(k)}$ for $i < j$. Moreover, ψ is onto if and only if α and β are linearly independent.*

Proof. (a) Since t fixes $(1, 1)$, z is central, and $\ker \pi = \{1, z\}$. The group is not abelian ($((1, 0), 0)t = ((1, 0), 1)$, $t((1, 0), 0) = ((0, 1), 1)$), so $Z(D) = \langle z \rangle$ has order 2 (a centre of order 4 would make $D/Z(D)$ cyclic and D abelian). As $D/\langle z \rangle \cong \mathbb{F}_2^2$, all squares and commutators lie in $\langle z \rangle$, and $[D, D] \neq 1$; so $\Phi(D) = D^2[D, D] = [D, D] = \langle z \rangle$. The five involutions $(v, 0)$ ($v \neq 0$), t and $((1, 1), 1)$ show that D is dihedral.

(b) By (a), $g^2 \in \langle z \rangle$, and by Lemma 3(ii), $g^2 = 1$ if and only if $\ell_1(g)\ell_2(g) = 0$. For the commutator, $[h, g]$ is central and $hg = gh[h, g]$, so $(gh)^2 = g(hg)h = g^2 h[h, g]h = g^2 h^2[h, g]$. Writing $q(g) = \ell_1(g)\ell_2(g) \in \mathbb{F}_2$ for the exponent of g^2 , the exponent of $[h, g]$ is $q(gh) - q(g) - q(h) = \ell_1(g)\ell_2(h) + \ell_2(g)\ell_1(h)$, which is symmetric in g and h ; and $[g, h] = [h, g]^{-1} = [h, g]$.

(c) The factors of $\psi(\rho_k)$ are central of order at most 2, so $\psi(\rho_k) = z^e$ with $e = \sum_i c_i^{(k)} \alpha_i \beta_i + \sum_{i < j} c_{ij}^{(k)} (\alpha_i \beta_j + \alpha_j \beta_i) = \alpha^\top S_k \beta$. By the Burnside basis theorem [Hup67, Kapitel III], ψ is onto if and only if the cosets $g_i \Phi(D)$ generate $D/\Phi(D)$. By (a), π induces an isomorphism $D/\Phi(D) \rightarrow \mathbb{F}_2^2$ with $g_i \Phi(D) \mapsto (\alpha_i, \beta_i)$. So ψ is onto if and only if the rows (α_i, β_i) of the 7×2 matrix $(\alpha | \beta)$ span $\mathbb{F}_2^2$, that is, if and only if its columns α and β are linearly independent. $\square$

Hence a group generated by elements $y_1, \dots, y_7$ that satisfy the relators (R2) has an epimorphism onto D only if some linearly independent pair (α, β) in $\mathbb{F}_2^7$ satisfies $\alpha^\top S_k \beta = 0$ for $k = 1, \dots, 12$. Over $\mathbb{F}_2$, the pair is linearly independent if and only if α and β are distinct and non-zero; there are $127 \cdot 126 = 16,002$ such ordered pairs.

Certificate C. *For no ordered pair of distinct non-zero vectors $\alpha, \beta \in \mathbb{F}_2^7$ is $\alpha^\top S_k \beta = 0$ for all $k = 1, \dots, 12$.* This finite statement is verified exhaustively in two ways. *Check A:* for $m = 0, 1, \dots, 12$, the number of ordered pairs of distinct non-zero vectors with $\alpha^\top S_k \beta = 0$ for all $k \leq m$ is 16002, 7938, 3938, 1954, 1002, 530, 274, 122, 46, 22, 8, 2, 0. *Check B,* which does not use Lemma 4: all 8^7 maps $x_i \mapsto g_i \in D$ are enumerated, the relators are evaluated by multiplication in D , and a map counts

rho1	0000000		001101	10011	0001	111	01	1	rho7	1010010		100000	00111	0010	011	00	0
rho2	0111101		000111	01000	1100	001	01	0	rho8	0000001		111010	10110	0100	001	00	0
rho3	1000000		110101	00110	0001	100	10	0	rho9	1000011		010110	01100	0100	110	11	1
rho4	0000010		010001	10011	1011	000	01	1	rho10	1000111		100010	01001	1111	000	11	0
rho5	0010010		101010	00110	1001	010	10	1	rho11	1100111		000011	11101	0100	011	01	1
rho6	1100101		011101	11101	0110	110	10	0	rho12	1110001		111001	00100	1011	101	11	1

Machine-readable copy in the package: `data/p2_relator_table.json`, SHA-256
`ae6e2e599e6af8742255beaa3a75042860c6f17a562b49fb12d828b8679c58e4`

Table 1: The exponents of the relators $\rho_1, \dots, \rho_{12}$ of (R2). For each k : the diagonal $c_1^{(k)} \dots c_7^{(k)}$, then the rows of the strict upper triangle, row i listing $c_{i,i+1}^{(k)} \dots c_{i,7}^{(k)}$ (groups of 6, 5, 4, 3, 2, 1 entries). The same data, in the same format, is the DATA string of `scripts/d8_certificate.py` in the package.

as onto if its image lies in none of the three maximal subgroups of D . For every m the number of epimorphisms killing $\rho_1, \dots, \rho_m$ is 2^7 times the number in check A (each tuple of cosets of $\langle z \rangle$ has 2^7 lifts): $2,048,256 = 8^7 - 3 \cdot 4^7 + 2 \cdot 2^7$ for $m = 0$, and 0 for $m = 12$. The second verification run found that 114 of the 336 single-entry changes of Table 1 destroy the certificate, so the exact table matters.

4 Bounded widths force small truncated group algebras

The following counting bound is the one used in [KU26, §3.33]; we prove it in full. For a finite p -group Q let I_Q be the augmentation ideal of the group algebra $\mathbb{F}_p[Q]$, and let $b_i = \dim_{\mathbb{F}_p} P_i(Q)/P_{i+1}(Q)$ (so $b_i = 0$ for large i).

Lemma 5. *Let c_n be the coefficient of t^n in $\prod_{i \geq 1} (1 + t^i + \dots + t^{(p-1)i})^{b_i}$. Then $\dim_{\mathbb{F}_p} \mathbb{F}_p[Q]/I_Q^N \leq \sum_{n < N} c_n$ for every $N \geq 1$. If $b_i \leq K$ for all i , then $c_n \leq c_n^{(K)}$, the coefficient of t^n in the power series*

$$P_K(t) = \prod_{i \geq 1} (1 + t^i + \dots + t^{(p-1)i})^K,$$

and $P_K(\tau) = \sum_n c_n^{(K)} \tau^n$ is finite for $0 < \tau < 1$.

Proof. (a) If $g \in P_i(Q)$, then $g - 1 \in I_Q^i$. We use induction on i ; for $i = 1$ this is the definition of I_Q . The set of $g \in Q$ with $g - 1 \in I_Q^{i+1}$ is a subgroup, because $gh - 1 = (g - 1)(h - 1) + (g - 1) + (h - 1)$ and $g^{-1} - 1 = -g^{-1}(g - 1)$. It contains h^p for $h \in P_i(Q)$, since $h^p - 1 = (h - 1)^p \in I_Q^{ip} \subseteq I_Q^{i+1}$. It contains $[h, g]$ for $h \in P_i(Q)$ and $g \in Q$, since $[h, g] - 1 = h^{-1}g^{-1}((h - 1)(g - 1) - (g - 1)(h - 1)) \in I_Q^{i+1}$. Hence it contains $P_{i+1}(Q)$.

(b) *Spanning set.* Choose $g_{ij} \in P_i(Q)$, $1 \leq j \leq b_i$, whose images form a basis of the elementary abelian group $P_i(Q)/P_{i+1}(Q)$, and order the pairs (i, j) lexicographically. Every element of Q is a product $\prod_{(i,j)} g_{ij}^{e_{ij}}$ in this order with $0 \leq e_{ij} < p$: we determine the exponents for $i = 1$ modulo $P_2(Q)$, divide them off, and continue in $P_2(Q)$; the series ends at 1. Expanding $g^e = ((g - 1) + 1)^e = \sum_{e' \leq e} \binom{e}{e'} (g - 1)^{e'}$ shows that each such product is a linear combination of the ordered products $\prod_{(i,j)} (g_{ij} - 1)^{e'_{ij}}$ with $0 \leq e'_{ij} < p$. Since Q spans $\mathbb{F}_p[Q]$, these products span $\mathbb{F}_p[Q]$.

(c) By (a), $\prod (g_{ij} - 1)^{e_{ij}} \in I_Q^{\sum_{ij} e_{ij}}$. So the products with $\sum_{ij} e_{ij} < N$ span $\mathbb{F}_p[Q]$ modulo I_Q^N , and their number is $\sum_{n < N} c_n$.

(d) If $b_i \leq K$ for all i , the coefficients of $\prod_i (1 + \dots + t^{(p-1)i})^{b_i}$ are at most those of $P_K(t)$, since the remaining factors $(1 + t^i + \dots + t^{(p-1)i})^{K-b_i}$ have non-negative coefficients and constant term 1.

Finally, $1 + \tau^i + \dots + \tau^{(p-1)i} \leq (1 - \tau^i)^{-1}$ and $-\log(1 - \tau^i) \leq \tau^i / (1 - \tau^i) \leq \tau^i / (1 - \tau)$, so every partial product $\prod_{i \leq M} (1 + \tau^i + \dots + \tau^{(p-1)i})^K$ is at most $\exp(K\tau / (1 - \tau)^2)$. The coefficients of the partial products are non-negative, do not decrease with M , and agree with $c_n^{(K)}$ for $M \geq n$; by monotone convergence $\sum_n c_n^{(K)} \tau^n \leq \exp(K\tau / (1 - \tau)^2) < \infty$. $\square$

5 The groups Q_N

Fix one of the pairs $(p, d) = (5, 3), (3, 5), (2, 7)$ with its relators (R5), (R3) or (R2), let $r = 6, 15, 12$ be their number, and let I and B_N be as in Section 2 for these relators.

Degrees. For p odd the relators are w^p with $w = x_i$ or $w = x_i x_j$, and $Z = w(1 + X) - 1$ lies in $\mathcal{F}_1$ ($Z = X_i$ or $Z = X_i + X_j + X_i X_j$); since 1 commutes with Z and $\binom{p}{k} \equiv 0 \pmod{p}$ for $0 < k < p$, the relation element is $(1 + Z)^p - 1 = Z^p \in \mathcal{F}_p$. For $p = 2$, let $m, n \in \mathcal{F}_1$. Then $(1 + m)^2 - 1 = m^2 \in \mathcal{F}_2$ in characteristic 2, and $[1 + m, 1 + n] - 1 = (1 + m)^{-1}(1 + n)^{-1}(mn - nm) \in \mathcal{F}_2$; products of elements of $1 + \mathcal{F}_2$ lie in $1 + \mathcal{F}_2$. So every relation element of (R2) lies in $\mathcal{F}_2$. Hence we may take $\delta_k = \delta$ for all k , with $\delta = 5, 3, 2$, and the polynomial of Lemma 2 is $P(t) = 1 - dt + rt^\delta$, namely $1 - 3t + 6t^5$, $1 - 5t + 15t^3$ and $1 - 7t + 12t^2 = (1 - 3t)(1 - 4t)$. At $\tau = 1/2, 1/3$ and $7/24$ these take the values

$$1 - \frac{3}{2} + \frac{6}{32} = -\frac{5}{16}, \quad 1 - \frac{5}{3} + \frac{15}{27} = -\frac{1}{9}, \quad 1 - \frac{49}{24} + \frac{12 \cdot 49}{576} = -\frac{1}{48}.$$

In all three cases $I \subseteq \mathcal{F}_2$.

For $N \geq 2$ let $A^{(N)} = \mathcal{F}/(\mathcal{F}_N + I)$, an algebra of dimension B_N ; let $M_N = \mathcal{F}_1/(\mathcal{F}_N + I)$, an ideal of $A^{(N)}$ with $M_N^N = 0$; let $u_i \in A^{(N)}$ be the image of $1 + X_i$, and let

$$Q_N = \langle u_1, \dots, u_d \rangle \leq A^{(N)\times}.$$

Proposition 6. *For every $N \geq 2$:*

- (a) Q_N is a finite p -group; it is a subgroup of the group $1 + M_N$, which has order p^{B_N-1} ;
- (b) $d(Q_N) = d$;
- (c) the relators hold in Q_N : $w_k(u_1, \dots, u_d) = 1$ for all k ;
- (d) Q_N has no epimorphic image isomorphic to $C_p \wr C_p$;
- (e) $B_N \leq \dim_{\mathbb{F}_p} \mathbb{F}_p[Q_N]/I_{Q_N}^N$.

Proof. (a) The set $1 + M_N$ is closed under products, and $(1 + m)^{-1} = \sum_{k < N} (-m)^k$ because $m^N = 0$; so it is a group, of order $|M_N| = p^{B_N-1}$. The u_i lie in it.

(b) As $\mathcal{F}_N + I \subseteq \mathcal{F}_2$, the algebra $A^{(N)}$ maps onto $\mathcal{F}/\mathcal{F}_2$. In $\mathcal{F}/\mathcal{F}_2$ the units $1 + a$ with a in the span of the X_i form a group isomorphic to $(\mathbb{F}_p^d, +)$, because $(1 + a)(1 + b) = 1 + a + b$ modulo $\mathcal{F}_2$, and the images of the u_i correspond to a basis. So Q_N maps onto $\mathbb{F}_p^d$, and $d(Q_N) \geq d$; it is generated by d elements.

(c) The map $\mathcal{F} \rightarrow A^{(N)}$ is a ring homomorphism, so it maps $w_k(1 + X) = 1 + f_k$ to $w_k(u_1, \dots, u_d)$, and $f_k \in I$.

(d) An epimorphism $\varphi: Q_N \rightarrow W_p$ would map $u_1, \dots, u_d$ to generators of W_p that satisfy the relators. For $p = 3, 5$ this is impossible by Lemma 3(iii). For $p = 2$ it is impossible by Lemma 4(c) and certificate C.

(e) Let $\theta: \mathbb{F}_p[Q_N] \rightarrow A^{(N)}$ be the algebra homomorphism that extends the inclusion $Q_N \subseteq A^{(N)\times}$. Its image is a subalgebra that contains 1 and the $X_i = u_i - 1$, and $A^{(N)}$ is spanned by the products of the X_i ; so θ is onto. For $q \in Q_N$ we have $\theta(q - 1) = q - 1 \in M_N$; so $\theta(I_{Q_N}) \subseteq M_N$ and $\theta(I_{Q_N}^N) \subseteq M_N^N = 0$. Thus θ induces a surjection $\mathbb{F}_p[Q_N]/I_{Q_N}^N \rightarrow A^{(N)}$. $\square$

6 Proof of Theorem 1

Fix (p, d) and the relators as in Section 5, and let $\tau = 1/2, 1/3$ or $7/24$, respectively. Then $P(\tau) < 0$, and Lemma 2 gives

$$\sum_{N \geq 1} B_N \tau^N = +\infty. \quad (1)$$

Suppose, to reach a contradiction, that there is a $K \geq 0$ such that for every $N \geq 2$ every factor $P_i(Q_N)/P_{i+1}(Q_N)$ has order at most p^K . Then $b_i(Q_N) \leq K$ for all i and all $N \geq 2$, and Proposition 6(e) and Lemma 5 give $B_N \leq \sum_{n < N} c_n^{(K)}$ for all $N \geq 2$. This also holds for $N = 1$, since $B_1 = 1 = c_0^{(K)}$. Hence

$$\sum_{N \geq 1} B_N \tau^N \leq \sum_{N \geq 1} \sum_{n < N} c_n^{(K)} \tau^N = \sum_{n \geq 0} c_n^{(K)} \frac{\tau^{n+1}}{1 - \tau} = \frac{\tau P_K(\tau)}{1 - \tau} < \infty$$

by Lemma 5, which contradicts (1).

So the supposition is false: for every $K \geq 0$ there are a level $N = N(K) \geq 2$ and an index $i = i(K)$ such that $|P_i(Q_N)/P_{i+1}(Q_N)| > p^K$. By Proposition 6(a), (b) and (d), $Q = Q_N$ is a finite p -group with $d(Q) = d$ and no epimorphic image isomorphic to $C_p \wr C_p$. This proves Theorem 1. The order of the quantifiers is: for every K there exist N and i . If a function f as in Problem 19.93 existed, then for K with $p^K \geq f(p, d)$ the group $Q_{N(K)}$ would violate the bound; so no such f exists for $p = 5, 3, 2$. $\square$

7 Remarks

Remark 7.1 (the Jennings–Zassenhaus series). For a finite p -group Q let $D_n(Q) = \{g \in Q : g - 1 \in I_Q^n\}$, the Jennings–Zassenhaus series [Jen41]. By the identities of step (a) in the proof of Lemma 5 and $x^{-1}gx - 1 = x^{-1}(g - 1)x$, the $D_n(Q)$ are normal subgroups with elementary abelian factors $(g^p - 1 = (g - 1)^p \in I_Q^{np}$ and $[g, h] - 1 \in I_Q^{2n}$ for $g, h \in D_n(Q)$), and $D_n(Q) = 1$ for large n , since I_Q is nilpotent: if $z \in Z(Q)$ has order p and $I_{Q/\langle z \rangle}^m = 0$, then I_Q^m lies in the kernel $(z - 1)\mathbb{F}_p[Q]$ of $\mathbb{F}_p[Q] \rightarrow \mathbb{F}_p[Q/\langle z \rangle]$, so $I_Q^{mp} \subseteq (z - 1)^p \mathbb{F}_p[Q] = 0$; induct on $|Q|$. The proof of Lemma 5 applies verbatim, with $b_i = \dim D_i(Q)/D_{i+1}(Q)$ and step (a) true by definition. So the Jennings–Zassenhaus factors of the Q_N are unbounded too, whichever series the Notebook means.

Remark 7.2 (every larger d). Let G be a finite p -group without an epimorphic image isomorphic to W_p , let $E = C_p^m$, and suppose that $\varphi: G \times E \rightarrow W_p$ is onto. Then $\varphi(E)$ commutes with $\varphi(G)$ and with itself, so $\varphi(E) \subseteq Z(W_p)$. An element (v, a) that commutes with t has a constant vector v , and one that commutes with all $(w, 0)$ has $a = 0$; so $Z(W_p) \subseteq \{((c, \dots, c), 0) : c \in \mathbb{F}_p\} \subseteq \ker \pi$. The kernel $\ker \pi = \{(v, 0) : s(v) = 0\}$ is generated by the commutators $[(e_k, 0), t] = (e_{k-1} - e_k, 0)$ (e_k the unit vectors), and $W_p/\ker \pi$ is abelian; so $Z(W_p) \subseteq \ker \pi = [W_p, W_p] \subseteq \Phi(W_p)$. Hence $W_p = \varphi(G)\Phi(W_p)$ and $\varphi(G) = W_p$, a contradiction. Moreover $d(G \times E) = d(G) + m$, as $\Phi(G \times E) = \Phi(G) \times 1$; and $P_i(G \times E) = P_i(G) \times P_i(E)$ with $P_2(E) = 1$, so the first factor of $G \times E$ is $G/P_2(G) \times E$ and the others are those of G . With $G = Q_N$ this gives the conclusion of Theorem 1 for $p = 5$ and every $d \geq 3$, for $p = 3$ and every $d \geq 5$, and for $p = 2$ and every $d \geq 7$.

Remark 7.3 (what is left open). For $d = 1$ every lower p -central factor has order at most p . The note does not decide whether a bound exists for $(p, d) = (5, 2), (3, 2), (3, 3), (3, 4)$ and $(2, 2), \dots, (2, 6)$; the conjecture as stated, with one function on $\mathbb{N} \times \mathbb{N}$, is refuted for $p = 2, 3, 5$ regardless. A heuristic, not proved here and not used: with two generators and relators that are p -th powers, at least three relators of degree p seem to be needed, with exponent-sum vectors covering three points of the projective line

over $\mathbb{F}_p$, since relators whose exponent-sum vectors lie on two lines admit an epimorphism onto W_p mapping these lines to the axes. This is the choice of [KU26], whose polynomial is positive on $(0, 1)$ for $p = 3, 5$ (Remark 7.6).

Remark 7.4 (the pro- p limit). The surjections $A^{(N+1)} \rightarrow A^{(N)}$ map Q_{N+1} onto Q_N ; the inverse limit Q_∞ is a pro- p group with d topological generators. Every continuous epimorphism $Q_\infty \rightarrow W_p$ factors through some Q_N , so Q_∞ has no such quotient (Proposition 6(d)), and its lower p -central factors map onto those of every Q_N and are unbounded. The theorem uses only the Q_N .

Remark 7.5 (a possible route to (5, 2); a sketch). This sketch was pointed out by the second verification run; it was not checked in detail, it is not part of Theorem 1, and we do not claim the case (5, 2). Let $\Delta = \langle x, y \mid x^5, y^5, (xy)^5 \rangle$, a cocompact hyperbolic triangle group. Every torsion element of a triangle group is conjugate to a power of x , y or xy , and these map to non-zero elements under $\Delta \rightarrow C_5^2$, $x \mapsto e_1, y \mapsto e_2$; so the kernel is torsion-free of index 25, a closed surface group of Euler characteristic $25 \cdot (-2/5) = -10$, that is, of genus 6. Its one-relator presentation on 12 generators has the Golod–Shafarevich polynomial $1 - 12t + t^2$, negative at $t = 1/2$. As the augmentation ideal of $\mathbb{F}_5[C_5^2]$ has nilpotency index 9, Lemmas 1, 2 and 5 would then show that the finite 5-group quotients of Δ , which are two-generated and by Lemma 3(iii) have no quotient isomorphic to W_5 , have unbounded lower 5-central factors. The same route would give the case $p \geq 7$ of [KU26] again (genus $(p-1)(p-2)/2$), and it fails for $p = 3$, since $\Delta(3, 3, 3)$ is a Euclidean triangle group.

Remark 7.6 (the polynomial of [KU26]). Let $f(t) = 1 - 2t + 3t^p$, the polynomial of two generators and three relators of degree p . For $p \geq 2$, f is convex on $[0, \infty)$ and attains its minimum there at the point t_0 with $3pt_0^{p-1} = 2$, where $f(t_0) = 1 - 2(1 - 1/p)t_0$. So $f > 0$ on $[0, \infty)$ if and only if $t_0 < p/(2(p-1))$, that is, if and only if $2/(3p) < (p/(2(p-1)))^{p-1}$. For $p = 3$ and $p = 5$ this reads $2/9 < 9/16$ and $2/15 < 625/4096$, which hold; so $f > 0$ on $(0, 1)$, and Lemma 2 does not apply. For $p = 7$ the inequality fails, and $f(2/3) = -115/729$ [KU26, §3.33]. For $p = 2$ the discriminant of f is negative, and anyway the relations $x^2 = y^2 = (xy)^2 = 1$ give an abelian group.

Remark 7.7 (Shalev’s theorem and powerful groups). By its zbMATH review (Zbl 0745.20029; the paper itself was not read), Shalev’s theorem [Sha92] says that a finitely generated pro- p group is p -adic analytic if and only if, for some $n \geq 1$, it has no closed subgroups $K \trianglelefteq H$ with $H/K \cong C_p \wr C_{p^n}$. It concerns sections, while Problem 19.93 excludes only quotients isomorphic to $C_p \wr C_p$, so there is no conflict. A p -adic analytic pro- p group has finite rank, hence bounded lower p -central factors [DdSMS99]; so Q_∞ is not p -adic analytic, and by Shalev’s theorem it has sections $C_p \wr C_{p^n}$ for every n . Likewise, a d -generated powerful p -group has lower p -central factors of order at most p^d [LM87], [DdSMS99, Chapter 2], so the Q_N are not powerful for large N .

8 Conclusion

By Theorem 1 and Remark 7.2, part b) is false: for each of $p = 2, 3, 5$ no bound $f(p, d)$ exists (for $p = 5$ with any $d \geq 3$, for $p = 3$ with any $d \geq 5$, for $p = 2$ with any $d \geq 7$). With the refutation of part a) in [KU26], whose method this note follows, the conjecture fails for every prime; the pairs of Remark 7.3 remain open. Under the Notebook’s rule, part b) should be regarded as open until this unrefereed, AI-assisted argument has been checked by experts.

9 Computations and reproducibility

Only certificate C rests on a computation; it is finite and exact. The reproducibility package (folder `reproducibility/` of the source archive) contains the four programs with which the results were first obtained, with their recorded outputs: `gs_polys.py` (the values of the polynomials);

`wreath_check.py`, which verifies Lemma 3(ii) for every element of W_p , $p = 2, 3, 5$, and the pair statement behind (iii) exhaustively over all pairs u, v with $u^p = v^p = 1$, which are 2,025 pairs for $p = 3$ and 31,640,625 for $p = 5$; `d8_certificate.py`, checks A and B; and `truncation_dims.py`, which computes B_N for small N (not used in the proof). It also contains the machine-readable table; `check_note.py`, a program written for this note in exact arithmetic (standard library only; a few seconds), which re-checks with code of its own the constants, Lemmas 3 and 4 in W_p and D , certificate C by both checks, the 114 destructive single-entry changes, the degrees of the relation elements, and the agreement of the printed table with the data; and, in `independent_run_2/`, the four programs of the third verification run (see below) with their recorded outputs. All checks pass; commands and running times are in `reproducibility/README.md`.

Verification. The results were first obtained, with proofs and with the four programs above. Two independent verification runs, both AI-assisted, followed; both examined the first write-up of the result, not this text, and each re-derived the proofs and wrote its own programs. The first re-derived all lemmas, Proposition 6 and the assembly, with the boundary cases; compared Lemma 1 with [Ers12, Theorem 2.6] and the account of [KU26] with that preprint; and recomputed certificate C (with D modelled as the symmetry group of a square), Lemma 3 (exhaustively for $p = 2, 3$, on random pairs for $p = 5$) and the values of the polynomials, with the same results. It required five editorial corrections, among them the marking of the heuristic in Remark 7.3 and the explicit dependence of N and i on K . The second wrote its programs before reading those above: W_p as a permutation group with $\Phi(W_p)$ computed by brute force, all pairs for $p = 3, 5$, and all 28,638,765 five-tuples in W_3 that satisfy (R3), none of which generates W_3 ; certificate C by bilinear forms and by brute force in three conventions, with mutation tests (without ρ_{12} there are 256 epimorphisms, and 114 of the 336 single-entry changes of Table 1 destroy the certificate); and Lemma 5 on eight small groups. It re-ran the four programs from a copy, with identical output, read v49 of the Notebook, the update file [KM26a], [KU26] and [Ers12], checked the bibliographic data and searched the literature. It required three editorial corrections and suggested the machine-readable table and Remark 7.5. Neither run found a gap in the proofs; all required corrections were made.

A third run, also AI-assisted, examined the final text of this note, whose proofs had been reworded since the first write-up. It re-derived every proof in the wording of the note, among them both well-definedness claims in Lemma 1 (including the case $N + 1 - \delta_k \leq 0$), the axis argument of Lemma 3(iii) with $i > j$, Lemma 5, Proposition 6, the order of the quantifiers and the remarks; read the sources again (v48 and v49 of the Notebook, the update file [KM26a], the introduction, the index and Section 3.33 of [KU26], Section 2.3 of [Ers12] and the review of [Sha92]); checked the references with Crossref and zbMATH; and repeated the literature search. Its four programs, written before those of the package were run, read Table 1 from the PDF of the note and compare it with the source text and the data files; verify certificate C by the bilinear count, by Gaussian elimination and by brute force over all 8^7 maps in two models of D , the unitriangular group $UT_3(\mathbb{F}_2)$ and the wreath product, with the prefix counts above, and Lemma 4(c) on all these maps; check Lemma 3 in W_p and, without the map π , in a permutation model of W_p for $p = 2, 3, 5$ (all 31,640,625 pairs for $p = 5$; for $p = 2, 3$ no set of pairwise compatible elements of exponent p generates W_p , so that (iii) holds for every d); and recompute the constants, the lowest degrees of all relation elements, the arithmetic of Remarks 7.5 and 7.6 (with the nilpotency index 9 computed in $\mathbb{F}_5[C_5 \times C_5]$) and, as a test, Lemma 5 on five small groups. It ran the package again from the source archive and obtained the recorded outputs (up to a printed running time). It found no mathematical error and required four corrections, which were made: the credit to [KU26, §3.33] for the argument behind Lemma 3, a more precise sentence on the relations used for $p \geq 7$, this account of the verification runs, and a clause in the proof of Lemma 4(a).

Scope and priority. The note shows that part b) of Problem 19.93 is false (Theorem 1 and Remark 7.2); the cases of Remark 7.3 are not decided. The conjecture is due to Spiga [KM26b]. The method is that of Kinyon and Urban [KU26, §3.33], who refuted part a); their preprint declares AI assistance. The filtered inequality is Vinberg’s [Vin65], in the form of [Ers12, Theorem 2.6 and Proposition 2.7(a)]; the original inequality is that of Golod and Shafarevich [GS64]. The argument behind Lemma 3 is that of [KU26, §3.33]. New here are the relators (R5) and (R3), to which it applies pair by pair, the relators (R2) with Lemma 4 and certificate C, and the observation of Remark 7.6.

What was read: the Notebook in arXiv v48 and v49 (Problem 19.93, its comment, the preface) and the update file [KM26a] (header and page 24); of [KU26], the introduction, the index of results and Section 3.33 in full; of [Ers12], in arXiv v1, Sections 1.1–1.2 and 2.3, with Theorem 2.6, Proposition 2.7 and the footnote on [Vin65]; of [Sha92], only the zbMATH review. We did not read [Vin65, GS64]; what we say about them is taken from [Ers12]. The results cited from [Hup67, DdSMS99, LM87, Jen41] were not re-read for this note.

The searches (21 October 2026): the arXiv API (wreath products with the lower exponent- p central series, Golod–Shafarevich groups or p -adic analytic groups; the proposer’s papers; the papers that mention the Kourovka Notebook, about 120, with the abstracts of those from 2025 and 2026 screened; triangle groups with p -group quotients), OpenAlex (keyword searches; the 20 works citing [Sha92]), Crossref, the zbMATH API, and a few web searches; the third verification run repeated them in part on the same day. Semantic Scholar refused automated access (HTTP status 429), and this was not worked around; the repository file of [KU26] named in its Section 3.33 was not located. No printed answer to the question was found in these sources; a search that finds nothing is not a proof of novelty, and we claim no priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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