

On Shumyatsky's Question on Coprime Commutators: Nineteen Sporadic Groups, the Tits Group and All Simple Groups of Order Less Than 10^9

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Abstract

Shumyatsky asked whether each element of a nonabelian finite simple group can be written as $[x, y]$ with x and y of coprime orders (Kourovka Notebook, Problem 18.117). The answer is known to be yes for the alternating groups (Shumyatsky), for the groups $\text{PSL}(2, q)$ (Pellegrini and Shumyatsky) and for the Suzuki groups (Zini); without the condition on the orders the statement is Ore's conjecture, proved by Liebeck, O'Brien, Shalev and Tiep. We give a partial answer; the proofs are computer-assisted. In the sporadic groups $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_2, J_3, HS, McL, He, Ru, Suz, Co_3, Co_2$ and Fi_{22} , in the Tits group and in 36 simple groups of Lie type, every element is a commutator $[x, y]$ with x a 2-element and y of odd order; the certificate is one stored pair per conjugacy class, re-checked by exact conjugacy tests in GAP. In $O'N, Ly$ and J_4 every element is a commutator of two elements of coprime orders, by short lemmas applied to the character tables of the GAP Character Table Library and explicit witnesses in five constructed subgroups, every step through a subgroup being checked against all possible class fusions. Consequently every nonabelian finite simple group of order less than 10^9 has the property, and so do 19 of the 26 sporadic groups. A withdrawn preprint of Lucchini and Spiga had reported a computer search, without published witnesses, for 19 groups, among them $O'N$ and Fi_{23} . The question remains open: for $HN, Th, Co_1, Fi_{23}, Fi'_{24}, B$ and M only some of the classes are certified here, and the groups of Lie type of order at least 10^9 , apart from $\text{PSL}(2, q)$, the Suzuki groups and five groups of the list above, are not treated. The results rest on GAP's exact algorithms, on the library character tables and on the ATLAS lists of maximal subgroups. This is an unrefereed note.

1 Introduction

1.1 The question

Throughout, $[x, y] = x^{-1}y^{-1}xy$, $\text{o}(x)$ is the order of x , and $\pi(m)$ is the set of prime divisors of an integer m . Following Pellegrini and Shumyatsky [PS12], an element g of a group G is a *coprime commutator* (in G) if $g = [x, y]$ with $x, y \in G$ and $\text{gcd}(\text{o}(x), \text{o}(y)) = 1$.

Problem 1.1 (Shumyatsky, [KM26, Problem 18.117]). *Is every element of every nonabelian finite simple group a coprime commutator?*

The question was first stated in [Shu15]; it is record KOU-18.117 of the collection [ula26], where it is listed as open. The identity is $[1, 1]$. Since $[x, y]^{-1} = [y, x]$ and $[x, y]^h = [x^h, y^h]$, the coprime

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commutators of G form a union of conjugacy classes, closed under inversion and under $\text{Aut}(G)$; the convention $xyx^{-1}y^{-1} = [x^{-1}, y^{-1}]$ gives the same notion. The elements x, y must lie in G .

Known cases: for $n \geq 5$, every element of A_n is a commutator of an element of odd order and an element of order dividing 4 (Shumyatsky [Shu15], Theorem 3.1 of the arXiv version); for every prime power $q > 3$, every element of $\text{PSL}(2, q)$ is a coprime commutator (Pellegrini and Shumyatsky [PS12, Theorem 1.1], who also report that computations in Magma confirmed the property for all nonabelian simple groups of order less than 10^7 , without giving data); and every element of a Suzuki group ${}^2B_2(q)$ is a coprime commutator (Zini [Zin21]). Without the condition on the orders the statement is Ore's conjecture, proved by Liebeck, O'Brien, Shalev and Tiep [LOST10]. In a preprint withdrawn one day after its first version, Lucchini and Spiga [LS25] proved the case of A_n again (the withdrawal notice says that this had been done in [Shu15]) and reported a random search in Magma, with $\text{o}(x) \in \{2, 4\}$ and $\text{o}(y)$ odd, by which they had verified the property for M_{11} , M_{12} , M_{22} , M_{23} , M_{24} , J_1 , J_2 , J_3 , ${}^2F_4(2)'$, HS , McL , He , Ru , Suz , $O'N$, Co_3 , Co_2 , Fi_{22} and Fi_{23} ; no witnesses or programs were published, and we do not use this report.

1.2 Results

Group names and class names are those of the ATLAS [CCN⁺85]; thus $L_n(q) = \text{PSL}(n, q)$, $U_n(q) = \text{PSU}(n, q)$, $S_{2m}(q) = \text{PSp}(2m, q)$, and $O_7(3)$, $O_8^\pm(q)$ are the simple orthogonal groups.

Theorem A. *Let G be one of the following 53 simple groups: the sporadic groups M_{11} , M_{12} , M_{22} , M_{23} , M_{24} , J_1 , J_2 , J_3 , HS , McL , He , Ru , Suz , Co_3 , Co_2 , Fi_{22} ; the Tits group ${}^2F_4(2)'$; and the 36 groups of Lie type*

$$L_3(q), U_3(q) \ (q = 3, 4, 5, 7, 8, 9, 11, 13), \quad L_4(3), L_4(4), L_5(2), U_4(2), U_4(3), U_5(2), U_6(2), \\ S_4(4), S_4(5), S_4(7), S_6(2), S_8(2), O_7(3), O_8^+(2), O_8^-(2), O_8^+(3), {}^3D_4(2), G_2(3), G_2(4), G_2(5).$$

Then every element of G is a commutator $[x, y]$ with x a 2-element and y an element of odd order.

Theorem B. *Every element of each of the sporadic groups $O'N$, Ly and J_4 is a coprime commutator.*

Corollary C. *Every element of every nonabelian finite simple group of order less than 10^9 is a coprime commutator.*

Corollary D. *Every element of each of the 19 sporadic groups other than HN , Th , Co_1 , Fi_{23} , Fi'_{24} , B and M , and of the Tits group, is a coprime commutator.*

All four statements are computer-assisted. Theorem A rests on one stored pair (x, y) for each conjugacy class and on GAP's exact algorithms for permutation groups (Section 3). Theorem B rests on four short lemmas applied to the character tables of the GAP Character Table Library [Bre25], on subgroups listed in the ATLAS, on the completeness of GAP's computation of possible class fusions, and on stored pairs in five subgroups that are constructed and identified explicitly (Section 4). Corollary C combines Theorem A with the three known families (Section 5).

Remark 1.2. The form of Theorem A is more than Problem 1.1 asks for, and it fails in $\text{PSL}(2, q)$ for $q \equiv 3 \pmod{8}$ (Remark 3.1). For six classes of Theorem B none of the certificates found here, through any subgroup of Table 2, has this form: for 31A, 31B of $O'N$, $g = [h, c]$ with $\text{o}(h) = 31$ and $\text{o}(c) = 5$ (and through $L_2(31)$, $\text{o}(c)$ divides 15); for 33A, 33B of Ly , $\text{o}(h) = 33$ and $\text{o}(c)$ divides $2^a 5^b$; for 35A, 35B of J_4 , $\text{o}(h) = 35$ and $\text{o}(c)$ divides $2^a 3^b$ (Lemma 2.1). The certificates printed in Appendix A for 11A, 11B of Ly and 7A, 7B of J_4 are not of this form either (Lemma 2.1 with $\text{o}(c) = 5$ and $\text{o}(c) = 3$), but these four classes lie in the image of M_{11} , respectively M_{22} , under every possible fusion, so Theorem A for these subgroups gives them certificates of this form. Every other class of Theorem B has a certificate of this form among those printed in Appendix A.

1.3 What remains open

Problem 1.1 remains open. The sporadic groups HN , Th , Co_1 , Fi_{23} , Fi'_{24} , B and M are not settled here (partial data in Section 6, not a proof), nor are the groups of Lie type of order at least 10^9 other than $PSL(2, q)$, the Suzuki groups and the five groups $S_8(2)$, $O_7(3)$, $U_6(2)$, $O_8^+(3)$, $G_2(5)$ of Theorem A. No counterexample was found; the classes that are not certified merely lack a certificate of the kinds used here. Section 2 contains the lemmas, Sections 3 to 5 the proofs, with the tables of $O'N$, Ly and J_4 in Appendix A, and Section 7 the programs, the verification runs and the literature search.

2 Lemmas

Let P and Q be disjoint sets of primes. A *certificate of type $P|Q$* for an element g of a group G is a pair $x, y \in G$ with $[x, y] = g$ such that $\pi(o(x)) \subseteq P$ and $\pi(o(y)) \subseteq Q$; then $\gcd(o(x), o(y)) = 1$. The type $\{2\}|\{\text{odd primes}\}$ is called $2|\text{odd}$. If (x, y) is a certificate of type $P|Q$ for g , then (y, x) is one of type $Q|P$ for g^{-1} , and (x^h, y^h) one of type $P|Q$ for g^h ; so certificates belong to conjugacy classes. A σ -*element*, for a set σ of primes, is an element whose order is a product of primes in σ .

Lemma 2.1 (rational classes of odd order). *Let $g \in G$ have odd order n , and let k be an integer such that g^k is conjugate to g , $\gcd(k-1, n) = 1$ and $d = o_n(k)$, the order of k modulo n , is prime to n . Then $g = [h, c] = [c, h']$ with $h, h' \in \langle g \rangle$ of order n and c a $\pi(d)$ -element. So g has certificates of types $\pi(n)|\pi(d)$ and $\pi(d)|\pi(n)$.*

Proof. Let $c_0^{-1}gc_0 = g^k$. Conjugation by c_0 induces on $\langle g \rangle$ the automorphism $s: u \mapsto u^k$, whose order is d ; hence d divides $o(c_0)$. Write $o(c_0) = uv$, where u is the $\pi(d)$ -part of $o(c_0)$ and $\gcd(v, d) = 1$, and choose e with $e \equiv 1 \pmod{u}$ and $e \equiv 0 \pmod{v}$. Then $c = c_0^e$ is a $\pi(d)$ -element, and it induces $s^e = s$ on $\langle g \rangle$, because d divides u . Choose j with $j(k-1) \equiv 1 \pmod{n}$; it is prime to n , and $[g^j, c] = g^{-j}(g^j)^c = g^{j(k-1)} = g$ and $[c, g^{-j}] = (g^j)^c g^{-j} = g^{j(k-1)} = g$. Finally $\pi(d) \cap \pi(n) = \emptyset$ since $\gcd(d, n) = 1$. $\square$

Lemma 2.2 (real classes of odd order). *Let $g \in G$ have odd order $n > 1$ and be conjugate to g^{-1} . Then $g = [h, c] = [c, h']$ with $h, h' \in \langle g \rangle$ and c a 2-element; so g has certificates of types $\pi(n)|\{2\}$ and $\{2\}|\pi(n)$.*

Proof. Apply Lemma 2.1 with $k = -1$: $\gcd(-2, n) = 1$, and $o_n(-1) = 2$ is prime to n . (Explicitly, $h = g^{(n-1)/2}$ and c is the 2-part of an element inverting g .) $\square$

In terms of the character table, g is real if and only if its class is fixed by the power map -1 , and g^k is conjugate to g if and only if the k -th power map fixes the class of g ; so the hypotheses of Lemmas 2.1 and 2.2 can be read off the table of G .

Lemma 2.3 (fixed-point-free coprime action). *Let N be a normal subgroup of a subgroup H of G , and let $c \in H$ with $\gcd(o(c), |N|) = 1$. (a) If $C_N(c) = 1$, then every element of N is of the form $[m, c]$ and of the form $[c, m']$ with $m, m' \in N$; so it has certificates of types $\pi(|N|)|\pi(o(c))$ and $\pi(o(c))|\pi(|N|)$. (b) $|C_N(c)| = |C_H(c)|/|C_{H/N}(cN)|$; so $C_N(c) = 1$ if and only if the centralizer orders of c in H and of cN in H/N are equal, which can be read off the character table of H and its factor table modulo N .*

Proof. (a) The map $N \rightarrow N$, $m \mapsto [m, c] = m^{-1}m^c$, is injective: if $m^{-1}m^c = m'^{-1}m'^c$ then $m'm^{-1} = (m'm^{-1})^c \in C_N(c) = 1$. As N is finite it is bijective. For $g \in N$ write $g^{-1} = [m, c]$; then $g = [c, m]$.

(b) Put $C = \langle c \rangle$. As $\gcd(|C|, |N|) = 1$, $C \cap N = 1$ and C is a complement of N in NC . The kernel of the map $C_H(c) \rightarrow C_{H/N}(cN)$, $h \mapsto hN$, is $C_N(c)$; it remains to show that the map is onto.

Let $hN \in C_{H/N}(cN)$, that is, $c^h \in cN$. Then $C^h \leq NC$ is a second complement of N in NC , so $C^h = C^m$ for some $m \in N$ by the Schur–Zassenhaus theorem (see [Isa08]; here C is cyclic, so the conjugacy part needs no further hypothesis). The complement C^h meets each coset of N in NC in exactly one element, and both c^h and $c^m = c[c, m]$ lie in $C^h \cap cN$. So $c^h = c^m$ and $hm^{-1} \in C_H(c)$, that is, $hN \in C_H(c)N/N$. $\square$

Lemma 2.4 (involutions in a subgroup A_4). *Let $s \in G$ be an involution in the class $2X$, and suppose that the class multiplication coefficient $a(2X, 3Y, 3Z)$ is positive for some classes $3Y, 3Z$ of elements of order 3, that is, some element of $3Z$ is a product $s't$ with $s' \in 2X$ and $t \in 3Y$. Then s has certificates of types $\{2\}|\{3\}$ and $\{3\}|\{2\}$.*

Proof. Let $s' \in 2X$ and $t \in 3Y$ with $s't \in 3Z$. Then $s'^2 = t^3 = (s't)^3 = 1$, so $K = \langle s', t \rangle$ is a quotient of $\langle a, b \mid a^2 = b^3 = (ab)^3 = 1 \rangle \cong A_4$. The proper quotients of A_4 have order 1 or 3 and contain no involution, so $K \cong A_4$. Its Klein four-group is $V = \{1, v_1, v_2, v_3\} \ni s'$, and t permutes v_1, v_2, v_3 cyclically by conjugation, say $v_i^t = v_{i+1}$ with indices modulo 3. Then $[v_i, t] = v_i v_i^t = v_i v_{i+1} = v_{i+2}$ and $[t, v_i] = [v_i, t]^{-1} = v_{i+2}$. So every involution of V , in particular s' , is $[v, t]$ and $[t, v']$ with v, v' involutions and $o(t) = 3$; and s is conjugate to s' . (This is Lemma 2.3(a) for $N = V$ in $H = K$.) $\square$

Lemma 2.5 (direct products). *Let $A, B \leq G$ commute elementwise with $A \cap B = 1$. If $a \in A$ has a certificate (x_1, y_1) of type $P_1|Q_1$ in A , $b \in B$ has a certificate (x_2, y_2) of type $P_2|Q_2$ in B , and $(P_1 \cup P_2) \cap (Q_1 \cup Q_2) = \emptyset$, then $(x_1 x_2, y_1 y_2)$ is a certificate of type $(P_1 \cup P_2)|(Q_1 \cup Q_2)$ for ab .*

Proof. As A and B commute, $[x_1 x_2, y_1 y_2] = [x_1, y_1][x_2, y_2] = ab$; and $(x_1 x_2)^m = x_1^m x_2^m = 1$ if and only if $x_1^m = 1 = x_2^m$, since $A \cap B = 1$, so $o(x_1 x_2) = \text{lcm}(o(x_1), o(x_2))$; likewise for $y_1 y_2$. $\square$

Lemma 2.5 is used only for the partial data of Section 6.

Lemma 2.6 (central elements). *Let H be a group and $z \in Z(H)$ with $z = [x, y]$ for some $x, y \in H$. Then $o(z)$ divides $o(x)$ and $o(y)$. Consequently a nontrivial central element of H is not a coprime commutator in H ; and if $N \trianglelefteq H$ and gN is a nontrivial central element of H/N , then g is not a coprime commutator in H .*

Proof. From $x^y = x[x, y] = xz$ and the centrality of z we get $x^{y^m} = xz^m$ for all $m \geq 0$; for $m = o(y)$ this gives $z^{o(y)} = 1$. Likewise $y^x = yz^{-1}$ gives $z^{o(x)} = 1$. If $g = [x, y]$ with coprime orders, then $gN = [xN, yN]$, where $o(xN)$ and $o(yN)$ divide $o(x)$ and $o(y)$; by the first part $o(gN) = 1$. $\square$

Lemma 2.6 explains why some classes of a subgroup cannot be certified inside that subgroup (Section 4.1). The step from a subgroup to the group is the following.

Lemma 2.7 (subgroups and class fusions). *Let $H \leq G$, let S be a set of classes of H consisting of coprime commutators of H , let $\mathcal{F}$ be a set of maps from the classes of H to those of G that contains the class fusion f of H into G , and let $\sigma \in \text{Aut}(G)$, possibly $\sigma = 1$. If a class C of G is such that C or $\sigma(C)$ lies in $f'(S)$ for every $f' \in \mathcal{F}$, then C consists of coprime commutators of G .*

Proof. A pair $x, y \in H$ is a pair in G with the same orders, so the classes in $f(S)$ consist of coprime commutators of G ; and σ maps coprime commutators to coprime commutators. Take $f' = f$. $\square$

We apply Lemma 2.7 with $\mathcal{F}$ the set of all *possible class fusions* from the library table of H to that of G , computed by GAP's function `PossibleClassFusions`: all maps between the classes that satisfy a list of necessary conditions for a class fusion, among them that element orders are preserved, that the power maps are respected and that the restriction of every irreducible character of G is a character of H . The true fusion of a subgroup H of G satisfies these conditions, so it lies in $\mathcal{F}$ as soon as the library table of H is the character table of the subgroup in question. Fusions stored in the library are not used in the proofs.

3 Explicit witnesses: Theorem A

The groups. Each group G of Theorem A is constructed in GAP [The26] as a permutation group (Table 1): the Mathieu groups in their natural actions (`MathieuGroup`); J_3 from the two generators of degree 6156 that are shipped with the AtlasRep package [WPN⁺26]; all other groups as `PrimitiveGroup( $d, k$ )` from the library of primitive groups of degree less than 4096 [HLRDR26, CQRD11]. Each group is checked to be simple and of the order of the intended group, and each group of Lie type also to be of the isomorphism type returned by GAP’s function `IsomorphismTypeInfoFiniteSimpleGroup` (the third verification run compared this type with that of the library table for all 51 groups that have one). By the classification, two nonabelian finite simple groups of the same order are isomorphic except for the pairs $\{A_8, L_3(4)\}$ and $\{B_n(q), C_n(q)\}$ with $n \geq 3$ and q odd [KLST90]; for the two groups of the list concerned, $L_3(4)$ and $O_7(3)$, the numbers of classes (10 and 58, checked below) differ from those of A_8 and $S_6(3)$ (14 and 74).

G	constr.	$k(G)$	G	constr.	$k(G)$	G	constr.	$k(G)$	G	constr.	$k(G)$
M_{11}^*	nat.	10	Co_2	2300, 1	60	$U_3(7)^*$	344, 1	58	$S_4(7)^*$	400, 1	52
M_{12}^*	nat.	15	Fi_{22}	3510, 1	65	$U_3(8)^*$	513, 1	28	$S_6(2)^*$	28, 6	30
M_{22}^*	nat.	12	${}^2F_4(2)^*$	1600, 20	22	$U_3(9)^*$	730, 1	92	$S_8(2)$	120, 15	81
M_{23}^*	nat.	17	$L_3(3)^*$	13, 7	12	$U_3(11)^*$	1332, 1	48	$O_7(3)$	351, 5	58
M_{24}^*	nat.	26	$L_3(4)^*$	21, 4	10	$U_3(13)^*$	2198, 1	184	$O_8^+(2)^*$	120, 16	53
J_1^*	266, 1	15	$L_3(5)^*$	31, 9	30	$L_4(3)^*$	40, 5	29	$O_8^-(2)^*$	119, 1	39
J_2^*	100, 1	21	$L_3(7)^*$	57, 2	22	$L_4(4)^*$	85, 3	84	$O_8^+(3)$	1080, 5	114
J_3^*	AR	21	$L_3(8)^*$	73, 13	72	$L_5(2)^*$	31, 10	27	${}^3D_4(2)^*$	819, 5	35
HS^*	100, 3	24	$L_3(9)^*$	91, 7	90	$U_4(2)^*$	27, 12	20	$G_2(3)^*$	351, 7	23
McL^*	275, 1	24	$L_3(11)^*$	133, 1	132	$U_4(3)^*$	112, 1	20	$G_2(4)^*$	416, 7	32
He	2058, 1	33	$L_3(13)^*$	183, 1	64	$U_5(2)^*$	165, 1	47	$G_2(5)$	3906, 1	44
Ru	4060, 1	36	$U_3(3)^*$	28, 4	14	$U_6(2)$	672, 1	46			
Suz	1782, 1	43	$U_3(4)^*$	65, 3	22	$S_4(4)^*$	85, 1	27			
Co_3	276, 3	42	$U_3(5)^*$	50, 1	14	$S_4(5)^*$	156, 1	34			

Table 1: The 53 groups of Theorem A, their constructions and their numbers $k(G)$ of classes: d, k is `PrimitiveGroup( $d, k$ )`, “nat.” the natural action of a Mathieu group, AR the AtlasRep representation of degree 6156 of J_3 . The 42 groups of order less than 10^9 are marked with *. There are 2273 classes, hence 2273 stored pairs.

The certificate. For every class of G one pair (x, y) of permutations is stored, with x a 2-element, y of odd order and $[x, y]$ in the class. The program `explicit/verify.g` checks that $x, y \in G$, that $o(x)$ is a power of 2 and $o(y)$ is odd, computes $z = x^{-1}y^{-1}xy$, locates the class of z among the conjugacy classes of G computed by GAP by exact conjugacy tests (`IsConjugate`), and checks that every class is met and that the number of classes equals that of the library table of G . For $L_3(13)$ and $U_3(13)$, which have no library table, the second verification run compared the numbers 64 and 184 of classes computed by GAP with the generic class numbers $(q^2 + q + 10)/3$ of $\text{PSL}(3, q)$, $3 \mid q - 1$, and $q^2 + q + 2$ of $\text{PSU}(3, q)$, $3 \nmid q + 1$, of the generic character tables [SF73]; the same formulas give the class numbers of the library tables of $L_3(4)$, $L_3(7)$, $U_3(3)$, $U_3(4)$, $U_3(7)$, $U_3(9)$. A second check avoids GAP’s computation of the classes: the commutators z are pairwise non-conjugate and $\sum_z |G|/|C_G(z)| = |G|$, so every class is met. It was made for all 53 groups by the second and by the third verification run, and for eleven of them by the program `check_note.g` of this note.

Proof of Theorem A. For each of the 53 groups both checks succeed (Section 7). So every element of G is a commutator $[x, y]$ with x a 2-element and y of odd order. $\square$

Example and search. GAP's M_{11} is generated by $(1, 2, \dots, 11)$ and $(3, 7, 11, 8)(4, 10, 5, 6)$, acting on the right. The pair stored for one of its two classes of elements of order 11 is $x = (1, 6, 8, 4, 10, 2, 3, 5)(9, 11)$, $y = (2, 9, 8, 3, 4)(5, 7, 6, 11, 10)$, with $\text{o}(x) = 8$, $\text{o}(y) = 5$ and $[x, y] = (1, 9, 3, 5, 2, 8, 6, 11, 4, 10, 7)$. Over all 53 groups, apart from the pairs $(1, 1)$ for the identity, the 2-elements x have orders 2, 4, 8, 16 in 507, 855, 670, 188 pairs. The pairs were found by a random search (x the 2-part of a random element, y the odd part of another), followed, for the classes not met, by a targeted search: if x is a 2-element and $x^{y_0} = xg$, then $[x, cy_0] = g$ for all $c \in C_G(x)$, and c was varied until $\text{o}(cy_0)$ was odd. The search is not part of the proof. As a test, the same programs met every class of $A_6, \dots, A_9$, of $\text{PSL}(2, q)$ for $q = 7, 8, 11, 13, 19, 23, 27$ and of ${}^2B_2(8)$; for $q = 11, 19, 27$ the two classes of elements of order p needed pairs of another type, as they must:

Remark 3.1. Let $q \equiv 3 \pmod{8}$ be a power of the prime p . The Sylow 2-subgroups of $\text{PSL}(2, q)$ have order 4, so every 2-element is an involution or 1. For an involution x , $[x, y] = x \cdot x^y$ and $[y, x] = x^y \cdot x$ are products of two involutions, hence real. As $q \equiv 3 \pmod{4}$, the two classes of elements of order p are not real, so they contain no commutator of a 2-element and an element of odd order, in either order; in [PS12] they are obtained from an element of order p and an element of the cyclic subgroup of odd order $(q - 1)/2$ of a Borel subgroup.

4 Character-table certificates: Theorem B

For $G \in \{O'N, Ly, J_4\}$ every class is certified either by Lemmas 2.1, 2.2 and 2.4 applied to the library table of G , or by Lemma 2.7 applied to one of the subgroups H of Table 2, with $\mathcal{F}$ the set of all possible class fusions and with the set S of classes of H given there. For Ly and J_4 , whose outer automorphism groups are trivial, $\sigma = 1$. For $O'N$, whose outer automorphism group has order 2, σ is an outer automorphism: all 24 possible class fusions from $O'N$ into $O'N.2$ induce the same permutation $(8A\ 8B)(16A\ 16C)(16B\ 16D)(20A\ 20B)(31A\ 31B)$ of the classes, and the true fusion is among them, so this is the action of σ on the classes.

The existence of these subgroups is taken from the ATLAS [CCN⁺85], and their character tables from the library [Bre25]; the extensions written with a colon are split, so M_{24} and $2 \times M_{11}$, hence M_{11} , are subgroups. For the simple subgroups the set S consists of all classes, by the cited theorem or by Theorem A. For $4^3.L_3(2)$, the normal subgroup N of order 64 consists of the classes $1A, 2A, 4A, 4B$ of H , and an element c of order 7 has $|C_H(c)| = 7 = |C_{H/N}(cN)|$; so $C_N(c) = 1$ by Lemma 2.3(b), and every element of N is $[m, c]$ with $m \in N$ by Lemma 2.3(a).

4.1 The constructed subgroups

Five subgroups are constructed as permutation groups, identified with the subgroups of the ATLAS, and searched for coprime commutators as in Section 3. All stored pairs are of type 2|odd, and the classes they do not meet are not coprime commutators in H (Lemma 2.6); so S is exactly the set of classes of coprime commutators of H . The facts quoted are checked by the program `identify_note.g`.

(a) $H = 4_2.L_3(4) < O'N$, the derived subgroup of the maximal subgroup $4_2.L_3(4):2_1$: a perfect central extension of $L_3(4)$ by a cyclic group of order 4, with library table $4_2.L_3(4)$. The library of perfect groups [HP89, HHPF26] contains all perfect groups of order $80640 = 4|L_3(4)|$; exactly two have a cyclic centre of order 4, numbers 3 and 4, with character tables isomorphic to $4_2.L_3(4)$ and $4_1.L_3(4)$. So $H \cong \text{PerfectGroup}(80640, 3)$, of degree 224. The 29 stored pairs give commutators in 29 classes, none central except the identity, whose sizes add up to $|H| - 3$: every non-central element is a coprime commutator in H , and the central classes $2A, 4A, 4B$ are not.

(b) $H = 2.A_{11} < Ly$, the Schur cover of A_{11} ; as the Schur multiplier of A_{11} has order 2, every perfect group with a central subgroup of order 2 and quotient A_{11} is isomorphic to H . The

G	no.	H	position of H in G	classes S of H	$ \mathcal{F} $
$O'N$	1	A_7	maximal (two classes)	all, [Shu15]	2
	2	$L_2(31)$	maximal (two classes)	all, [PS12]	16
	3	$4^3.L_3(2)$	maximal	$1A, 2A, 4A, 4B$, Lemma 2.3	4
	4	M_{11}	maximal (two classes)	all, Theorem A	2
	5	J_1	maximal	all, Theorem A	6
	6	$L_3(7)$	index 2 in maximal $L_3(7):2$ (two classes)	all, Theorem A	36
	7	$4_2.L_3(4)$	index 2 in maximal $4_2.L_3(4):2_1$	non-central, (a)	8
Ly	1	$G_2(5)$	maximal	all, Theorem A	20
	2	M_{11}	subgroup of maximal $3^5:(2 \times M_{11})$	all, Theorem A	8
	3	$2.A_{11}$	maximal	non-central, (b)	64
J_4	1	$L_2(32)$	index 5 in maximal $L_2(32):5$	all, [PS12]	6
	2	$L_2(23)$	index 2 in maximal $L_2(23):2$	all, [PS12]	3
	3	$U_3(11)$	index 2 in maximal $U_3(11):2$	all, Theorem A	6
	4	M_{22}	index 2 in maximal $M_{22}:2$	all, Theorem A	4
	5	$U_3(3)$	maximal	all, Theorem A	6
	6	M_{24}	complement in maximal $2^{11}:M_{24}$	all, Theorem A	4
	7	$2^{11}:M_{24}$	maximal	all, (c)	16
	8	$2^{10}:L_5(2)$	maximal	all, (d)	6
	9	$11^{1+2}:(5 \times 2S_4)$	maximal	11 classes of H in H' , (e)	4

Table 2: The subgroups used for Theorem B. The positions are those of the ATLAS lists of maximal subgroups; $|\mathcal{F}|$ is the number of possible class fusions from the library table of H into that of G . The letters (a)–(e) refer to Section 4.1.

group obtained from the basic spin representation of the double cover of S_{11} over $\mathbb{F}_3$ (GAP's `BasicSpinRepresentationOfSymmetricGroup`), acting on an orbit of 60480 vectors, is perfect of order $2|A_{11}|$ with a centre of order 2 and quotient of type A_{11} , so it is H . The 48 stored pairs meet the identity and the 47 non-central classes.

(c) $H = 2^{11}:M_{24} < J_4$. The library table of H (`mx1j4`) has exactly three normal subgroups, of orders 1, 2^{11} , $|H|$. So $V = O_2(H)$ is a faithful irreducible $\mathbb{F}_2 M_{24}$ -module (a proper submodule, or the kernel of the action, would be another normal subgroup), and $H = V:M_{24}$, acting on V by affine maps, is a primitive group of degree 2048. The library [HLRDR26, CQRD11] contains all primitive groups of this degree; exactly two have order $2^{11}|M_{24}|$, numbers 10 and 11, and the point stabilizer M_{24} has orbits of lengths 1, 759, 1288 in number 10 (the Golay code modulo the all-one vector) and 1, 276, 1771 in number 11 (the even part of the cocode). The classes of H inside V are the M_{24} -orbits on V , of sizes 1, 276, 1771 in the table of H . So $H \cong \text{PrimitiveGroup}(2048, 11)$ (its 72 classes also agree with the table in element orders and centralizer orders; number 10 has 80 classes). The 72 stored pairs meet all 72 classes.

(d) $H = 2^{10}:L_5(2) < J_4$. Likewise the table of H has exactly the normal subgroups of orders 1, 2^{10} , $|H|$, so H is an affine primitive group of degree 1024, and exactly one primitive group of this degree has order $2^{10}|L_5(2)|$: $H \cong \text{PrimitiveGroup}(1024, 83)$ (point stabilizer orbits 1, 155, 868, as are the class sizes inside 2^{10} in the table; 81 classes as in the table). The 81 stored pairs meet all 81 classes.

(e) $H = 11^{1+2}:(5 \times 2S_4) < J_4$. Here $Q = O_{11}(H)$ is extraspecial of order 11^3 and exponent 11, and $H/Q \cong C_5 \times T$ with $|T| = 48$ and $T/Z \cong S_4$ for a central subgroup Z of order 2. By the Schur–Zassenhaus theorem $H = Q:K$ with $K \cong H/Q$. (i) K acts faithfully on Q , since no element of order 2, 3 or 5 of J_4 has a centralizer order divisible by 11^2 . (ii) A coprime automorphism of Q that is trivial on $Q/\Phi(Q) = Q/Z(Q)$ is trivial (Burnside; see [Isa08]), so K embeds into $\text{GL}(Q/Z(Q)) = \text{GL}(2, 11)$.

(iii) $\text{GL}(2, 11)$ has exactly two classes of subgroups of order 48, isomorphic to `SmallGroup(48, 6)` and `SmallGroup(48, 29)` $\cong \text{GL}(2, 3)$, and only the second has a normal subgroup of order 2 with quotient S_4 ; so $T \cong \text{GL}(2, 3)$, embedded uniquely up to conjugacy (the binary octahedral group has no faithful representation of degree 2 over $\mathbb{F}_{11}$, as 2 is not a square modulo 11). (iv) C_5 centralizes the absolutely irreducible T , so it is scalar. (v) The automorphisms of Q trivial on $Q/Z(Q)$ are the inner ones, and every element of $\text{GL}(2, 11)$ is induced by an automorphism of Q ; so subgroups of order 240 of $\text{Aut}(Q)$ with conjugate images in $\text{GL}(2, 11)$ become, after conjugation, complements of $\text{Inn}(Q)$ in a common subgroup, and are conjugate by the Schur–Zassenhaus theorem. So H is determined up to isomorphism. The constructed group is $Q:K$ on the 1331 elements of $Q = \mathbb{F}_{11}^2 \times \mathbb{F}_{11}$, with $(v, c)(w, d) = (v + w, c + d + (v_1 w_2 - v_2 w_1)/2)$, where $A \in \text{GL}(2, 11)$ acts by $(v, c) \mapsto (vA, \det(A)c)$ and K is generated by the scalar of order 5 and a subgroup $\text{GL}(2, 3)$. It has order 319440, O_{11} extraspecial of exponent 11 with $C_H(O_{11}) = Z(O_{11})$, quotient $C_5 \times \text{GL}(2, 3)$, and a character table isomorphic to the library table. Its derived subgroup $H' = Q:\text{SL}(2, 3)$ is the union of 13 classes of H , of element orders 1, 2, 3, 4, 6, 11, 11, 22, 33, 33, 44, 66, 66 (as a group, H' has 82 classes); the classes of H outside H' contain no commutators. The 11 stored pairs meet all of these 13 classes except those of orders 2 and 22 (the classes 2A, 22A of the library table), which are not coprime commutators in H : they map to the central involution of H/Q , the only involution of $(H/Q)' \cong \text{SL}(2, 3)$, and Lemma 2.6 applies. As the orders 11, 33, 66 occur twice among the 13 classes and the others once, S is defined by element orders alone.

4.2 Examples

In $O'N$: $a(2A, 3A, 3A) = 1620 > 0$ certifies 2A (Lemma 2.4); 19A is real, so $g = [g^9, c]$ with c a 2-element inverting g (Lemma 2.2); 31A is not real, but the second power map fixes it and $\text{o}_{31}(2) = 5$, so $g = [g, c]$ with c a 5-element such that $c^{-1}gc = g^2$ (Lemma 2.1); 4A and 4B meet the normal subgroup 4^3 of $4^3.L_3(2)$ under each of its 4 possible fusions (Lemma 2.3); both possible fusions of A_7 map its classes of elements of orders 4 and 6 to 4B and 6A (a published theorem). The outer automorphism is needed for 16A–16D: $L_2(31)$ has four classes of elements of order 16, and each of its 16 possible fusions maps them two to one onto $\{16A, 16B\}$ or onto $\{16C, 16D\}$, which σ exchanges. Constructed subgroups: the classes 22A, 22B of Ly meet non-central classes of $2.A_{11}$ under each of the 64 possible fusions, and 66A, 66B of J_4 meet the two classes of elements of order 66 of $11^{1+2}:(5 \times 2S_4)$ under each of its 4 possible fusions.

Proof of Theorem B. Tables 4 to 6 in Appendix A give a certificate for each of the 30, 53 and 62 classes of $O'N$, Ly and J_4 , computed by the program `note_tables.g` from Lemmas 2.1, 2.2 and 2.4 and the subgroup steps described above. By Lemma 2.7 every element of these groups is a coprime commutator. $\square$

5 Proof of the corollaries

Proof of Corollary C. GAP’s `SimpleGroupsIterator`, which enumerates the nonabelian finite simple groups by order on the basis of the classification, lists those of order less than 10^9 . Apart from the alternating groups $A_5, \dots, A_{12}$, the groups $\text{PSL}(2, q)$ and the Suzuki groups ${}^2B_2(8)$ and ${}^2B_2(32)$, there are 42 of them: the sporadic groups $M_{11}, M_{12}, J_1, M_{22}, J_2, M_{23}, HS, J_3, M_{24}, McL$, the Tits group, and the 31 groups of Lie type marked in Table 1. (The second and the third verification run obtained the same 42 groups again from the order formulas of all families of simple groups.) All 42 are covered by Theorem A, and the others by [Shu15], [PS12, Theorem 1.1] and [Zin21]. $\square$

Proof of Corollary D. Theorem A covers 16 sporadic groups and the Tits group, and Theorem B the sporadic groups $O'N$, Ly and J_4 . $\square$

6 The seven remaining sporadic groups

For HN , Th , Co_1 , Fi_{23} , Fi'_{24} , B and M the methods of Section 4 certify only part of the classes (Table 3). The third column uses only the table of G (Lemmas 2.1, 2.2 and 2.4) and is a proof for these classes, on the same basis as Theorem B. The fourth column comes from the program `certify_all.g` with which the results were first obtained. It finds subgroup tables through the class fusions stored in the library (to depth 3, for Fi'_{24} , B , M to depth 2), uses Lemmas 2.3 and 2.5 inside them, and recognizes as simple subgroups the three known families and 40 of the 53 groups of Theorem A (the sporadic groups, the Tits group and 23 of the groups of Lie type; the other 13 are not used). For HN and Th each subgroup step is checked against all possible fusions, for the other groups only the stored fusions are used, and the existence of the subgroups behind the stored fusions was not examined. So the fourth and fifth columns are partial data, not a proof; the verification runs did not examine them.

G	classes	by R, K, A4	in total	not certified	subgroup steps
HN	54	23	38	16	all possible fusions
Th	48	20	30	18	all possible fusions; $8B$ fails the check
Co_1	101	32	86	15	stored fusions
Fi_{23}	98	27	85	13	stored fusions
Fi'_{24}	108	43	87	21	stored fusions, depth 2
B	184	30	84	100	stored fusions, depth 2
M	194	63	92	102	stored fusions, depth 2

Table 3: Partial data for the seven sporadic groups not settled here. The column “by R, K, A4” counts the trivial class and the classes certified by Lemmas 2.1, 2.2 and 2.4 from the table of G alone.

Apart from $27B$ and $27C$ of Th , which satisfy neither Lemma 2.2 nor Lemma 2.1, the classes that are not certified have even order. They would need further constructions, as in Section 4.1, or computations in G itself; none of them is known to be a counterexample.

7 Computations and reproducibility

All computations use GAP 4.16.1 [The26] with CTblLib 1.3.11 [Bre25], PrimGrp 4.0.3 [HLRDR26], PerfGrp 1.0.0 [HHPF26] and the two generator files of J_3 shipped with AtlasRep 2.1.12 [WPN⁺26]; no data were downloaded. The proofs rely on GAP’s exact algorithms for permutation groups and character tables, on the library tables of the groups of Table 2 and of $O'N$, $O'N.2$, Ly , J_4 , on the ATLAS data on maximal subgroups as encoded in the library, and, for the identifications of Section 4.1, on the completeness of the libraries of perfect groups of order 80640 and of primitive groups of degrees 1024 and 2048 [HP89, HHPF26, HLRDR26, CQRD11].

The reproducibility package contains, first, the programs with which the results were first obtained, with all recorded outputs: the constructions, the search, the verification program and the stored pairs (`explicit/`, `covers/`, `locals/`), and the table-level certification `certify_all.g` (`tables/`). That program chooses its subgroups automatically and checks each step against all possible fusions; it certifies all classes of $O'N$, Ly and J_4 , but for $4B$ of $O'N$ it used a subgroup table, $(4^2 \times 2).2^3.S_3$, that is not among the maximal subgroups of the ATLAS and whose origin is not documented in the library; the certificates of this note use only Table 2. Second, the programs written with this note: `note_tables.g` computes Tables 2 and 4 to 6 (one second), `identify_note.g` checks the facts

of Section 4.1 (ten seconds), `check_note.g` re-checks the stored pairs of M_{11} , M_{12} , J_1 , J_2 , HS , ${}^2F_4(2)'$, $L_3(4)$, $U_3(5)$, $L_3(7)$, $G_2(3)$, $U_3(11)$ and $4_2.L_3(4)$ by the class equation and the certificates of $O'N$ (two seconds), and `run_full_check.sh` repeats `verify.g` for all 53 groups (two minutes). All checks passed. When the package was run again from its own folder on 27 October 2026, the 53 verifications reproduced the recorded outputs up to the order in which GAP lists the classes, and `certify_all.g` reproduced its outputs for $O'N$, Ly , J_4 byte for byte and the certified classes of Table 3. Third, the programs and outputs of the three verification runs, in the folders `verification_run_A/`, `verification_run_B/` and `independent_run_2/` (the second stage of the verification).

Verification. The results were first obtained with the programs of `explicit/`, `covers/`, `locals/` and `tables/`. Three independent verification runs, all AI-assisted, followed; each wrote its own programs.

The first two runs examined the first results, and neither saw the other. The first run re-derived the proofs of all lemmas, read the certification programs, recomputed the certification of every class of $O'N$, Ly and J_4 with its own program (all possible fusions, the outer automorphism of $O'N$), checked the stored pairs and identifications of the five constructed subgroups, re-verified Theorem A for ten groups, reproduced the list of Corollary C and compared the statements quoted from the literature with the sources. The second run re-checked all 2273 stored pairs of Theorem A by the class equation, with the class invariants compared to the library tables; recomputed every subgroup step against all possible fusions and found the further routes through simple maximal subgroups of Appendix A; rebuilt $11^{1+2}:(5 \times 2S_4)$ in a different way and found it isomorphic to the constructed group; obtained the list of Corollary C by two enumerations; and repeated the literature search. Neither run found an error in a theorem or a lemma. They required fourteen corrections of the text and of the package, all of which were made (they are listed in the verification report); the mathematical ones are that the certificate of $4B$ of $O'N$ now uses maximal subgroups only, that the identifications (c) and (d) of Section 4.1 are derived from the library tables with the orbit lengths of the primitive groups, that the certificates not of type $2|\text{odd}$ are named (Remark 1.2; the classes $31A$, $31B$ of $O'N$ were added when this note was written), and that the class numbers of $L_3(13)$ and $U_3(13)$ are compared with the generic formulas. For this note the certificates of Theorem B were recomputed from the subgroups of Table 2 alone, and the facts of Section 4.1 were checked again; the results agree with those of the two runs.

The third run examined the final text of this note and the programs written with it. It compared the statement with the notebook and the corpus record, and every sentence about [Shu15, PS12, Zin21, LS25] with the texts and records named below; re-derived the proofs of Section 2 line by line and tested Lemmas 2.1 to 2.4 and 2.6 by brute force on small groups; and wrote programs of its own from the statements of the note. These programs re-checked all 2273 stored pairs of Theorem A by the class equation, with the constructions, orders and isomorphism types of Table 1; recomputed Theorem B with the real classes, the power maps and the class multiplication coefficients obtained from the character values and with every possible class fusion tested again, and compared Tables 2 and 4 to 6 with the computation entry by entry; re-checked the stored pairs and the facts of Section 4.1, and built $11^{1+2}:(5 \times 2S_4)$ again from the formulas printed in (e), obtaining a group isomorphic to the constructed one; enumerated the simple groups of order less than 10^9 ; and recomputed the third column of Table 3. The run also executed the programs of the package again from the archive (all commands of its README, the identification programs of `tables/aux/` and the two constructions); all recorded outputs were reproduced. It found no wrong theorem, lemma or table entry, and required six corrections of the text and of the package, all of which were made; the mathematical ones concern Remark 1.2 (the classes $11A$, $11B$ of Ly and $7A$, $7B$ of J_4), the position of M_{11} in Ly in Table 2 and the description of H' in Section 4.1(e).

Scope and priority. The question is Shumyatsky’s [KM26, Problem 18.117], [Shu15]. This note settles it for the 53 groups of Theorem A and for $O’N$, Ly and J_4 , and, together with the known results, for all simple groups of order less than 10^9 ; it remains open in general, in particular for the seven sporadic groups of Section 6 and for the groups of Lie type of order at least 10^9 other than $PSL(2, q)$, the Suzuki groups and the five groups $S_8(2)$, $O_7(3)$, $U_6(2)$, $O_8^+(3)$, $G_2(5)$. The results for A_n , $PSL(2, q)$ and the Suzuki groups are those of Shumyatsky [Shu15], Pellegrini and Shumyatsky [PS12] and Zini [Zin21], which are used, not reproved. Pellegrini and Shumyatsky report a Magma computation for the simple groups of order less than 10^7 without data [PS12], and Lucchini and Spiga reported, in a withdrawn preprint, a random search for 19 groups, among them $O’N$ and Fi_{23} , without witnesses [LS25]; Fi_{23} is not settled here. The searches were made on 27 October 2026, when the results were first obtained, and repeated in part by the three verification runs: queries to arXiv (for coprime commutators, for commutators of elements of coprime orders and for the authors of the known results), to zbMATH Open and to Crossref, the lists of works citing [Shu15], [PS12] and [Zin21] in OpenAlex, and three web searches. No printed answer to the question beyond A_n , $PSL(2, q)$ and ${}^2B_2(q)$ was found in these sources; the works citing [Shu15, PS12, Zin21] that were found treat other questions (conciseness of commutator words, criteria for nilpotency and solubility, profinite groups, products of conjugacy classes, symmetric groups). MathSciNet and Google Scholar were not searched. A search that finds nothing is not a proof of novelty, and no priority is claimed.

What was read: [KM26] in version 49, at Problem 18.117; [Shu15] in the arXiv version (v2), Theorem 3.1 and the statement of the question; [PS12] in the arXiv version (v1), Theorem 1.1, its proof in Section 3 and the remark on the Magma computation; of [Zin21] the Crossref and zbMATH records with the abstract; [LS25] in versions 1 and 2. The proofs of the cited theorems were not checked. The other works in the bibliography are cited for standard facts or for the software and data used, and were not read for this note; the ATLAS data are used through the library [Bre25].

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

A The classes of $O'N$, Ly and J_4

For every class of G the tables give its ATLAS name, the order of the centralizer of an element g of the class, and the certificates: the trivial class; R for Lemma 2.2; K k, d for Lemma 2.1 with the smallest k and $d = o_n(k)$; A4 3Y, 3Z for Lemma 2.4 with $a(2X, 3Y, 3Z) > 0$; or the numbers of the subgroups of Table 2 through which the class is certified (Lemma 2.7 with all possible fusions), the first one before the semicolon and the others after it. An asterisk marks a step that needs the outer automorphism of $O'N$. Further subgroups are listed only for the classes that are not certified by a lemma. The tables are the output of the program `note_tables.g`.

class	$ C_G(g) $	certificate	class	$ C_G(g) $	certificate	class	$ C_G(g) $	certificate
1A	$ G $	trivial	8B	32	7; 2*, 4*, 6*	16D	16	2*; 6*
2A	161280	A4 3A, 3A	10A	20	5; 7	19A	19	R
3A	3240	R	11A	11	R	19B	19	R
4A	80640	3; 7	12A	36	7	19C	19	R
4B	256	1; 2, 3, 4, 6, 7	14A	28	6; 7	20A	20	7
5A	180	R	15A	45	R	20B	20	7
6A	72	1; 4, 5, 6, 7	15B	45	R	28A	28	7
7A	1372	R	16A	16	2*; 6*	28B	28	7
7B	49	R	16B	16	2*; 6*	31A	31	K 2, 5
8A	32	7; 2*, 4*, 6*	16C	16	2*; 6*	31B	31	K 2, 5

Table 4: Certificates for the 30 classes of $O'N$.

class	$ C_G(g) $	certificate	class	$ C_G(g) $	certificate	class	$ C_G(g) $	certificate
1A	$ G $	trivial	12A	288	1; 3	30B	90	1; 3
2A	39916800	A4 3B, 3B	12B	72	1; 3	31A	31	R
3A	2694384000	R	14A	168	3	31B	31	R
3B	174960	R	15A	2250	R	31C	31	R
4A	20160	1; 2, 3	15B	90	R	31D	31	R
5A	2250000	R	15C	75	R	31E	31	R
5B	3750	R	18A	18	3	33A	33	K 5, 10
6A	120960	1; 3	20A	40	1; 3	33B	33	K 5, 10
6B	2160	1; 3	21A	42	R	37A	37	R
6C	324	1; 3	21B	42	R	37B	37	R
7A	168	R	22A	22	3	40A	40	3
8A	480	1; 3	22B	22	3	40B	40	3
8B	96	1; 3	24A	24	1; 3	42A	42	3
9A	54	R	24B	24	1; 3	42B	42	3
10A	3600	1; 3	24C	24	1; 3	67A	67	R
10B	50	1; 3	25A	25	R	67B	67	R
11A	66	K 3, 5	28A	28	3	67C	67	R
11B	66	K 3, 5	30A	90	1; 3			

Table 5: Certificates for the 53 classes of Ly .

class	$ C_G(g) $	certificate	class	$ C_G(g) $	certificate	class	$ C_G(g) $	certificate
1A	$ G $	trivial	12B	192	3; 7, 8	31A	31	R
2A	21799895040	A4 3A,3A	12C	48	8; 7	31B	31	R
2B	1816657920	A4 3A,3A	14A	84	6; 7, 8	31C	31	R
3A	2661120	R	14B	84	6; 7, 8	33A	66	R
4A	5406720	3; 7, 8, 9	14C	56	8; 7	33B	66	R
4B	98304	4; 3, 5, 6, 7, 8	14D	56	8; 7	35A	35	K 2, 12
4C	43008	8; 7	15A	30	R	35B	35	K 2, 12
5A	6720	R	16A	32	7	37A	37	R
6A	2661120	9; 7, 8	20A	160	3; 7	37B	37	R
6B	2304	4; 3, 5, 6, 7, 8	20B	160	3; 7	37C	37	R
6C	2304	8; 7	21A	42	6; 7, 8	40A	40	3
7A	840	K 2, 3	21B	42	6; 7, 8	40B	40	3
7B	840	K 2, 3	22A	264	3	42A	42	8
8A	1280	3; 7, 8	22B	22	7	42B	42	8
8B	768	8; 7	23A	23	R	43A	43	R
8C	512	4; 6, 7, 8	24A	48	7	43B	43	R
10A	960	3; 7	24B	48	7	43C	43	R
10B	80	8; 7	28A	28	7	44A	44	3; 9
11A	31944	R	28B	28	7	66A	66	9
11B	242	R	29A	29	R	66B	66	9
12A	192	3; 7, 8	30A	30	7			

Table 6: Certificates for the 62 classes of J_4 .

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