

On Shumyatsky's Question on Soluble Subgroups and the Generalized Fitting Height: A Partial Answer for Groups with Chief Factors of Type $\mathrm{PSL}(2, q)$ and $\mathrm{Sz}(q)$

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Abstract

Let $h^*(G)$ be the generalized Fitting height of a finite group G , and $h(K)$ the Fitting height of a soluble group K . P. Shumyatsky asked (Kourovka Notebook, Problem 19.82) whether every finite group G has a soluble subgroup K with $h^*(G) = h(K)$, that is, whether $h^*(G)$ never exceeds the largest Fitting height $s(G)$ of a soluble subgroup; Khukhro and Shumyatsky proved $\lambda(G) \leq s(G)$ for the nonsoluble length $\lambda(G)$, which is at most $h^*(G)$ (Israel J. Math. 2015). We give a partial answer: yes for every finite group whose nonabelian chief factors are of type $\mathrm{PSL}(2, q)$ ($q \geq 4$) or $\mathrm{Sz}(q)$, provided that no $\mathrm{PSL}(2, q)$ -factor with q odd and $q \equiv \pm 1 \pmod{8}$ induces a group containing $\mathrm{PGL}(2, q)$; the general question stays open. For these groups, if $O_p(G) = 1$, some soluble K has $h(K/O_p(K)) \geq h^*(G)$. Extending the method of Khukhro and Shumyatsky, we exclude a minimal counterexample by a *witness system*, a condition on the automorphisms induced on a minimal normal subgroup S^m , and construct witness systems for $\mathrm{PSL}(2, q)$ and $\mathrm{Sz}(q)$ from unipotent subgroups, Zsigmondy primes and the Lang–Steinberg theorem. The excluded case is a limit of the method, not a counterexample; the criterion is only sufficient and fails for some simple groups; A_n ($n \geq 7$), the other groups of Lie type and the sporadic groups are not treated. No printed answer to the question was found. This is an unrefereed note.

1 Introduction

1.1 The question

All groups are finite. $F(G)$ is the Fitting subgroup, $E(G)$ the layer (the product of the subnormal quasisimple subgroups, the components) and $F^*(G) = F(G)E(G)$ the generalized Fitting subgroup; G is *quasinilpotent* if $G = F^*(G)$. Put $F_0^*(G) = 1$ and let $F_{i+1}^*(G)$ be the inverse image of $F^*(G/F_i^*(G))$; the *generalized Fitting height* $h^*(G)$ is the least k with $F_k^*(G) = G$. For soluble G it is the Fitting height $h(G)$. (The Kourovka Notebook starts with F_1^* ; this matters only for $G = 1$, where $K = 1$ answers the question.) P. Shumyatsky asked in 2018 (Problem 19.82 of [KM26]) whether every finite group G contains a soluble subgroup K with $h^*(G) = h(K)$. In the versions of the Notebook of 6 and 20 October 2026 the problem has no comment, and the corpus [ula26] lists it as open.

Let $s(G)$ be the largest Fitting height of a soluble subgroup of G . The heights of the soluble subgroups of G are $0, 1, \dots, s(G)$ (Lemma 2.1), so the question asks whether $h^*(G) \leq s(G)$ for every finite group G . Khukhro and Shumyatsky [KS15, Corollary 1.2] proved $\lambda(G) \leq s(G)$ for the nonsoluble length $\lambda(G)$, the least number of nonsoluble factors in a normal series whose factors are

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soluble or direct products of nonabelian simple groups; refining the generalized Fitting series by Fitting subgroups shows $\lambda(G) \leq h^*(G)$.

Remark 1.1 (the trivial range). If $h^*(G) \leq 2$, the answer is yes: for $G = 1$ take $K = 1$; if $h^*(G) = 1$, a subgroup of prime order; if $h^*(G) = 2$, then G is not nilpotent, and a subgroup minimal among the non-nilpotent subgroups of G is soluble by a theorem of O. Yu. Schmidt (see [Hup67, Kap. III]), so $s(G) \geq 2$. The question concerns the groups with $h^*(G) \geq 3$.

1.2 Results

For a prime p and a soluble K put $h_p(K) = h(K/O_p(K))$, and let $s_p(G)$ be the largest $h_p(K)$, $K \leq G$ soluble; so $s_p(G) \leq s(G)$, with equality if $p \nmid |G|$. For a nonabelian chief factor $H/L = S_1 \times \cdots \times S_m$ of G ($S_i \cong S$ simple), the *induced group* is the group $A \leq \text{Aut}(S_1)$ of automorphisms induced on S_1 by $N_G(S_1)$ (Definition 2.8). For $S = \text{PSL}(2, q)$, q odd, let $\delta \in \text{Out}(S)$ be the class of $\text{PGL}(2, q) \setminus \text{PSL}(2, q)$; then $\text{PGL}(2, q) \leq A$ if and only if $\delta \in A/\text{Inn}(S)$.

Theorem 1.2. *Let G be a finite group such that every nonabelian chief factor of G is a direct power of a group $S \cong \text{PSL}(2, q)$, $q \geq 4$ a prime power, or $S \cong \text{Sz}(q)$, and such that for every chief factor with $S \cong \text{PSL}(2, q)$, q odd and $q \equiv \pm 1 \pmod{8}$, the induced group A does not contain $\text{PGL}(2, q)$, i.e. $\delta \notin A/\text{Inn}(S)$. Then $s_p(G) \geq h^*(G)$ for every prime p with $O_p(G) = 1$. In particular (take $p \nmid |G|$), G contains a soluble subgroup K with $h(K) = h^*(G)$.*

Status: Theorem 1.2 is a *partial answer*: yes for every finite group whose nonabelian chief factors are of type $\text{PSL}(2, q)$ ($q \geq 4$) or $\text{Sz}(q)$, provided that no $\text{PSL}(2, q)$ -factor with q odd and $q \equiv \pm 1 \pmod{8}$ induces a group containing $\text{PGL}(2, q)$; the general question stays open. The proof is complete modulo the standard facts listed, with their sources, in Sections 2.1, 3 and 4.

Corollary 1.3. *The conclusion of Theorem 1.2 holds (a) for every finite group whose nonabelian composition factors are of type $\text{PSL}(2, 2^f)$, $\text{PSL}(2, q)$ with $q \equiv \pm 3 \pmod{8}$, or $\text{Sz}(q)$, with no condition on the induced groups, for instance $\text{PGL}(2, q) \wr H$ and $\text{Aut}(\text{Sz}(q)) \wr H$ for such q and H , and mixed extensions such as $\text{Aut}(\text{Sz}(32)) \wr \text{PGL}(2, 8)$; (b) in particular for every finite group all of whose nonabelian composition factors are isomorphic to $A_5 \cong \text{PSL}(2, 4) \cong \text{PSL}(2, 5)$ (Example 3.8); (c) for groups with chief factors of type $\text{PSL}(2, q)$, $q \equiv \pm 1 \pmod{8}$ odd, whose induced groups do not contain $\text{PGL}(2, q)$, such as $S_6 \wr H$, $M_{10} \wr H$ and $\text{PSL}(2, 7) \wr H$ with H as in (a).*

The limits of the result have the same weight as the result (Section 5.1). (i) The method fails for chief factors of type $\text{PSL}(2, q)$, q odd, $q \equiv \pm 1 \pmod{8}$, with induced group containing $\text{PGL}(2, q)$, at the defining characteristic: exhaustive searches over all soluble subgroups for $q = 7, 9, 17, 23, 25, 31, 41, 47, 49, 71, 73$, made by the verification runs, find no witness, while in the groups of this kind that were tested other soluble subgroups still reach $h^*(G)$; this is not a counterexample. (ii) The witness criterion is a sufficient condition only and is not available for every simple group: it fails for A_6 with the class of $\text{PGL}(2, 9)$, and for $\text{PSL}(2, 7)$, $\text{PSL}(2, 17)$, $\text{PSL}(2, 23)$, $\text{PSL}(2, 25)$ at the defining prime with the diagonal class. (iii) Chief factors of type A_n ($n \geq 7$), of the other groups of Lie type and of the sporadic groups are not treated. (iv) The hypothesis $O_p(G) = 1$ cannot be dropped from the p -form: $s_2(S_4) = 2 < 3 = h^*(S_4)$.

1.3 Method, credit and related work

Section 2 proves a reduction theorem (Theorem 2.10): $s_p(G) \geq h^*(G)$ holds in a class of groups defined by the induced groups of the nonabelian chief factors as soon as every pair (S, A) that occurs has a *witness system*, a condition on the cosets of prime order in $A/\text{Inn}(S)$; Sections 3 and 4 construct

witness systems, and Section 5 completes the proof. The strategy is that of Khukhro and Shumyatsky [KS15, Section 3]; taken from there are the lift of soluble subgroups of quotients through Sylow 2-normalizers (the proof of [KS15, Lemma 3.1], which is stated there for the $(2 \times 2')$ -length; our Lemma 2.4), and the treatment of a minimal counterexample with a nonabelian minimal normal subgroup $S_1 \times \cdots \times S_m$ of trivial centralizer through the kernel $\bigcap_i N_G(S_i)$ and Sylow normalizers [KS15, Lemma 3.2] (our Lemmas 2.11 and 2.12). Not in [KS15] are the parameter s_p , which records the bottom prime and lets the induction pass the abelian layers, the residual series, the witness systems and their construction. Unlike [KS15], which uses Schreier's conjecture, we do not use the classification of finite simple groups.

Related work, none of which answers Problem 19.82: Murashka and Vasil'ev [MV25] prove $h^*(G) - h^*(M) \leq 2$ for maximal subgroups M ; Khukhro and Shumyatsky [KS26] bound the non- p -soluble length by the generalized Fitting height of a Hall π -subgroup; Guralnick and Tracey [GT20] prove two conjectures of Khukhro and Shumyatsky on Engel-type subgroups; Guo and Vdovin [GV18] relate Fitting heights to Carter subgroups; Gong, Yan and Cosme Llópez [GYL26] treat soluble groups only.

2 The reduction

2.1 Standard facts and lemmas

We use the following facts on components and quasinilpotent groups (see [HB82, Ch. X, §13], [Asc00, Section 31], [KS04, Section 6.5]; we did not check the numbering inside these books). (Q1) Distinct components commute; a component L and a normal subgroup M satisfy $L \leq M$ or $[L, M] = 1$; $[F(G), E(G)] = 1$; and $E(G)/Z(E(G))$ is the direct product of the simple groups $LZ(E(G))/Z(E(G))$. (Q2) $C_G(F^*(G)) \leq F^*(G)$; $F^*(G)$ contains every normal quasinilpotent subgroup; normal subgroups and quotients of quasinilpotent groups are quasinilpotent; soluble quasinilpotent groups are nilpotent. (Q3) (The chief-factor criterion [HB82, Ch. X, §13].) G is quasinilpotent if and only if $G = H C_G(H/L)$ for every chief factor H/L of G .

Lemma 2.1. *If K is soluble and $M \trianglelefteq K$, then $F_i(M) = M \cap F_i(K)$ for all i (with $F_0 = 1$, $F_{i+1}/F_i = F(\cdot/F_i)$). Hence $h(F_j(K)) = j$ for $0 \leq j \leq h(K)$, the Fitting heights of the soluble subgroups of G are $0, 1, \dots, s(G)$, and G has a soluble subgroup of height $h^*(G)$ if and only if $h^*(G) \leq s(G)$.*

Proof. $F(M)$ is characteristic in M , hence nilpotent and normal in K , and $M \cap F(K)$ is nilpotent and normal in M ; so $F(M) = M \cap F(K)$. Applied to $MF_i(K)/F_i(K) \cong M/(M \cap F_i(K))$, normal in $K/F_i(K)$, this gives by the Dedekind law and induction $F_{i+1}(M) = M \cap F_{i+1}(K)$. For $M = F_j(K)$ we get $F_i(M) = F_i(K)$ for $i \leq j$. $\square$

Lemma 2.2. *If $M_1, M_2 \trianglelefteq X$ and $X/M_1, X/M_2$ are quasinilpotent, so is $X/(M_1 \cap M_2)$. Hence X has a smallest normal subgroup $X^\mathcal{Q}$ with quasinilpotent quotient. Put $R_0^*(X) = X$ and $R_{j+1}^*(X) = R_j^*(X)^\mathcal{Q}$; then $R_j^*(X/M) = R_j^*(X)M/M$ for $M \trianglelefteq X$.*

Proof. Let $M_1 \cap M_2 = 1$ and take a chief series of X through M_1 . Its factors above M_1 satisfy (Q3). For a factor H/L with $H \leq M_1$ we have $H \cap LM_2 = L(H \cap M_2) = L$, so H/L is X -isomorphic to the chief factor HM_2/LM_2 of X/M_2 , and (Q3) gives $X = HM_2 C_X(H/L) = H C_X(H/L)$, as $[H, M_2] \leq M_1 \cap M_2 = 1$. By Jordan–Hölder and the invariance of (Q3) under X -isomorphism, X is quasinilpotent. For $M \trianglelefteq X$, $X/X^\mathcal{Q}M$ is quasinilpotent, and if $(X/M)/(N/M)$ is quasinilpotent then $X^\mathcal{Q} \leq N$; so $(X/M)^\mathcal{Q} = X^\mathcal{Q}M/M$, and induction on j (applied to $R_j^*(X)$ and $R_j^*(X) \cap M$) gives the last formula. $\square$

Lemma 2.3. *(a) $h^*(X)$ is the least length k of a series $1 = X_0 \leq \cdots \leq X_k = X$ of normal subgroups with quasinilpotent factors, and $h^*(X) = \min\{k : R_k^*(X) = 1\}$; if $h = h^*(X) \geq 1$, then*

$R_{h-1}^*(X)$ is a nontrivial normal quasinilpotent subgroup. (b) $h^*(X/M) \leq h^*(X)$ for $M \trianglelefteq X$, and $h^*(X) = 1 + h^*(X/F^*(X))$ if $X \neq 1$. (c) If $M_i \trianglelefteq X$ and $\bigcap_{i=1}^t M_i = 1$, then $h^*(X) \leq \max_i h^*(X/M_i)$.

Proof. (a) For such a series, $X_i F_{i-1}^*(X)/F_{i-1}^*(X)$ is normal in $X/F_{i-1}^*(X)$ and a quotient of X_i/X_{i-1} , so by (Q2) and induction $X_i \leq F_i^*(X)$, and $h^*(X) \leq k$. Also $R_j^*(X) \leq X_{k-j}$ by induction, as $R_j^*(X)/(R_j^*(X) \cap X_{k-j-1})$ is isomorphic to a normal subgroup of X_{k-j}/X_{k-j-1} . The generalized Fitting series and the reversed residual series are series of this kind. (b) Take images of a series; and $F_{i+1}^*(X)/F^*(X) = F_i^*(X/F^*(X))$. (c) For $k = \max_i h^*(X/M_i)$, Lemma 2.2 gives $R_k^*(X) \leq \bigcap_i M_i = 1$. $\square$

Lemma 2.4 (the argument of [KS15, Lemma 3.1]). *If $M \trianglelefteq X$, every soluble subgroup K/M of X/M is the image of a soluble subgroup L of X ; hence $s_p(X/M) \leq s_p(X)$ for every prime p .*

Proof. For $T \in \text{Syl}_2(M)$ and $L = N_K(T)$ we have $K = ML$ (Frattini), $L \cap M = N_M(T)$ is soluble as $N_M(T)/T$ has odd order [FT63], and $L/(L \cap M) \cong K/M$. As $O_p(L)$ maps into $O_p(K/M)$, $h_p(K/M) \leq h_p(L)$. $\square$

Lemma 2.5. *Let P be an ℓ -group and $1 = P_0 \leq \dots \leq P_k = P$ normal subgroups. An automorphism a of P of order prime to ℓ that leaves each P_i invariant and acts trivially on each P_i/P_{i-1} is trivial; so a group of automorphisms with this property is an ℓ -group.*

Proof. By induction a centralizes P_{k-1} . For $g \in P$, $g^a = gz$ with $z \in P_{k-1}$, so $z^a = z$ and $g = g^{a^n} = gz^n$ for the order n of a ; hence $z = 1$. $\square$

In the proof of Lemma 2.7 this lemma replaces P. Hall's theorem on stability groups [Hal58].

Lemma 2.6. *Let $R, Q \trianglelefteq T$, with R quasinilpotent and Q a q -group. Then every q' -element x of R centralizes Q .*

Proof. $Q_0 = Q \cap R \leq O_q(R)$ and $[Q, x] \leq Q_0$. Write $x = fe$ with $f \in F(R)$, $e \in E(R)$. By (Q1) e centralizes $F(R)$, and the q' -part of f centralizes $O_q(R)$; so x induces on Q_0 the automorphism induced by the q -part of f , which has q -power order dividing the order of x , hence is trivial. Lemma 2.5 for $1 \leq Q_0 \leq Q$ completes the proof. $\square$

Lemma 2.7. *Let $F = F^*(X)$, $E = E(X)$, $Z = Z(E)$, let $1 = U_0 < \dots < U_n = F(X)$ be part of a chief series of X , and let $E_1, \dots, E_t$ be the products of the X -orbits of components, so that $E/Z = \prod_j E_j Z/Z$ with $E_j Z/Z$ minimal normal in X/Z (the chief factors of X between $F(X)$ and F are X -isomorphic to these sections, as $E \cap F(X) = Z$). Then F is the intersection of the normal subgroups $C_X(U_i/U_{i-1})$ and $E_j C_X(E_j Z/Z)$, each of which contains F . So, if $F < X$, one of them, C , satisfies $h^*(X/C) = h^*(X/F)$.*

Proof. Let $U/W = U_i/U_{i-1}$, an elementary abelian r -group. $O_r(X)$ acts trivially on the irreducible $\mathbb{F}_r X$ -module U/W , $O_{r'}(F(X))$ centralizes $O_r(U) \leq O_r(X)$ while $U = O_r(U)W$, and E centralizes $F(X)$; so $F \leq C_X(U/W)$. By (Q1) $F(X)$ and the E_i , $i \neq j$, centralize E_j , so $F \leq E_j C_X(E_j Z/Z)$. Conversely, let y lie in all these subgroups, write $y = e_j c_j$ with $e_j \in E_j$, $c_j \in C_X(E_j Z/Z)$, and put $e = e_1 \dots e_t$. Then $y' = ye^{-1}$ centralizes every $E_j Z/Z$, so $[E, y'] \leq Z$, and $[E, y'] = 1$ by the three subgroups lemma (E is perfect); and y' centralizes every U_i/U_{i-1} . Let $Y \trianglelefteq X$ consist of the elements centralizing E and all U_i/U_{i-1} . By Lemma 2.5, applied to the series $(U_i \cap O_\ell(X))_i$, $Y/C_Y(O_\ell(X))$ is an ℓ -group for every prime ℓ , so $Y/C_Y(F(X))$ is nilpotent; and $A = C_Y(F(X)) \leq C_X(F) \leq Z(F) \leq F(X)$ by (Q2), with $[A \cap U_i, Y] \leq A \cap U_{i-1}$, so $A \leq Z_n(Y)$. Hence Y is nilpotent, $Y \leq F(X)$ and $y = y'e \in F$. The last statement follows from Lemma 2.3(b),(c) for X/F . $\square$

2.2 Induced pairs and witness systems

Definition 2.8. A *pair* (S, A) is a nonabelian simple group S with a group A , $\text{Inn}(S) \leq A \leq \text{Aut}(S)$; $(S, A) \cong (S', A')$ if $\psi A \psi^{-1} = A'$ for an isomorphism $\psi: S \rightarrow S'$. The *induced pair* of a nonabelian chief factor $H/L = S_1 \times \cdots \times S_m$ of G is (S_1, A_1) , where A_1 is induced on S_1 by $N_G(S_1)$. As G permutes the S_i transitively and G -isomorphisms of chief factors respect simple factors and their stabilizers, it depends only on the G -isomorphism class of H/L ; by Jordan–Hölder the induced pairs of G do not depend on the chief series. For a set Σ of pairs, closed under isomorphism, $\mathcal{C}(\Sigma)$ is the class of finite groups all of whose nonabelian chief factors have induced pairs in Σ ; it contains the soluble groups and is closed under quotients (a chief factor of G/M is one of G , with the same induced pair).

Definition 2.9. A *witness system* for a pair (S, A) assigns to every prime p and every coset $c \in A/\text{Inn}(S)$ of prime order r in $\text{Out}(S)$ a subgroup X , $1 < X < S$, whose S -class is A -invariant, such that, with $D_S = N_S(X)$ and β running through the automorphisms of S in c : (b1) X is a q -group for a prime $q \notin \{p, r\}$, and no $\beta \in c$ centralizes X ; or (b2) $O_p(D_S) = 1$, and every $\beta \in c$ with $\beta(X) = X$ acts nontrivially on $D_S/F^*(D_S)$. A set Σ of pairs satisfies (Cl) if for every $(S, A) \in \Sigma$ the normalizer $N_S(X)$ is soluble for every Sylow subgroup X of S and every subgroup X used by the witness system of (S, A) .

2.3 The reduction theorem

Theorem 2.10. *Let Σ satisfy (Cl) , and let every pair in Σ have a witness system. Then $s_p(G) \geq h^*(G)$ for every $G \in \mathcal{C}(\Sigma)$ and every prime p with $O_p(G) = 1$; in particular (Lemma 2.1, $p \nmid |G|$) every $G \in \mathcal{C}(\Sigma)$ has a soluble subgroup K with $h(K) = h^*(G)$.*

Proof. Suppose not, and choose a pair (G, p) with $G \in \mathcal{C}(\Sigma)$, $O_p(G) = 1$ and $s_p(G) < h^*(G)$, with $|G|$ minimal over all such pairs; the prime is not fixed, and minimality is applied to $(G/C, r)$ with $r \neq p$ (Step 2), to $(\bar{G}, p)$ (Step 3) and to $(\bar{T}, p)$ (Lemma 2.11).

Step 0. $G \neq 1$. If G were quasinilpotent, an element of prime order $\ell \neq p$ (it exists, as $O_p(G) = 1$) would give $s_p(G) \geq 1 = h^*(G)$. So $F := F^*(G) < G$ and $h^*(G) = 1 + h^*(G/F) \geq 2$.

Step 1. By Lemma 2.7 some C listed there has $F \leq C$ and $h^*(G/C) = h^*(G) - 1$; as $F \neq 1$, $|G/C| < |G|$.

Step 2: $C = C_G(U/W)$, $U/W \leq F(G)$ an elementary abelian r -chief factor. Then $O_r(G) \neq 1$, so $r \neq p$, and $O_r(G/C) = 1$ as G/C acts faithfully and irreducibly on U/W . By minimality for $(G/C, r)$ there is a soluble $\bar{K} \leq G/C$ with $h(\bar{K}/O_r(\bar{K})) \geq h^*(G) - 1$, and by Lemma 2.4 a soluble $L \leq G$ with $LC/C = \bar{K}$. Put $W_0 = LU$ (soluble), $W_1 = W_0/W$ and $V = U/W$, a normal r -subgroup of W_1 with $W_1/C_{W_1}(V) \cong W_0C/C = \bar{K}$ (as $U \leq C$). As $V \cap O_p(W_1) = 1$, V maps isomorphically onto a normal r -subgroup V_2 of $W_2 = W_1/O_p(W_1)$, with $W_2/C_{W_2}(V_2) \cong \bar{K}$. Under $W_2 \rightarrow \bar{K}$, $O_r(W_2)$ goes into $O_r(\bar{K})$ and $O_{r'}(F(W_2))$, which centralizes $O_r(W_2) \geq V_2$, goes to 1; so $\bar{K}/O_r(\bar{K})$ is a quotient of $W_2/F(W_2)$ and $h(W_2) \geq 1 + h(\bar{K}/O_r(\bar{K})) \geq h^*(G)$. As W_2 is a quotient of $W_0/O_p(W_0)$, $h_p(W_0) \geq h^*(G)$: a contradiction.

Step 3: $C = E_j C_G(X_j)$, $X_j = E_j Z/Z \cong S^m$. Let $\bar{G} = G/C_G(X_j) \leq \text{Aut}(X_j)$. The image $\bar{E}_j$ of E_j is $\text{Inn}(X_j)$; as $C_{\text{Aut}(X_j)}(\text{Inn}(X_j)) = 1$ and $\bar{G}$ permutes the factors transitively, $\bar{E}_j$ is the unique minimal normal subgroup of $\bar{G}$. So $O_p(\bar{G}) = F(\bar{G}) = 1$, $F^*(\bar{G}) = \bar{E}_j$ by (Q1), $\bar{G}/F^*(\bar{G}) \cong G/C$ and $h^*(\bar{G}) = h^*(G)$. If $C_G(X_j) \neq 1$, minimality and Lemma 2.4 give $s_p(G) \geq s_p(\bar{G}) \geq h^*(G)$, a contradiction. So $C_G(X_j) = 1$; then Z , $F(G)$ and the E_i , $i \neq j$, are trivial, $N := E_j \cong S^m$ is the unique minimal normal subgroup, $C_G(N) = 1$, $F^*(G) = N$, and the induced pair of N is isomorphic to some $(S, A) \in \Sigma$.

Step 4. Write $N = S_1 \times \cdots \times S_m$, $h' = h^*(G/N) = h^*(G) - 1 \geq 1$, $\bar{R} = R_{h'-1}^*(G/N)$, a nontrivial normal quasinilpotent subgroup (Lemma 2.3(a)), and $K = \bigcap_i N_G(S_i) \geq N$. By Lemma 2.12 this situation is impossible. $\square$

Lemma 2.11. *In Step 4, let $X = X_1 \times \cdots \times X_m$ with $1 < X_i < S_i$ and $N_{S_i}(X_i)$ soluble, such that the N -class of X is G -invariant. Put $T = N_G(X)$, $D = N_N(X) = \prod_i N_{S_i}(X_i)$ and $D_0 = D \cap O_p(T)$. Then every $\bar{y} \in \bar{R}$ is (i) zN for some $z \in T$ acting trivially on $(D/D_0)/F^*(D/D_0)$, and (ii) if X is a q -group, $q \neq p$, and $\bar{y}$ is a q' -element, also zN for some $z \in C_G(X)$.*

Proof. By the Frattini argument $G = NT$; $T < G$, as $1 < X < N$ and N is the only minimal normal subgroup; $D = T \cap N$ and $T/D \cong G/N$. As D is soluble, the nonabelian chief factors of T lie above D and correspond, with their induced pairs, to those of G/N ; so $T \in \mathcal{C}(\Sigma)$, and $\tilde{T} = T/O_p(T) \in \mathcal{C}(\Sigma)$ with $O_p(\tilde{T}) = 1$ and $|\tilde{T}| < |G|$. By minimality, Lemma 2.4 and $T \leq G$, $h^*(\tilde{T}) \leq s_p(\tilde{T}) \leq s_p(T) \leq s_p(G) \leq h'$. Let $\bar{O} = O_p(T)N/N$, a normal p -subgroup of G/N , and $\tilde{D} = DO_p(T)/O_p(T) \cong D/D_0$; then $\tilde{T}/\tilde{D} \cong (G/N)/\bar{O}$ and $R_{h'-1}^*((G/N)/\bar{O}) = \bar{RO}/\bar{O}$ (Lemma 2.2). Elements of $O_p(T)$ act trivially on D/D_0 , and centralize X if X is a q -group, $q \neq p$ (as $X \cap O_p(T) = 1$). If $\bar{R} \leq \bar{O}$, we are done. Otherwise $h' \geq h^*(\tilde{T}) \geq h^*(\tilde{T}/\tilde{D}) = h^*((G/N)/\bar{O}) \geq h'$, and $\tilde{R} = R_{h'-1}^*(\tilde{T})$, a normal quasinilpotent subgroup, maps onto the subgroup of $\tilde{T}/\tilde{D}$ corresponding to $\bar{RO}/\bar{O}$. For $\bar{y} \in \bar{R}$ take $\tilde{z} \in \tilde{R}$ over $\bar{y}\bar{O}$ and a preimage $z \in T$; then $\bar{y} = zz'N$ with $z' \in O_p(T)$. (i) $\tilde{D} \cap \tilde{R}$ is normal in $\tilde{D}$ and quasinilpotent, so $[\tilde{D}, \tilde{R}] \leq F^*(\tilde{D})$, and zz' acts trivially on $(D/D_0)/F^*(D/D_0)$. (ii) Choose $\tilde{z}$ as the q' -part of an element of $\tilde{R}$ over the q' -element $\bar{y}\bar{O}$. The image of X is a normal q -subgroup of $\tilde{T}$, so $\tilde{z}$ centralizes it (Lemma 2.6), and $[X, z] \leq X \cap O_p(T) = 1$; so $zz' \in C_G(X)$. $\square$

Lemma 2.12. *The situation of Step 4 is impossible.*

Proof. If $g \in C_G(X)$, $X = \prod_i X_i$ with all $X_i \neq 1$, and $S_i^g = S_j$, then $X_i = X_i^g \leq S_i \cap S_j$, so $i = j$; thus $C_G(X) \leq K$.

(α) $\bar{R} \not\leq K/N$. As $\bar{R}$ is generated by elements of prime power order, some $\bar{y} \in \bar{R} \setminus K/N$ has order a power of a prime r . By Burnside's $p^a q^b$ -theorem there is a prime $q \mid |S|$ with $q \notin \{p, r\}$; let $X = Q \in \text{Syl}_q(N)$, so $Q = \prod_i Q_i$ with $Q_i \in \text{Syl}_q(S_i)$, its N -class is G -invariant, and the $N_{S_i}(Q_i)$ are soluble by (Cl'). Lemma 2.11(ii) gives $\bar{y} \in C_G(Q)N/N \leq K/N$, a contradiction.

(β) $\bar{R} \leq K/N$. Take $\bar{y} = yN \in \bar{R}$ of prime order r . Then $y \in K \setminus N$ induces $\alpha_i \in \text{Aut}(S_i)$; not all α_i are inner, since $C_G(N) = 1$, and some class $c_i = \alpha_i \text{Inn}(S_i)$ has order r . Choose $t_j \in G$ with $S_1^{t_j} = S_j$ ($t_1 = 1$), an isomorphism $\psi_1: S \rightarrow S_1$ with $\psi_1^{-1}A_1\psi_1 = A$ (A_1 induced by $N_G(S_1)$), and put $\psi_j = \kappa(t_j) \circ \psi_1$, with $\kappa(g): x \mapsto x^g$. If $S_j^g = S_k$, then $\psi_k^{-1}\kappa(g)\psi_j = \psi_1^{-1}\kappa(t_j g t_k^{-1})\psi_1$ with $t_j g t_k^{-1} \in N_G(S_1)$, so these transported automorphisms lie in A ; for $j = k$ this gives $\psi_j^{-1}A_j\psi_j = A$ (A_j induced by $N_G(S_j) = t_j^{-1}N_G(S_1)t_j$). So $c = \psi_i^{-1}c_i\psi_i \in A/\text{Inn}(S)$ has order r . Let X_S be the subgroup assigned to (p, c) and $X = \prod_j \psi_j(X_S)$. For $S_j^g = S_k$, $\psi_j(X_S)^g = \psi_k(\gamma(X_S))$ with $\gamma \in A$, which is S_k -conjugate to $\psi_k(X_S)$; so the N -class of X is G -invariant, and Lemma 2.11 applies by (Cl'). It gives $\bar{y} = zN$ with $z \in N_G(X)$ as in (i), and in case (b1), where $q \neq p$ and $\bar{y}$ is a q' -element, with $z \in C_G(X)$ as in (ii). Then $z \in yN \subseteq K$ normalizes S_i and X_i and induces $\beta_i \in c_i$ with $\beta_i(X_i) = X_i$, and $\beta = \psi_i^{-1}\beta_i\psi_i \in c$ normalizes X_S . In case (b1) β centralizes X_S , a contradiction. In case (b2), $D_0 \leq O_p(D) = \prod_j O_p(N_{S_j}(X_j)) = 1$ and $F^*(D) = \prod_j F(N_{S_j}(X_j))$ (D is soluble); z acts trivially on $D/F^*(D)$ and normalizes $N_{S_i}(X_i)$, so β acts trivially on $D_S/F^*(D_S)$, a contradiction. $\square$

3 Witness systems for $\text{PSL}(2, q)$

Let $q = l^f \geq 4$, l prime, $d = \gcd(2, q-1)$, $S = \text{PSL}(2, q)$, acting on $\mathbb{F}_q \cup \{\infty\}$ by $x \mapsto (ax+b)/(cx+e)$ with $ae-bc$ a nonzero square; $\text{PGL}(2, q)$ consists of all such maps with $ae \neq bc$, σ is $x \mapsto x^l$, and $\text{P}\Gamma\text{L}(2, q) = \text{PGL}(2, q) \rtimes \langle \sigma \rangle$. Standard facts (for (P2)–(P4) see [Hup67, Kap. II, §8], in particular Dickson's list [Hup67, II.8.27]; numbering not checked): (P1) $\text{Aut}(S) = \text{P}\Gamma\text{L}(2, q)$ [SvdW28], and $\text{Out}(S) = \langle \delta \rangle \times \langle \sigma \rangle \cong C_d \times C_f$, δ the image of $\text{PGL}(2, q) \setminus S$. A coset $c \in \text{Out}(S)$ is *field-type* if

its image in $\text{Out}(S)/\langle \delta \rangle$ is nontrivial; a coset of prime order r is δ ($r = 2$) or field-type (σ^j of order r , or $\delta\sigma^{f/2}$ for q odd, f even, $r = 2$). (P2) An odd prime $t \neq l$ dividing $|S|$ divides exactly one of $(q \mp 1)/d$; S has cyclic subgroups $T_{\mp}$ of these orders, $T_{\mp} = T_{\mp}^* \cap S$ for cyclic subgroups $T_{\mp}^* \leq \text{PGL}(2, q)$ of orders $q \mp 1$; a Sylow t -subgroup Q_t is conjugate to a subgroup of the corresponding $T_{\mp}$, and if $Q_t \leq T_{\mp}$ then $C_S(Q_t) = T_{\mp}$. (P3) Let q be odd. If $q \equiv \pm 3 \pmod{8}$, a Sylow 2-subgroup V of S is a Klein four-group, $N_S(V) \cong A_4$ and $N_{\text{PGL}(2, q)}(V) \cong S_4$. If $q \equiv \pm 1 \pmod{8}$, S has two classes of Klein four-groups, interchanged by $\text{PGL}(2, q)$, with normalizers S_4 , and its Sylow 2-subgroups are dihedral of order ≥ 8 and self-normalizing. (P4) (Dickson) The nonsoluble subgroups of S are isomorphic to A_5 , $\text{PSL}(2, q_0)$ or $\text{PGL}(2, q_0)$, $q_0 \geq 4$ a power of l , and have no nontrivial soluble normal subgroup; so $N_S(X)$ is soluble for every nontrivial soluble $X \leq S$.

Lemma 3.1 (Lang–Steinberg). *Let $q = q_0^r$, r prime, $\varphi_0 = \sigma^{f/r}$ ($x \mapsto x^{q_0}$), $G = \text{PGL}(2, q)$ and $r \nmid k$. Every element of order r of $G\varphi_0^k$ is G -conjugate to φ_0^k , and $C_G(\varphi_0^k) = \text{PGL}(2, q_0)$. If l is odd and $r = 2$, every involution of $G\varphi_0$ lies in $S\varphi_0$; so the coset $\delta\sigma^{f/2}$ contains no automorphism of order 2.*

Proof. Let $\overline{G} = \text{PGL}_2(\overline{\mathbb{F}}_l)$, F_0 the map raising matrix entries to the power q_0 , and $F = F_0^r$. A fixed point of F lifts to a matrix M with $F(M) = \lambda M$, and μM is fixed if $\mu^{q-1} = \lambda^{-1}$; so $\overline{G}^F = G$, and likewise $\overline{G}^{F_0} = \text{PGL}(2, q_0)$. In $G\langle \varphi_0 \rangle$, $\varphi_0 g \varphi_0^{-1} = F_0(g)$. Let $x = g\varphi_0$ have order r . By Lang’s theorem [Lan56] for the connected group $\overline{G}$, $g = h^{-1}F_0(h)$ with $h \in \overline{G}$, and $1 = x^r = gF_0(g) \cdots F_0^{r-1}(g) = h^{-1}F(h)$; so $h \in G$ and $x = h^{-1}\varphi_0 h$. For x of order r in $G\varphi_0^k$ take m with $km \equiv 1 \pmod{r}$; then $x^m = h^{-1}\varphi_0 h$ ($h \in G$) and $x = (x^m)^k = h^{-1}\varphi_0^k h$. Also $C_G(\varphi_0^k) = \overline{G}^{F_0^k} \cap \overline{G}^{F_0} = \overline{G}^{F_0}$. If l is odd and $r = 2$, a lift $\hat{h} \in \text{GL}(2, q)$ of h has $\det(\hat{h}^{-1}F_0(\hat{h})) = \det(\hat{h})^{q_0-1}$, a square; so $h^{-1}F_0(h) \in S$. $\square$

The proof does not need $r \neq l$. (For groups of Lie type in general see [GLS98, Section 4.9]; we did not check that reference and do not use it.) Let $U = \{x \mapsto x + b\} \in \text{Syl}_l(S)$ and $B = N_S(U)$. As U fixes only ∞ , its normalizer in $\text{PGL}(2, q)$ consists of the maps $x \mapsto ax^{l^j} + b$, and $B = U \rtimes H$, $H = \{x \mapsto a^2x\} \cong C_{(q-1)/d}$. Conjugating $x \mapsto x + b$ by $x \mapsto ax^{l^j} + b'$ gives $x \mapsto x + ab^{l^j}$.

Lemma 3.2. $C_{\text{PGL}(2, q)}(U) = U$. So no non-inner automorphism of S centralizes U , $C_B(U) = U = F(B)$, and $O_p(B) = 1$ for $p \neq l$.

Proof. $x \mapsto ax^{l^j} + b'$ centralizes U if and only if $ab^{l^j} = b$ for all b , i.e. $a = 1$ and $f \mid j$. Then $O_{l'}(F(B)) \leq C_B(U) = U$. $\square$

Lemma 3.3 (field-type, $p \neq l$). *If c is field-type, $X = U$ is a (b2)-witness for every prime $p \neq l$.*

Proof. The class of U is $\text{Aut}(S)$ -invariant, $D_S = B$, $F^*(B) = U$, $O_p(B) = 1$. A $\beta \in c$ with $\beta(U) = U$ is $x \mapsto ax^{l^j} + b$ with $0 < j < f$; it maps $x \mapsto sx$ to $x \mapsto s^{l^j}x$ modulo U , so it acts on $B/U \cong C_{(q-1)/d}$ as $y \mapsto y^{l^j}$, which is not trivial as $0 < l^j - 1 \leq l^{f-1} - 1 < (q-1)/2$. $\square$

Lemma 3.4 (field-type, $p = l$). *Let c be field-type of prime order r , $q = q_0^r$. Let t be a primitive prime divisor of $l^{2f} - 1$ if $r = 2$ or $r \nmid q_0 + 1$, and of $l^f - 1$ if r is odd and $r \mid q_0 + 1$. Then t exists, $t \notin \{l, r\}$, and $X = Q_t \in \text{Syl}_l(S)$ is a (b1)-witness for $p = l$.*

Proof. By Zsigmondy’s theorem [Zsi92], $l^n - 1$ ($n \geq 2$) has a primitive prime divisor unless $n = 2$ and $l + 1$ is a power of 2, or $(l, n) = (2, 6)$. Here $f \geq 2$; for $n = 2f$ the only exception is $q = 8$, where $r = 3 \mid q_0 + 1$ and the second rule applies; for $n = f$, r odd, the only exception is $q = 64$, $r = 3$, where $r \nmid q_0 + 1$ and the first rule applies. As $t \equiv 1 \pmod{n}$, $t > n \geq r$, t is odd, $t \neq l$. First rule: $t \mid (q+1)/d$ (as $t \nmid q-1$), and by (P2) we may take $Q_t \leq T_+ = C_S(Q_t)$; $r \nmid |T_+|$, as $q+1 \equiv q_0+1 \pmod{r}$ for r odd, and $(q+1)/d$ is odd for $r = 2$ ($l = 2$, or $q = q_0^2 \equiv 1 \pmod{4}$). Second rule: $t \mid (q-1)/d$, $Q_t \leq T_- = C_S(Q_t)$, and $r \nmid q-1 \equiv q_0-1 \pmod{r}$ as r is odd and $r \mid q_0+1$. Let $\beta \in c$ centralize Q_t ,

and let $\beta' = \beta^e$ be its r -part, $e \equiv 1 \pmod{r}$; then $\beta' \in c$, and β'^r is the conjugation by an r -element of $C_S(Q_t) = T_\pm$, hence trivial; so β' has order r . If $c = \delta\sigma^{f/2}$, this contradicts Lemma 3.1. Otherwise $c \subseteq \text{PGL}(2, q)\varphi_0^k$, $r \nmid k$, and by Lemma 3.1 $|C_S(\beta')|$ divides $|\text{PGL}(2, q_0)| = q_0(l^{2f/r} - 1)$, which t does not divide ($2f/r < 2f$, resp. $2f/r < f$); but $Q_t \leq C_S(\beta')$. $\square$

Lemma 3.5 (the diagonal coset). *Let q be odd and $c = \delta$. (a) For $p \neq l$, $X = U$ is a (b1)-witness (with the prime $l \notin \{p, 2\}$; Lemma 3.2). (b) For $p = l$ and $q \equiv \pm 3 \pmod{8}$, $X = V \in \text{Syl}_2(S)$ is a (b2)-witness: its class is $\text{Aut}(S)$ -invariant, by (P3) $D_S \cong A_4$, $F^*(D_S) = V$, $O_l(D_S) = 1$, and the elements of δ normalizing V form $N_{\text{PGL}(2, q)}(V) \setminus N_S(V) \cong S_4 \setminus A_4$; they invert $N_S(V)/V \cong C_3$.*

Remark 3.6 (the excluded pairs). Let q be odd, $q \equiv \pm 1 \pmod{8}$, $p = l$, $c = \delta$. Then (b1) fails for every X : a t -group X with $t \notin \{l, 2\}$ lies in a conjugate of some $T_\pm$, and the abelian group $T_\pm^*$ centralizes X and meets δ . For (b2) the obvious candidates fail: the two classes of Klein four-groups (hence of A_4 , S_4) are interchanged by δ , Sylow 2-subgroups are self-normalizing, torus normalizers have a quotient of order ≤ 2 by their Fitting subgroup, and l -local subgroups have $O_l \neq 1$. This is a check of candidates, not a proof. Exhaustive computations of the verification runs found no subgroup satisfying Definition 2.9 for (l, δ) : among all subgroups for $q = 7, 9, 17, 23, 25, 31$, and among all soluble subgroups (the only ones allowed by (Cl')) for $q = 7, 9, 17, 23, 25, 31, 41, 47, 49, 71, 73$ (for $q = 5, 11, 13, 19, 27, 29$ the same searches find exactly V and A_4). So the method does not cover these pairs, e.g. A_6 with $A \supseteq \text{PGL}(2, 9)$. This is not a counterexample: $h^*(\text{PGL}(2, q)) = 2$, and $S_4 \leq S$ has height 3.

Proposition 3.7. *Let Σ_L be the set of pairs $(\text{PSL}(2, q), A)$, $q \geq 4$, except those with q odd, $q \equiv \pm 1 \pmod{8}$ and $\delta \in A/\text{Inn}(S)$. Every pair in Σ_L has a witness system, and Σ_L satisfies (Cl').*

Proof. By (P1) a coset of prime order in $A/\text{Inn}(S)$ is field-type, covered by Lemmas 3.3 and 3.4 (including $\delta\sigma^{f/2}$, e.g. the coset of M_{10} for $q = 9$), or δ , covered by Lemma 3.5 unless $q \equiv \pm 1 \pmod{8}$, in which case $\delta \notin A/\text{Inn}(S)$. The witness classes are $\text{Aut}(S)$ -invariant. The witnesses U , Q_t , V and the Sylow subgroups are soluble and nontrivial, so (P4) gives (Cl'). $\square$

Example 3.8 (A_5). For $A_5 \cong \text{PSL}(2, 5)$ the witnesses are a Sylow 5-subgroup for $p \neq 5$ (b1) and a Klein four-group, or the point stabilizer A_4 , for $p = 5$ (b2); Theorem 2.10 for $\{(A_5, A_5), (A_5, S_5)\}$ gives Corollary 1.3(b).

4 Witness systems for the Suzuki groups

Let $S = \text{Sz}(q)$, $q = 2^n$, $n = 2e + 1 \geq 3$, $r_0 = 2^{e+1}$. Standard facts [Suz62], [HB82, Ch. XI, §3] (numbering not checked): (Z1) $|S| = q^2(q-1)(q^2+1)$, $q^2+1 = (q+r_0+1)(q-r_0+1)$; $\text{Aut}(S) = S \rtimes \langle \sigma \rangle$, σ the field automorphism induced by $x \mapsto x^2$, of order n ; so $\text{Out}(S) \cong C_n$ and every coset of prime order r has $r \mid n$ odd. (Z2) A Sylow 2-subgroup U has order q^2 and is nonabelian, and $N_S(U) = B = U \rtimes H$, H cyclic of order $q-1$ acting fixed-point-freely on U ; σ normalizes U and $H = \{h_\lambda : \lambda \in \mathbb{F}_q^\times\}$, with $\sigma(h_\lambda) = h_{\lambda^2}$. (Z3) An odd prime $t \mid |S|$ divides exactly one of the pairwise coprime numbers $q-1$, $q \pm r_0 + 1$; S has cyclic subgroups T of these orders with $C_S(x) = T$ for $1 \neq x \in T$, $N_S(T) \cong D_{2(q-1)}$ or $C_{q \pm r_0 + 1} \rtimes C_4$, and a Sylow t -subgroup lies in a conjugate of the corresponding T .

Lemma 4.1 (Lang–Steinberg). *Let $r \mid n$ be prime and $q_0 = 2^{n/r}$. There is $\varphi \in \text{Aut}(S)$ with $\varphi^r = 1$ and $C_S(\varphi) \cong \text{Sz}(q_0)$ whose image generates the subgroup of order r of $\text{Out}(S)$, such that for $r \nmid k$ every element of order r of $S\varphi^k$ is S -conjugate to φ^k , and $C_S(\varphi^k) = C_S(\varphi)$.*

Proof. Let $\overline{G} = \text{Sp}_4(\overline{\mathbb{F}}_2)$ and τ an endomorphism of $\overline{G}$ whose square is the Frobenius map $x \mapsto x^2$ on matrix entries, so that $\text{Sz}(2^m) = \overline{G}^{\tau^m}$ for odd m (see [Ste68]). Put $F' = \tau^{n/r}$, $F = F'^r$, so $S = \overline{G}^F$,

$\overline{G}^{F'} \cong \text{Sz}(q_0)$, and let $\varphi = F'|_S$. The proof of Lemma 3.1, with the Lang–Steinberg theorem [Ste68] for the endomorphism F' of the connected group $\overline{G}$ (finitely many fixed points), gives the conjugacy statement and $C_S(\varphi^k) = \overline{G}^{F'^k} \cap \overline{G}^{F'^r} = \overline{G}^{F'}$. φ is not inner: an $s \neq 1$ with $C_S(s) \cong \text{Sz}(q_0)$ would be central in $C_S(s)$, while $Z(\text{Sz}(q_0)) = 1$; as $\text{Out}(S)$ is cyclic, the image of φ generates its subgroup of order r . $\square$

Lemma 4.2 (p odd). *For every coset $c \neq 1$ and every odd prime p , $X = U$ is a (b2)-witness.*

Proof. If $b = uh$ ($u \in U$, $h \in H$) centralizes U , then h acts on U as conjugation by u^{-1} , fixes $Z(U) \neq 1$ pointwise, and is trivial; so $C_B(U) \leq U$ (indeed $C_B(U) = Z(U)$), $F(B) = U$ as $O_{2'}(F(B)) \leq C_B(U)$, and $O_p(B) = 1$. Let $c = S\sigma^j$, $0 < j < n$, and $\beta \in c$ with $\beta(U) = U$. Then β is σ^j composed with the conjugation by an element of $N_S(U) = B$, which acts trivially on the abelian group B/U , and σ^j acts on $B/U \cong H$ as $h_\lambda \mapsto h_{\lambda^{2^j}}$, not trivially, as $0 < 2^j - 1 < q - 1$. $\square$

Lemma 4.3 ($p = 2$). *Let c have prime order r and $q = q_0^r$. Let t be a primitive prime divisor of $2^n - 1$ if $r \nmid q - 1$, and of $2^{4n} - 1$ if $r \mid q - 1$. Then t exists, $t \notin \{2, r\}$, and $X = Q_t \in \text{Syl}_t(S)$ is a (b1)-witness for $p = 2$.*

Proof. t exists by Zsigmondy ($n \geq 3$ odd, $4n \geq 12$), and $t \equiv 1$ modulo n , resp. $4n$, so $t > n \geq r$ and t is odd. If $r \nmid q - 1$, then Q_t lies in a torus T of order $q - 1$, prime to r . If $r \mid q - 1$, then $t \mid q^2 + 1$ (as $t \nmid q^2 - 1$), Q_t lies in a torus T of order $q \pm r_0 + 1$, and from $q \equiv q_0 \pmod{r}$ we get $r \mid q_0 - 1$, $q^2 + 1 \equiv 2 \pmod{r}$, so $r \nmid |T|$. In both cases $C_S(Q_t) = T$ by (Z3). If $\beta \in c$ centralizes Q_t , its r -part $\beta' \in c$ centralizes Q_t and has order r (as in Lemma 3.4), so $|C_S(\beta')| = |\text{Sz}(q_0)| = q_0^2(q_0 - 1)(q_0^2 + 1)$ by Lemma 4.1. But the order of 2 modulo t is n , resp. $4n$, and divides neither n/r nor $4n/r$, while $q_0 - 1$ and $q_0^2 + 1$ divide $2^{4n/r} - 1$; so $t \nmid |C_S(\beta')|$, although $Q_t \leq C_S(\beta')$. $\square$

Proposition 4.4. *Every pair $(\text{Sz}(q), A)$, $q = 2^{2e+1} \geq 8$, has a witness system, and the set Σ_{Sz} of these pairs satisfies (Cl').*

Proof. Lemma 4.2 covers the odd p and Lemma 4.3 the prime 2; the witness classes are classes of Sylow subgroups. (Cl'): $N_S(U) = B$ is soluble, and $N_S(Q_t) \leq N_S(C_S(Q_t)) = N_S(T)$ is soluble by (Z3). $\square$

5 Proof of the main theorem, and the limits of the method

Proof of Theorem 1.2 and Corollary 1.3. Let $\Sigma_0 = \Sigma_L \cup \Sigma_{\text{Sz}}$; it is closed under isomorphism, as $\text{Out}(\text{PSL}(2, q))$ is abelian, so δ does not depend on the identification of S with $\text{PSL}(2, q)$ (and $A_5 \cong \text{PSL}(2, 4) \cong \text{PSL}(2, 5)$ is excluded under neither). For G as in Theorem 1.2 the induced pair of a nonabelian chief factor lies in Σ_{Sz} , or in Σ_L because q is even, or $q \equiv \pm 3 \pmod{8}$, or $\delta \notin A/\text{Inn}(S)$; so $G \in \mathcal{C}(\Sigma_0)$, and Propositions 3.7 and 4.4 and Theorem 2.10 give Theorem 1.2. In Corollary 1.3(a),(b) the simple factors of the nonabelian chief factors are composition factors of the listed types; in (c), the stabilizer of a coordinate of $S_6 \wr H$ induces $S_6 = \text{PSL}(2, 9)$ on its simple factor, and $S_6 \cap \text{PGL}(2, 9) = A_6$; similarly for M_{10} (the extension of A_6 by $\delta\sigma$) and $\text{PSL}(2, 7)$.

5.1 The limits of the method

(i) *The excluded pairs* (Remark 3.6). In all groups of this kind tested by the verification runs, $s_p(G) \geq h^*(G)$ holds: exactly for $\text{PGL}(2, q)$ ($q = 7, 9, 17, 23, 25, 31$), $\text{PFL}(2, 9)$, $\text{PFL}(2, 25)$ and $2^6:\text{PGL}(2, 7)$, and through explicit soluble subgroups for $\text{PGL}(2, 7) \wr C_3$, $\text{PGL}(2, 7) \wr S_3$, a group $\text{PSL}(2, 7)^3.S_3$ with twisted complement, and $G = \langle \text{PSL}(2, 7)^3, (1\ 2\ 3), (\delta, 1, 1)(2\ 3) \rangle$ (with $h^* = 3, 4, 3, 4$, respectively). In

the last one the diagonal automorphisms form the term $\overline{R}$ of Step 4, the case not handled at $p = 7$, and still $K = N_G(Q)$, $Q \in \text{Syl}_3(\text{PSL}(2, 7)^3)$, has $h_7(K) = 4 = h^*(G)$. This says nothing beyond the groups tested.

(ii) *The criterion is only sufficient.* For A_6 with the coset of $\text{PGL}(2, 9)$ ($p = 3$) and for $\text{PSL}(2, 7)$, $\text{PSL}(2, 17)$, $\text{PSL}(2, 23)$, $\text{PSL}(2, 25)$ with the diagonal coset ($p = 7, 17, 23, 5$), exhaustive computations of the verification runs show that no subgroup X of S at all satisfies (b1) or (b2) at the defining prime, already for $A = \text{PGL}(2, q)$ (the same holds for $q = 31$). So Definition 2.9 is not available for every simple group, and the general case needs a different local condition, for instance (not explored here) one that uses a pair of classes interchanged by δ , such as the two classes of S_4 .

(iii) *Other simple groups* are not treated; the following remarks are not proved here. A verification run found a witness system for A_7 by computation; for larger n the Sylow normalizers of A_n need not be soluble (a Sylow 7-normalizer of A_{12} involves A_5), so (Cl') fails. For groups of Lie type the witnesses of Sections 3 and 4 have natural analogues, but diagonal and graph automorphisms were not examined.

(iv) *The hypothesis* $O_p(G) = 1$. The soluble subgroups K of S_4 have $h(K/O_2(K)) \leq 2$ (equality for S_4 and S_3), while $h^*(S_4) = 3$. Nonsoluble examples, computed exactly by a verification run, are $2^6:\Sigma L(2, 8)$ ($\Sigma L(2, 8)$ acting on $\mathbb{F}_8^2$) and $2^6:\text{PGL}(2, 7)$ ($\text{PGL}(2, 7)$ on $W \oplus W^*$, W the natural $\mathbb{F}_2\text{PSL}(3, 2)$ -module, the outer automorphism interchanging the summands): $O_2(G) = 2^6$, $h^*(G) = 3$, $s_2(G) = 2$, $s(G) = 3$.

Conclusion. The note gives a *partial answer: yes for every finite group whose nonabelian chief factors are of type* $\text{PSL}(2, q)$ ($q \geq 4$) *or* $\text{Sz}(q)$, *provided that no* $\text{PSL}(2, q)$ -*factor with* q *odd and* $q \equiv \pm 1 \pmod{8}$ *induces a group containing* $\text{PGL}(2, q)$; *the general question stays open.* Theorem 2.10 applies to every class for which witness systems and (Cl') are available, and (i)–(iv) show where the present method stops. No counterexample to the question is known to us.

6 Computations and reproducibility

No theorem, proposition or lemma depends on a computation; computations describe the limits of the method (Remark 3.6, Section 5.1) and serve as checks. The package has four parts. (1) `check_note.py`, written with this note (standard library, 6 seconds, 2667 checks, all passed), checks the inequality in Lemma 3.3 for $q \leq 2^{20}$; the choices of t in Lemmas 3.4 and 4.3 for 731 classes of $\text{PSL}(2, l^f)$ and 150 of $\text{Sz}(2^n)$; Zsigmondy's theorem in this range; Lemmas 3.1 to 3.5, (P3) and (Cl') for $4 \leq q \leq 32$ by brute force; and the groups S_4 (Section 5.1(iv)) and S_5 . (2) The programs with which the results were first obtained: `check_A5_witness.py` (Example 3.8), `check_psl2_witness.py` (Lemmas 3.2 to 3.5 and Remark 3.6 for 16 values $q \leq 32$) and `check_zsigmondy_choices.py` (65 classes). (3) The programs and recorded outputs of the verification runs: among them exact values of h^* , s , s_p over all soluble subgroups of 43 groups, lower bounds for 12 larger groups, the witness lemmas for all prime powers $4 \leq q \leq 128$ and for $\text{Sz}(8)$, built from scratch, and the searches of Remark 3.6. (4) The programs and recorded outputs of the run on the final text (folder `independent_run_2/`; see the next paragraph). The recorded outputs that were rerun from the package were reproduced, the second time from the archive.

Verification. The results were first obtained, with proofs and the programs of part (2), in two stages: the reduction with the case of A_5 , then the relative witness systems with $\text{PSL}(2, q)$ and $\text{Sz}(q)$. Each stage was examined by two independent verification runs, all AI-assisted, each with its own programs. The runs re-derived all proofs line by line, including the boundary cases (a quasinilpotent G , $m = 1$, $\overline{R} \leq \overline{O}$, the change of prime in Step 2), and derived Lemmas 3.1 and 4.1 from the theorems

of Lang and Steinberg. They computed h^* , s , s_p exactly for 43 groups, tested Lemma 2.7 on 13 groups and Theorem 2.10 on 169 classes of nonsoluble subgroups, checked the witness lemmas for all 42 prime powers $4 \leq q \leq 128$ (for $q \leq 32$ also directly against the definitions), built $\text{Sz}(8)$ in two independent ways, and made the searches of Remark 3.6 and Section 5.1. No run found a gap in a proof. Their required corrections were all made: the hypothesis of Theorem 1.2 (an earlier text presented it as equivalent to “the induced group lies in $\text{PSL}(2, q)$ ”, which differs for f even, e.g. for M_{10}); the status of Remark 3.6; the Sylow 2-normalizers in (P3); the wording on the general case; Remark 1.1; the references. This text was written after these runs, with Lemmas 2.5, 3.1 and 4.1 in self-contained form.

A further verification run, also AI-assisted, examined the final text as if nothing had been checked. It re-derived all proofs of Sections 2 to 5 line by line, compared the question and the definitions with the versions 48 and 49 of [KM26] and with the corpus record, read Corollary 1.2 and Lemmas 3.1 and 3.2 of [KS15] again, checked all DOIs with Crossref and repeated the literature search. Its own programs (part (4) above) checked Definition 2.9 literally for the witnesses of Lemmas 3.3 to 3.5, together with Lemmas 3.1 and 3.2, (P2), (P3) and (C1’), for all 16 prime powers $4 \leq q \leq 32$ (295 checks); enumerated all subgroups of $\text{PSL}(2, q)$ for the 12 odd prime powers $5 \leq q \leq 31$ and found no witness for (l, δ) when $q \equiv \pm 1 \pmod{8}$, and exactly V and A_4 otherwise; built $\text{Sz}(8)$ once more, as the group of linear maps preserving the 65 points of the Suzuki–Tits ovoid, and checked (Z1)–(Z3) and Lemmas 4.1 to 4.3 for it (34 checks); checked the arithmetic of Lemmas 3.3, 3.4 and 4.3 by complete factorization of cyclotomic values (3674 checks); and confirmed $s_2(S_4) = 2$ and the values $h^*(G) = h_7(N_G(Q)) = 4$ of Section 5.1(i). It re-ran the package from the archive and reproduced the recorded outputs. It found no error and no gap in a proof. Its required corrections, all made, concern the text: the attribution of Lemma 2.4 to the proof of [KS15, Lemma 3.1]; the coverage of the searches stated in Remark 3.6 and Section 5.1(ii); two formulations in the abstract and in Section 5.1(i); the description of the package; and this record.

Scope and priority. The note gives a partial answer to Problem 19.82 (Theorem 1.2); the general question stays open, with the limits of Section 5.1. The question is due to P. Shumyatsky [KM26]. The method extends that of Khukhro and Shumyatsky [KS15]: Lemma 2.4 is the argument of the proof of [KS15, Lemma 3.1], the treatment of the minimal counterexample follows the proof of [KS15, Lemma 3.2], and [KS15, Corollary 1.2] is the inequality $\lambda(G) \leq s(G)$. The standard tools are listed with their sources in Sections 2.1, 3 and 4: the Feit–Thompson theorem [FT63], Burnside’s $p^a q^b$ -theorem, Schmidt’s theorem, facts on quasipotent groups [HB82, Asc00, KS04], $\text{Aut}(\text{PSL}(2, q))$ [SvdW28], Dickson’s list [Hup67], Suzuki’s results [Suz62, HB82], Zsigmondy’s theorem [Zsi92] and the theorems of Lang [Lan56] and Steinberg [Ste68]; the classification of finite simple groups is not used. Numbering inside the books [Asc00, GLS98, HB82, Hup67, KS04] was not checked (the titles of the cited chapters of [HB82, Hup67] were checked with Crossref). We read [KS15] (arXiv version 3, Sections 1–3), Problem 19.82 in the versions 48 and 49 of [KM26], the arXiv abstracts of [KS26, MV25, GT20, GYL26] and the zbMATH review of [MV25]; two verification runs read the introduction of [GT20], and the run on the final text read Corollary 1.2 and Lemmas 3.1–3.2 of [KS15] and Problem 19.82 in both versions again. Of [GV18] we saw only the bibliographic data (the publisher’s page refused our automated request); a verification run read its abstract, by which it bounds the generalized Fitting height in terms of a Carter subgroup; Semantic Scholar and zbMATH give no abstract. The searches (21 October 2026, by the verification runs and at the writing) used the arXiv API (among others, all nine abstracts containing “generalized Fitting height” and the recent papers of P. Shumyatsky), OpenAlex (the 40 works citing [KS15], titles read), the zbMATH API, Semantic Scholar (the 51 works citing arXiv:1310.2434, titles read), Crossref and five web searches. No printed answer to the question was found in the sources searched; a search that finds nothing is not a proof

of novelty, and we claim no priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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