

A Gap of Three Between the Derived Lengths of a Finite Soluble Group and of a Maximal Subgroup (Kourovka Notebook Problem 17.91)

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Abstract

Problem 17.91 of the Kourovka Notebook, posed by V. S. Monakhov and also Question 9 of a survey of Monakhov and Trofimuk (2010), asks whether $d(G) - d(M) \leq k$ for an absolute constant k , for all finite soluble groups G and maximal subgroups M , where d is the derived length, and asks for the least such k . We answer neither part. We show that $k \geq 3$ if k exists: the split extension $G = 3_+^{1+2} \rtimes \mathrm{GL}(2, 3)$ of order 1296 has derived length 6, and its maximal subgroup $3_+^{1+2} \rtimes B$ of index 4, with B a Borel subgroup, has derived length 3. By an exhaustive computer search in the SmallGroups library, reduced by elementary lemmas, 1296 is the least order of a finite soluble group with a maximal subgroup M such that $d(G) - d(M) \geq 3$, and exactly three groups of order 1296 have one. We prove that $d(G) - d(M) \leq 1 + \min\{e, d(M_G)\}$, where M_G is the core of M and $e = d(M/M_G)$, and deduce that a difference 4 needs $|G| \geq 5184$; we also give a formula for split extensions which explains why the construction does not iterate in the simplest cases. In a second pair with difference 3, of order 41472, the group is a primitive permutation group of degree 64. The group G itself is a term of an extraspecial tower of Glasby and Howlett. Related results of Doerk, of Gol’chuk and Monakhov, and of Murashka and Vasil’ev concern other lengths or special cases and do not answer the question. The contribution is modest. This is an unrefereed note.

1 Introduction

1.1 The problem

All groups are finite. For a soluble group X put $X^{(0)} = X$ and $X^{(i+1)} = [X^{(i)}, X^{(i)}]$; the *derived length* $d(X)$ is the least $n \geq 0$ with $X^{(n)} = 1$. Thus $d(1) = 0$, and $d(X) = 1$ if and only if X is abelian and non-trivial. If $M \leq G$, then $d(M) \leq d(G)$.

Problem 1.1 (Monakhov [KM26, Problem 17.91], [MT10, Question 9]). (a) *Is there an absolute constant k such that $d(G) - d(M) \leq k$ for every soluble group G and every maximal subgroup M of G ?* (b) *Find the least such k .*

The problem appeared in the 17th issue (2010) of the Notebook; in version 49 (20 October 2026) it has no comment and no mark of a solution [KM26], and it is record KOU-17.91, listed as open, of the UnsolvedMath collection [ula26]. It is also Question 9 of the survey [MT10, p. 78], which asks for the least natural number k “if it exists”. The survey poses it after the theorem of Doerk [Doe94, Satz 1], [MT10, Theorem 4.6] (if U is a maximal subgroup of a soluble group G , then the nilpotent length n

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satisfies $n(U) = n(G) - i$ with $i \in \{0, 1, 2\}$, and $i = 2$ occurs for a Sylow 2-subgroup of S_4 [MT10, Example 15]) and its analogues for the $\mathfrak{F}$ -length and the π -lengths [MT10, Theorems 4.7–4.9]; it gives no result and no example on $d(G) - d(M)$. (The $\mathfrak{F}$ -length analogue, Theorem 4.7 of the survey, does not hold for every hereditary saturated formation $\mathfrak{F}$, by a theorem of Murashka and Vasil'ev [MV25, Theorem 4].)

We write $\text{gap}(G, M) = d(G) - d(M)$, and $M_G = \bigcap_{g \in G} M^g$ for the core of M . Small values are easy to find: $\text{gap}(S_3, C_2) = 1$, and the maximal subgroup C_6 of $\text{SL}(2, 3)$ gives $\text{gap} = 3 - 1 = 2$.

1.2 Results

Let $E = 3_+^{1+2}$ be the extraspecial group of order 27 and exponent 3. In Section 3 we realise E on $\mathbb{F}_3^2 \times \mathbb{F}_3$ and let $\text{GL}(2, 3)$ act on it, naturally on $E/Z(E) = \mathbb{F}_3^2$ and by the determinant on $Z(E)$; let B be the group of upper triangular matrices in $\text{GL}(2, 3)$ (the Borel subgroup, of order 12).

Theorem 1.2. *The group $G = E \rtimes \text{GL}(2, 3)$ has order 1296 and derived length 6, and $M = E \rtimes B$ is a maximal subgroup of index 4 and derived length 3. Hence $\text{gap}(G, M) = 3$, and if a constant k as in Problem 1.1(a) exists, then $k \geq 3$. The core of M is $E \rtimes \langle -I \rangle$, of order 54 and derived length 3, and $G/M_G \cong S_4$. In the SmallGroups library, $G \cong \text{SmallGroup}(1296, 2891)$.*

Theorem 1.3 (computer-assisted). *Let G be a soluble group with a maximal subgroup M such that $\text{gap}(G, M) \geq 3$. Then $|G| \geq 1296$. The soluble groups of order 1296 with such a maximal subgroup are exactly $\text{SmallGroup}(1296, i)$ for $i \in \{2889, 2890, 2891\}$. Each of them has derived length 6, a normal subgroup $O_3(G) \cong 3_+^{1+2}$ with $G/O_3(G) \cong \text{GL}(2, 3)$, and exactly one conjugacy class of maximal subgroups M with $\text{gap}(G, M) \geq 3$; these have index 4 and $\text{gap}(G, M) = 3$. The extension of $O_3(G)$ by $\text{GL}(2, 3)$ is split for $i = 2891$ (this is the group of Theorem 1.2) and non-split for $i = 2889, 2890$.*

The proof (Section 4) rests on the completeness of the SmallGroups library [BEO02, BEO26] and on three functions of GAP [GAP26].

Theorem 1.4. *Let G be a soluble group, M a maximal subgroup, $N = M_G$ and $e = d(M/N)$.*

- (a) $\text{gap}(G, M) = d(G^{(d(M))})$.
- (b) $d(G/N) = e + 1$; in particular $G^{(e+1)} \leq N$.
- (c) $\text{gap}(G, M) \leq 1 + \min\{e, d(N)\}$.
- (d) *If $\text{gap}(G, M) \geq 3$, then $e \geq 2$, $d(N) \geq 2$, $d(G) \geq 5$ and $|G| \geq 144$. If $\text{gap}(G, M) \geq 4$, then $e \geq 3$, $d(N) \geq 3$, $d(G) \geq 7$ and $|G| \geq 5184$.*

The pair of Theorem 1.2 attains (c), with $e = 2$ and $d(N) = 3$. Section 5 explains how it arises, by adding a layer at the bottom of a pair with difference 2 (Lemmas 5.1 and 5.3); the same mechanism gives a second pair with difference 3, of order 41472, in which the bottom layer is abelian (Proposition 5.4). Every extension of the pair of Theorem 1.2 by a normal subgroup inside the core has difference at most 3 (Remark 5.2), and in the construction of Lemma 5.3 a second abelian layer on which the first, irreducible, layer has no non-zero fixed points cannot raise the difference again; the two layers then have different characteristics (Proposition 5.6). Computations in several families found no difference 4 (Remark 5.7); this is not a theorem.

1.3 What is known, what is new, and what stays open

We found no printed answer to Problem 1.1, and no example with difference 3, in the sources searched (see “Scope and priority” at the end). The closest results concern other lengths or special cases. Gol’chuk and Monakhov [GM24, Theorem 1(a)] proved that a soluble group G with a maximal subgroup that is a Miller–Moreno group (a non-abelian group all of whose proper subgroups are abelian) satisfies $|\pi(G)| \leq 3$ and $d(G) \leq 4$, and $d(G) \leq 3$ with a Sylow tower if $|\pi(G)| = 3$ (we read the statement in the review of zbMATH Open, Zbl 1544.20031); as $d(M) = 2$ here, the difference is at most 2. Murashka and Vasil’ev [MV25] proved that $h^*(G) - h^*(M) \leq 2$ for the generalised Fitting height and $\lambda_p(G) - \lambda_p(M) \leq 1$ for the non- p -soluble length, for all finite G and maximal M (Theorem 1), and that for some hereditary saturated formation $\mathfrak{F}$ the nilpotent length of the $\mathfrak{F}$ -residual can drop from G to M by any given number (Theorem 4); in their construction M is normal in G , so the derived length drops by at most 1 (Corollary 2.4(i)). Glasby and Howlett [GH92, Section 1] construct *extraspecial towers* (groups with a series of normal subgroups whose factors are extraspecial), beginning with $\mathrm{GL}(2, 3) = S_3 \rtimes Q_8$ and $(S_3 \rtimes Q_8) \rtimes 3_+^{1+2}$ (the normal subgroup written on the right); the latter is isomorphic to the group G of Theorem 1.2 (a subgroup of $\mathrm{Aut}(3_+^{1+2})$ isomorphic to $\mathrm{GL}(2, 3)$ meets the inner automorphisms trivially, as $O_3(\mathrm{GL}(2, 3)) = 1$, and all such complements are conjugate, by a computation in Section 6), and the construction of Lemma 5.3 is one step of such a tower. They do not consider the derived lengths of maximal subgroups. None of these results answers Problem 1.1. Lemmas 2.1 to 2.3 and 5.5 are standard, and the group G is not new. To our knowledge the pairs with difference 3 of Theorem 1.2 and Proposition 5.4, Theorem 1.3, Theorem 1.4(c),(d), Lemma 5.3 and Proposition 5.6 are not in print; the bound 3 may nevertheless be known, and we claim no priority. Both parts of Problem 1.1 remain open; in particular we do not know whether a difference 4 occurs.

2 Primitive quotients and general restrictions

We write $[x, y] = x^{-1}y^{-1}xy$ and $[A, B] = \langle [a, b] : a \in A, b \in B \rangle$ for subgroups A, B ; recall that $[A, B]$ is normalised by A and by B . An elementary abelian normal p -subgroup V on which a group H acts by conjugation is written additively and regarded as an $\mathbb{F}_p H$ -module; then $[V, H]$ is the submodule spanned by the vectors $v - v^h$.

Lemma 2.1. *Let A and B be subgroups of a group, with A normalised by B . Then $(AB)' = A'[A, B]B'$.*

Proof. A' and $[A, B]$ are normal in AB , and B' normalises their product, so the right-hand side is a subgroup of $(AB)'$. Modulo $A'[A, B]$ the image of A is abelian and commutes with the image of B , hence is central, and the commutator of two elements of AB is congruent to the commutator of their components in B . $\square$

The next lemma collects the standard facts on primitive soluble groups that we use [DH92, Chapter A, §15], [Hup67, Kapitel II, §3]; we include the short proof, so that no numbered statement of these books is needed.

Lemma 2.2. *Let $P \neq 1$ be a soluble group with a maximal subgroup H such that $H_P = 1$, and let V be a minimal normal subgroup of P . Then V is an elementary abelian p -group for a prime p , $P = V \rtimes H$, $C_P(V) = V$, V is the only minimal normal subgroup of P , H acts faithfully and irreducibly on V , and $O_p(H) = 1$.*

Proof. As P is soluble, V is elementary abelian. Since $H_P = 1$, $V \not\leq H$, so $P = VH$. The group $V \cap H$ is normalised by H and by the abelian group V , hence normal in P and contained in H , so

$V \cap H = 1$. By Dedekind's law $C_P(V) = V(C_P(V) \cap H)$, and $C_P(V) \cap H$ is normalised by H and centralised by V , hence normal in P and contained in H , hence trivial. So $C_P(V) = V$ and $H \cong P/V$ acts faithfully on V . An H -invariant subgroup of V is normal in $VH = P$; so V is irreducible. A second minimal normal subgroup V_2 would satisfy $V \cap V_2 = 1$, hence $V_2 \leq C_P(V) = V$, which is impossible. Finally $C_V(O_p(H))$ is non-zero, as $O_p(H)$ is a p -group acting on a non-trivial p -group, and H -invariant, as $O_p(H) \trianglelefteq H$; so it is V , and $O_p(H) = 1$ by faithfulness. $\square$

Lemma 2.3. *In Lemma 2.2 let $e = d(H)$. Then $P^{(i)} = VH^{(i)}$ for $0 \leq i \leq e$. In particular $P^{(e)} = V$ and $d(P) = e + 1$.*

Proof. Let $P^{(i)} = VH^{(i)}$ with $i < e$. By Lemma 2.1, $P^{(i+1)} = V'[V, H^{(i)}](H^{(i)})' = [V, H^{(i)}]H^{(i+1)}$. As $H^{(i)} \neq 1$ is normal in H , $[V, H^{(i)}]$ is an H -invariant subgroup of V , and it is non-trivial because $H^{(i)}$ acts faithfully; so it is V . Hence $P^{(e)} = VH^{(e)} = V \neq 1$ and $P^{(e+1)} = V' = 1$. $\square$

Now let G be soluble, M a maximal subgroup, $N = M_G$, $P = G/N$ and $H = M/N$. Then H is a maximal subgroup of P with trivial core, so Lemmas 2.2 and 2.3 apply; put $e = d(H) = d(M/N)$.

Proof of Theorem 1.4(a)–(c). (a) For $0 \leq t \leq d(X)$ we have $(X^{(t)})^{(j)} = X^{(t+j)}$, hence $d(X^{(t)}) = d(X) - t$; apply this with $X = G$ and $t = d(M) \leq d(G)$. (b) $d(G/N) = d(P) = e + 1$ by Lemma 2.3, and $G^{(e+1)}N/N = P^{(e+1)} = 1$. (c) By (b), $G^{(e+1+d(N))} \leq N^{(d(N))} = 1$, so $d(G) \leq e + 1 + d(N)$. On the other hand $d(M) \geq d(N)$, as $N \leq M$, and $d(M) \geq e$, as $M/N = H$. Hence $d(G) - d(M) \leq e + 1 + d(N) - \max\{e, d(N)\} = 1 + \min\{e, d(N)\}$. $\square$

Corollary 2.4. *Let G, M, N, e be as above.*

- (i) *If $M \trianglelefteq G$, then $\text{gap}(G, M) \leq 1$. In particular $\text{gap}(G, M) \leq 1$ if G is nilpotent, for instance a p -group.*
- (ii) *If $|G : M|$ is a prime, then $e \leq 1$ and $\text{gap}(G, M) \leq 2$.*
- (iii) *If $\text{gap}(G, M) \geq 3$, then $e \geq 2$, $d(N) \geq 2$, $d(M) \geq 2$ and $d(G) \geq 5$. If $\text{gap}(G, M) \geq 4$, then $e \geq 3$, $d(N) \geq 3$, $d(M) \geq 3$ and $d(G) \geq 7$.*
- (iv) *If M is a Miller–Moreno group, then $\text{gap}(G, M) \leq 2$ and $d(G) \leq 4$, the bound for $d(G)$ in [GM24, Theorem 1(a)].*

Proof. (i) G/M has prime order, so $G' \leq M$ and $d(G) = 1 + d(G') \leq 1 + d(M)$; in a nilpotent group all maximal subgroups are normal. (ii) By Lemma 2.2, $|V| = |P : H| = |G : M|$ is a prime, so H embeds in the abelian group $\text{Aut}(V)$ and $e \leq 1$; apply Theorem 1.4(c). (iii) By Theorem 1.4(c), $\min\{e, d(N)\} \geq \text{gap}(G, M) - 1$; moreover $d(M) \geq \max\{e, d(N)\}$ and $d(G) = \text{gap}(G, M) + d(M)$. (iv) The derived subgroup M' is a proper subgroup of the soluble group M , hence abelian, so $d(M) = 2$. If $N = M$, then (i) gives $d(G) \leq 3$. Otherwise N is a proper subgroup of M , hence abelian, and Theorem 1.4(c) gives $\text{gap}(G, M) \leq 2$, so $d(G) \leq 4$. $\square$

Lemma 2.5. *A soluble group X with $d(X) \geq 3$ has order at least 24.*

Proof. Suppose $|X| < 24$. As X' is non-abelian and proper, $6 \leq |X'| \leq 11$, so X' is one of S_3, D_8, Q_8, D_{10} . If $X' \in \{D_8, Q_8\}$, then $|X| = 16$ and $|X : X'| = 2$; since $X' \leq \Phi(X)$ in a 2-group, $X/\Phi(X)$ is cyclic, so X is cyclic, a contradiction. If $X' \in \{S_3, D_{10}\}$, the normal Sylow subgroup R of X' of order 3 or 5 is characteristic in X' , hence normal in X ; as $X/C_X(R)$ embeds in the abelian group $\text{Aut}(R)$, X' centralises R , which is false in S_3 and in D_{10} . $\square$

Lemma 2.6. *Let $P = V \rtimes H$ be as in Lemma 2.2, with $e = d(H)$. If $e \geq 2$, then $|P| \geq 24$. If $e \geq 3$, then $|P| \geq 216$.*

Proof. If $e \geq 2$, then H is non-abelian, so $|H| \geq 6$, and $|V|$ is not a prime (otherwise H embeds in the abelian group $\text{Aut}(V)$); so $|V| \geq 4$ and $|P| \geq 24$. Let $e \geq 3$. Then $|H| \geq 24$ by Lemma 2.5, and $|V| = p^n$ with $n \geq 2$. If $|V| = 4$, then $|H| \leq |\text{GL}(2, 2)| = 6$, which is impossible; if $|V| = 9$ or $|V| \geq 16$, then $|P| \geq 216$. Let $|V| = 8$. Then $H \leq \text{GL}(3, 2)$, of order 168, and $O_2(H) = 1$, so the Fitting subgroup F of H has order dividing 21, and $C_H(F) \leq F$. If $7 \nmid |F|$, then $|F| = 3$ and $|H| \leq |F| |\text{Aut}(F)| = 6$. Otherwise H has a normal subgroup $Q = \langle q \rangle$ of order 7. Then q leaves no subspace U with $0 \neq U \neq V$ invariant (it would act trivially on U and V/U , through groups of order 1 or 6, and have 2-power order), so by Schur's lemma its centraliser K in $\text{End}(V)$ is a division ring; the field $\mathbb{F}_2[q] \subseteq K$ has at least 8 elements, as q has order 7, and $|K| \leq |V| = 8$. So $K = \mathbb{F}_2[q] \cong \mathbb{F}_8$, $C_{\text{GL}(V)}(Q) = K^\times$ has order 7, and $N_{\text{GL}(V)}(Q)/C_{\text{GL}(V)}(Q)$ embeds in $\text{Gal}(\mathbb{F}_8/\mathbb{F}_2)$, of order 3. Hence $|H| \leq 21$, a contradiction. $\square$

Proof of Theorem 1.4(d). The statements on e , $d(N)$ and $d(G)$ are Corollary 2.4(iii). If $\text{gap}(G, M) \geq 3$, then $|G/N| \geq 24$ by Lemma 2.6 and $|N| \geq 6$, as N is non-abelian. If $\text{gap}(G, M) \geq 4$, then $|G/N| \geq 216$ by Lemma 2.6 and $|N| \geq 24$ by Lemma 2.5. $\square$

The bounds of Lemma 2.6 are attained by $S_4 = \mathbb{F}_2^2 \rtimes S_3$ and $\text{ASL}(2, 3) = \mathbb{F}_3^2 \rtimes \text{SL}(2, 3)$. The bounds 144 and 5184 of Theorem 1.4(d) are only lower bounds: by Theorem 1.3 the least order for a difference 3 is 1296.

3 A pair of order 1296

Let $\omega(v, w) = v_1 w_2 - v_2 w_1$ for $v, w \in \mathbb{F}_3^2$. On $E = \mathbb{F}_3^2 \times \mathbb{F}_3$ define

$$(v, z)(w, y) = (v + w, z + y + \tfrac{1}{2}\omega(v, w)), \quad \tfrac{1}{2} = 2 \in \mathbb{F}_3.$$

Bilinearity of ω gives associativity, $(0, 0)$ is the identity and $(-v, -z)$ the inverse of (v, z) . As ω is alternating, the commutator of (v, z) and (w, y) is $(0, \omega(v, w))$, and $(v, z)^3 = (0, 0)$. Hence $E' = Z(E) = \{(0, z)\}$ has order 3 (ω is non-degenerate), $E/Z(E) \cong \mathbb{F}_3^2$, E has exponent 3, so $E \cong 3_+^{1+2}$, and $Z(E) = \Phi(E)$. For $g \in \text{GL}(2, 3)$ the map $\varphi_g(v, z) = (gv, \det(g)z)$ is an automorphism of E , because $\omega(gv, gw) = \det(g)\omega(v, w)$, and $g \mapsto \varphi_g$ is injective. Let $G = E \rtimes \text{GL}(2, 3)$ for this action; we identify $\text{GL}(2, 3)$ with a complement of E in G . Let B be the group of upper triangular matrices (of order 12), $M = E \rtimes B$, $U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\} \cong C_3$, and $Q = \text{SL}(2, 3)'$. Then $|G| = 1296$ and $|G : M| = 4$.

We use the following well-known facts: $\text{GL}(2, 3)' = \text{SL}(2, 3)$, $\text{SL}(2, 3)' = Q \cong Q_8$ and $Q' = \langle -I \rangle$; and $B' = U$, $B'' = 1$, as $B = \langle -I \rangle \times \langle \text{diag}(1, -1), U \rangle$ with $\langle \text{diag}(1, -1), U \rangle \cong S_3$. (The quotients by $\text{SL}(2, 3)$ and Q are abelian; an element r of order 3 acts without fixed points on $Q/\langle -I \rangle = Q/\Phi(Q)$, so $[Q, r] = Q$; and $[\text{diag}(-1, 1), t] = t^{-1}$ for $t \in U$.)

Lemma 3.1. *If $L \leq \text{GL}(2, 3)$ contains $-I$, then $[E, L] = E$ and $(EL)' = EL'$.*

Proof. As $\varphi_{-I}(v, z) = (-v, z)$, we get $(v, z)^{-1}\varphi_{-I}(v, z) = (-v, -z)(-v, z) = (v, 0)$; the elements $(v, 0)$ generate E . So $[E, L] = E$, and $(EL)' = E'[E, L]L' = EL'$ by Lemma 2.1. $\square$

Proof of Theorem 1.2. Derived series of G . Since $\text{GL}(2, 3)$, $\text{SL}(2, 3)$, Q and $\langle -I \rangle$ contain $-I$, Lemma 3.1 gives

$$G' = E \text{SL}(2, 3), \quad G'' = EQ, \quad G''' = E\langle -I \rangle, \quad G^{(4)} = E, \quad G^{(5)} = E' = Z(E), \quad G^{(6)} = 1,$$

of orders 648, 216, 54, 27, 3, 1. So $d(G) = 6$.

Derived series of M . As $-I \in B$, Lemma 3.1 gives $M' = EB' = EU$, of order 81. By Lemma 2.1, $M'' = E'[E, U]U' = Z(E)[E, U]$. For $u = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ we have $uv - v = (bv_2, 0)$, so $(v, z)^{-1}\varphi_u(v, z)$ lies in

$L = \{(ae_1, z) : a, z \in \mathbb{F}_3\}$, and for $v = e_2$, $b = 1$ its first component is e_1 . Since ω vanishes on $\mathbb{F}_3e_1$, L is an abelian subgroup of order 9, and $M'' = Z(E)[E, U] = L$. Hence $M''' = 1$ and $d(M) = 3$.

Maximality. A subgroup K with $M < K < G$ would have index 2, so it would be normal and contain G' . But $MG' = E B \text{SL}(2, 3) = G$, since B contains $\text{diag}(-1, 1)$, of determinant -1 .

Core. The core of M contains the normal subgroup E of G , and modulo E it is the core of B in $\text{GL}(2, 3)$. Now B is the stabiliser of the line $\mathbb{F}_3e_1$ in the transitive action of $\text{GL}(2, 3)$ on the four lines of $\mathbb{F}_3^2$, so its core is the kernel $\{\pm I\}$ of this action. Hence $M_G = E\langle -I \rangle$, of order 54, with derived series $E\langle -I \rangle > E > Z(E) > 1$ by Lemma 3.1; and $G/M_G \cong \text{PGL}(2, 3) \cong S_4$, $M/M_G \cong B/\langle -I \rangle \cong S_3$.

The identification with $\text{SmallGroup}(1296, 2891)$, and $M \cong \text{SmallGroup}(324, 39)$, is a computation (Section 6). $\square$

Thus $e = 2$ and $d(M_G) = 3$, and the bound of Theorem 1.4(c) is attained. The program of Section 6 rebuilds G and M from these definitions and checks every subgroup displayed above.

4 The least order

Proof of Theorem 1.3. Let G be soluble with a maximal subgroup M such that $\text{gap}(G, M) \geq 3$. By Theorem 1.4(d), $|G| \geq 144$ and $d(G) \geq 5$, and by Corollary 2.4(i) $|G|$ is not a prime power. The SmallGroups library contains, up to isomorphism, all groups of order at most 2000 except those of order 1024 [BEO02, BEO26], and 1024 is a prime power. So it suffices to examine, for every order n with $144 \leq n \leq 1296$ that is not a prime power, every group $\text{SmallGroup}(n, i)$ of the library: to skip the non-soluble ones, to compute the derived length, and, when it is at least 5, to compute representatives of the conjugacy classes of maximal subgroups and their derived lengths (conjugate subgroups have the same derived length). This was done in GAP 4.16.1 with SmallGrp 1.7.0 and the functions `IsSolvableGroup`, `DerivedLength` and `MaximalSubgroupClassReps`. The result is Table 1: among the groups examined, exactly six of order below 1296 have derived length at least 5, and for each of them the difference is at most 2; of the nine groups of order 1296 with derived length at least 5, exactly the three groups $\text{SmallGroup}(1296, i)$, $i = 2889, 2890, 2891$, have a maximal subgroup with difference 3, and none has one with difference 4. For these three groups the program also found $O_3(G) \cong 3_+^{1+2}$ with quotient $\text{GL}(2, 3)$, three classes of complements of $O_3(G)$ for $i = 2891$ and none for $i = 2889, 2890$, and one class of maximal subgroups with difference 3, of index 4, containing $O_3(G)$, with core of order 54 and quotient S_4 . Finally $\text{SmallGroup}(1296, 2891)$ is the group of Theorem 1.2. $\square$

Dependence on the computation. Theorem 1.3 is computer-assisted: it depends on the completeness and correctness of the SmallGroups library for the orders searched and on the three GAP functions named above, applied to polycyclic presentations. The search examined, among others, all 1,090,235 groups of order 768, 1,116,461 of order 1280, 157,877 of order 1152 and 3609 of order 1296 (none of order 768, 1152 or 1280 has derived length 5 or more), in about 1.4 hours of CPU time. It was repeated by the three independent verification runs of Section 6: the first re-ran these programs (except the one for order 1296), and the second and the third ran their own programs on all 2,496,072 library groups of non-prime-power order at most 1296 and recomputed the maximal subgroups of the groups of Table 1 by other algorithms; and by the program of this note for all orders except 768, 1152 and 1280. All results agree.

5 Split layers, and a second pair

Lemma 5.1. *Let X be a soluble group with a maximal subgroup Y , $D = d(X)$, $s = d(Y)$, and let X act on a soluble group F . Put $\widehat{G} = F \rtimes X$ and $\widehat{M} = F \rtimes Y$, and for $L \in \{X, Y\}$ define $F_0^L = F$ and*

SmallGroup(n, i)	$d(G)$	$F(G)$	$G/F(G)$	split	maximal subgroups ($ G : M , d(M)$)	gap
(432, 734)	5	3^2	GL(2, 3)	yes	(2, 4), (3, 3), (4, 3), (9, 4)	2
(648, 531), (648, 532)	5	3_+^{1+2}	SL(2, 3)	no	(3, 4), (4, 3), (9, 3)	2
(648, 533)	5	3_+^{1+2}	SL(2, 3)	yes	(3, 4), (4, 3), (9, 3)	2
(864, 4053)	5	$C_6 \times C_3$	GL(2, 3)	no	(2, 4), (3, 3), (4, 3), (9, 4)	2
(864, 4661)	5	$C_6 \times C_3$	GL(2, 3)	yes	(2, 4), (2, 5) ² , (3, 3), (4, 3), (9, 4)	2
(1296, 2889), (1296, 2890)	6	3_+^{1+2}	GL(2, 3)	no	(2, 5), (3, 4), (4, 3), (9, 4)	3
(1296, 2891)	6	3_+^{1+2}	GL(2, 3)	yes	(2, 5), (3, 4), (4, 3), (9, 4)	3
(1296, 2892)	5	3^3	GL(2, 3)	no	(2, 4), (3, 3), (4, 3), (9, 4)	2
(1296, 2893), (1296, 2894)	5	$C_2 \times 3_+^{1+2}$	SL(2, 3)	no	(2, 5), (3, 4), (4, 3), (9, 3)	2
(1296, 2895)	5	$C_2 \times 3_+^{1+2}$	SL(2, 3)	yes	(2, 5), (3, 4), (4, 3), (9, 3)	2
(1296, 3487)	5	3^3	GL(2, 3)	yes	(2, 4), (3, 3), (3, 5), (4, 3), (9, 4)	2
(1296, 3488)	5	3^3	GL(2, 3)	yes	(2, 4), (3, 3), (3, 5) ³ , (4, 3), (9, 4)	2

Table 1: The soluble groups of non-prime-power order n , $144 \leq n \leq 1296$, with derived length at least 5 (the third verification run searched all non-prime-power orders $n \leq 1296$ and found no other). p^m denotes an elementary abelian group of order p^m . “Split” says whether the Fitting subgroup $F(G)$ has a complement; the maximal subgroups are listed by conjugacy classes, with an exponent for repeated pairs; “gap” is $d(G) - \min d(M)$. Here $\text{SmallGroup}(432, 734) \cong \text{AGL}(2, 3)$, $\text{SmallGroup}(864, 4661) \cong C_2 \times \text{AGL}(2, 3)$, and $\text{SmallGroup}(648, 533) \cong 3_+^{1+2} \rtimes \text{SL}(2, 3)$ is the derived subgroup of the group of Theorem 1.2.

$F_{j+1}^L = (F_j^L)' [F_j^L, L^{(j)}]$. Then $\widehat{M}$ is maximal in $\widehat{G}$, $(F \rtimes L)^{(j)} = F_j^L \rtimes L^{(j)}$ for all $j \geq 0$, and

$$d(\widehat{G}) = D + d(F_D^X), \quad d(\widehat{M}) = s + d(F_s^Y), \quad \text{gap}(\widehat{G}, \widehat{M}) = \text{gap}(X, Y) + d(F_D^X) - d(F_s^Y).$$

If $[F, X^{(D-1)}]F' = F$, then $F_D^X = F$ and $\text{gap}(\widehat{G}, \widehat{M}) = \text{gap}(X, Y) + d(F) - d(F_s^Y)$. This holds, for instance, if F is abelian and an element of $X^{(D-1)}$ fixes only the identity of F , or if F is an elementary abelian r -group and a subgroup $A \leq X^{(D-1)}$ of order prime to r fixes only the identity.

Proof. Suppose that F_j^L is normalised by $L^{(j)}$, as it is for $j = 0$. By Lemma 2.1,

$$(F_j^L L^{(j)})' = (F_j^L)' [F_j^L, L^{(j)}] (L^{(j)})' = F_{j+1}^L L^{(j+1)},$$

and $F_{j+1}^L \leq F$ is normalised by $L^{(j)} \geq L^{(j+1)}$; this proves the formula for $(F \rtimes L)^{(j)}$ by induction. As $X^{(j)} \neq 1$ for $j < D$ and $X^{(D)} = 1$, it gives $d(\widehat{G}) = D + d(F_D^X)$, and similarly for $\widehat{M}$. $\widehat{M}$ is maximal because it contains F and $\widehat{M}/F \cong Y$ is maximal in $\widehat{G}/F \cong X$. If $[F, X^{(D-1)}]F' = F$, then inductively $F_{j+1}^X = F'[F, X^{(j)}] \geq F'[F, X^{(D-1)}] = F$ for $j < D$. If F is abelian and $x \in X^{(D-1)}$ fixes only the identity, then $f \mapsto f^{-1}f^x$ is an injective endomorphism of the finite group F , so $[F, x] = F$. In the last case, Maschke’s theorem gives an A -invariant complement C of $[F, A]$ in F ; A acts trivially on $C \cong F/[F, A]$, so $C = 1$ and $[F, A] = F$. $\square$

Remark 5.2. If a normal subgroup F of a soluble group $\widehat{G}$ lies in a maximal subgroup $\widehat{M}$, then the core of $\widehat{M}$ is the preimage of the core of $\widehat{M}/F$ in $\widehat{G}/F$, so $(\widehat{G}, \widehat{M})$ and $(\widehat{G}/F, \widehat{M}/F)$ have the same e , and $\text{gap}(\widehat{G}, \widehat{M}) \leq 1 + e$ by Theorem 1.4(c). For the pair of Theorem 1.2, $e = 2$; so no group $\widehat{G}$ with $\widehat{G}/F \cong G$, split over F or not, has difference more than 3 for the maximal subgroup corresponding to M , and a difference 4 needs $e \geq 3$ at the top (Theorem 1.4(d)).

The two pairs with difference 3 come from the following construction.

Lemma 5.3. *Let E be an extraspecial p -group and S a soluble group acting on E , trivially on $Z(E)$, such that $V = E/Z(E)$ is a faithful irreducible $\mathbb{F}_p S$ -module. Let $t = d(S) \geq 1$, $X = E \rtimes S$ and $Y = Z(E) \times S$. Then Y is maximal in X , $X^{(j)} = ES^{(j)}$ for $0 \leq j \leq t$, $X^{(t+1)} = Z(E)$, $d(X) = t + 2$, $d(Y) = t$ and $\text{gap}(X, Y) = 2$. If moreover X acts on a soluble group F with $[F, Z(E)]F' = F$, then*

$$\text{gap}(F \rtimes X, F \rtimes Y) = 2 + d(F) - d(F_t),$$

where $F_1 = F$ and $F_{j+1} = F'_j [F_j, S^{(j)}]$ for $j \geq 1$.

Proof. If $Y \leq K \leq X$, then $K = (K \cap E)S$ by Dedekind's law, and $(K \cap E)/Z(E)$ is an S -submodule of V ; so $K \in \{Y, X\}$. For $1 \leq j \leq t$ the normal subgroup $S^{(j-1)} \neq 1$ of S acts non-trivially on V , so $[V, S^{(j-1)}] = V$; hence $[E, S^{(j-1)}]Z(E) = E$, and $[E, S^{(j-1)}] = E$ as $Z(E) = \Phi(E)$. By Lemma 2.1, $(ES^{(j-1)})' = E'[E, S^{(j-1)}]S^{(j)} = ES^{(j)}$. So $X^{(t)} = E$, $X^{(t+1)} = E' = Z(E)$, $X^{(t+2)} = 1$; and $d(Y) = d(S) = t$. For the last statement apply Lemma 5.1 with $D = t + 2$: $X^{(D-1)} = Z(E)$, so $F_D^X = F$; and $F_1^Y = F'[F, Y] \geq F'[F, Z(E)] = F$, while $Y^{(j)} = S^{(j)}$ for $j \geq 1$, so $F_j^Y = F_j$ for $j \geq 1$. $\square$

For a non-trivial abelian layer F the formula reads $\text{gap} = 3$ if $F_t = 1$ and $\text{gap} = 2$ otherwise; a difference 4 would need a non-abelian F whose derived length drops by 2 along $F_1, \dots, F_t$.

The pair of Theorem 1.2. Take $p = 2$, $E = Q \cong Q_8$ and $S = \langle \text{diag}(1, -1), U \rangle \cong S_3$, so that $X = Q \rtimes S = \text{GL}(2, 3)$, $Y = \langle -I \rangle \times S = B$ and $t = 2$; here S centralises $Z(Q) = \langle -I \rangle$ and acts on $Q/Z(Q) \cong \mathbb{F}_2^2$ as $\text{GL}(2, 2) \cong S_3$. Let $F = 3_+^{1+2}$ with the action of Section 3. By Lemma 3.1, $[F, Z(Q)] = F$; and $F_2 = F'[F, S'] = Z(F)[F, U]$ is abelian (Section 3). So the difference is $2+2-1 = 3$. The group $F \rtimes X$, written $(S_3 \rtimes Q_8) \rtimes 3_+^{1+2}$ in [GH92, Section 1], is a term of the extraspecial tower of Glasby and Howlett.

A second pair. Take $p = 3$, $E = 3_+^{1+2}$ and $S = \text{SL}(2, 3)$, acting as in Section 3; then $t = 3$, $X = E \rtimes \text{SL}(2, 3)$ is the group G' of Theorem 1.2, of order 648, and $Y = Z(E) \times \text{SL}(2, 3)$ has index 9.

Proposition 5.4. *Up to isomorphism there are exactly three irreducible $\mathbb{F}_2 X$ -modules W of dimension 6 on which $Z(E)$ has no non-zero fixed vector. For each of them let $G_2 = W \rtimes X$ and $M_2 = W \rtimes Y$. Then $|G_2| = 41472$, M_2 is a maximal subgroup of index 9, $d(G_2) = 6$ and $d(M_2) = 3$. The core of M_2 is $W \rtimes Z(E)$, of order 192 and derived length 2, and $G_2/(M_2)_{G_2} \cong \text{ASL}(2, 3)$, so $e = 3 = d(M_2)$. Acting by affine maps on the 64 vectors of W , G_2 is permutation isomorphic to the primitive group $\text{PrimitiveGroup}(64, 37)$ of [HLRDR26], whose maximal subgroups of index 9 form one conjugacy class.*

Proof. The number of modules, the facts $\dim[W, Q] = 4$ and $[[W, Q], -I] = 0$, and the identification with $\text{PrimitiveGroup}(64, 37)$ and its maximal subgroups are computations (Section 6). Granted them, Lemma 5.3 applies to the abelian layer $F = W$, since $[W, Z(E)] = W$ by Lemma 5.1 (a generator of $Z(E)$ fixes only 0), and gives $F_1 = W$, $F_2 = [W, S'] = [W, Q]$ and $F_3 = [F_2, Q'] = [[W, Q], \langle -I \rangle] = 0$, so the difference is 3, and Lemma 5.1 gives $d(G_2) = 5 + 1$, $d(M_2) = 3 + 0$ and the maximality. A normal subgroup K of X contained in Y satisfies $[E, K] \leq Z(E)$, so it acts trivially on V , and by faithfulness $K \leq Z(E)$; hence the core of Y in X is $Z(E)$ and the core of M_2 is $W \rtimes Z(E)$, which is non-abelian because $[W, Z(E)] = W$. Finally $X/Z(E) \cong \mathbb{F}_3^2 \rtimes \text{SL}(2, 3) = \text{ASL}(2, 3)$. $\square$

Here the bottom layer is abelian and $d(M_2) = e > d((M_2)_{G_2}) = 2$, whereas in the pair of Theorem 1.2 it is non-abelian and $d(M) > e$; both attain the bound of Theorem 1.4(c). The next two results show that, in the setting of Lemma 5.3, the construction does not iterate by adding a second abelian layer on which the first one has no non-zero fixed points.

Lemma 5.5. *Let a group L act faithfully on a finite-dimensional vector space W over a field of characteristic $r > 0$, let $L = L_1 \geq L_2 \geq \dots$ be subgroups, $W_1 = W$ and $W_{j+1} = [W_j, L_j]$. If $W_k = 0$ for some $k \geq 2$, then L_{k-1} is an r -group of derived length at most $\lceil \log_2(k-1) \rceil$.*

Proof. By induction W_j is L_{j-1} -invariant, hence L_j -invariant, and L_j acts trivially on $W_j/[W_j, L_j]$. So $L_{k-1} \leq L_j$ ($j \leq k-1$) acts trivially on all factors of the flag $W = W_1 \geq \dots \geq W_k = 0$. The linear maps with this property form a group T ; let T_a be the set of $g \in T$ with $(g-1)W_j \leq W_{j+a}$ for all j (where $W_i = 0$ for $i \geq k$). Since $[g, h] - 1 = g^{-1}h^{-1}((g-1)(h-1) - (h-1)(g-1))$, we have $[T_a, T_b] \leq T_{a+b}$, hence $T^{(i)} \leq T_{2^i}$, which is trivial once $2^i \geq k-1$. Every $g \in T$ satisfies $(g-1)^{k-1} = 0$, so its order is a power of r . By faithfulness L_{k-1} embeds in T . $\square$

Proposition 5.6. *In Lemma 5.3 let W_1 be an irreducible $\mathbb{F}_{r_1}X$ -module on which $Z(E)$ fixes only 0, $X_1 = W_1 \rtimes X$ and $Y_1 = W_1 \rtimes Y$; let W_2 be a non-zero finite-dimensional $\mathbb{F}_{r_2}X_1$ -module on which W_1 fixes only 0, $X_2 = W_2 \rtimes X_1$ and $Y_2 = W_2 \rtimes Y_1$. Then $r_1 \neq r_2$ and $\text{gap}(X_2, Y_2) \leq 3$.*

Proof. An r -group acting on a non-zero vector space in characteristic r fixes a non-zero vector; so $r_1 \neq p$ and $r_2 \neq r_1$. Let $W_{1,j}$ and $W_{2,j}$ be the series of Lemma 5.1 for the layers W_1 of (X, Y) and W_2 of (X_1, Y_1) ; as the layers are abelian, $W_{1,j+1} = [W_{1,j}, Y^{(j)}]$ and $W_{2,j+1} = [W_{2,j}, Y_1^{(j)}]$. By Lemma 5.3, $\text{gap}(X_1, Y_1) = 3 - d(W_{1,t}) \leq 3$, $d(X_1) = t + 3$ and $X_1^{(t+2)} = W_1$, and $d(Y_1) = t + d(W_{1,t})$. As the r_1 -group W_1 fixes only 0 on W_2 , Lemma 5.1 applies to the layer W_2 with $D = t + 3$ and $A = W_1$, and gives $\text{gap}(X_2, Y_2) = \text{gap}(X_1, Y_1) + 1 - d(W_{2,d(Y_1)})$. So a difference 4 forces $W_{1,t} = 0$, hence $d(Y_1) = t$, and $W_{2,t} = 0$; as $W_{1,1} = [W_1, Y] \geq [W_1, Z(E)] = W_1 \neq 0$, also $t \geq 2$.

X acts faithfully on W_1 : the kernel K is normal in X , and $K \cap E = 1$, since a non-trivial normal subgroup of E contains $Z(E)$, which acts non-trivially; so $[K, E] = 1$, and $K \leq C_X(E) = Z(E)$ (an element es with $e \in E$, $s \in S$ centralising E acts on V as s and trivially, so $s = 1$), whence $K = 1$. X_1 acts faithfully on W_2 : the kernel K_1 meets W_1 in an X -submodule different from W_1 , hence in 0; so $[K_1, W_1] = 1$ and $K_1 \leq C_{X_1}(W_1) = W_1$, whence $K_1 = 1$.

Apply Lemma 5.5 to Y acting on W_1 with $L_j = Y^{(j-1)}$: the spaces of the lemma are $W_{1,j-1}$, so $W_{1,t} = 0$ shows that $Y^{(t-1)} = S^{(t-1)}$ is an r_1 -group. Apply it to Y_1 acting on W_2 with $L_j = Y_1^{(j-1)}$: $W_{2,t} = 0$ shows that $Y_1^{(t-1)}$ is an r_2 -group; it contains $S^{(t-1)}$, by Lemma 5.1. So the non-trivial group $S^{(t-1)}$ is both an r_1 -group and an r_2 -group, a contradiction. $\square$

Remark 5.2 and Proposition 5.6 say nothing about non-split extensions with $e \geq 3$, non-abelian layers, layers in the characteristic of the previous one, or groups built otherwise; they give no bound for k .

Remark 5.7 (computation, not a theorem). In Lemma 5.3 a difference 4 needs a layer F with $d(F_t) = d(F) - 2$, for instance an extraspecial layer with $F/Z(F) \cong W$. Then $F_t = 1$ requires $W_t = 0$, where $W_1 = W$ and $W_{j+1} = [W_j, S^{(j)}]$, and requires W_{t-1} to be totally isotropic for the commutator form of F , an invariant alternating form on W . The programs of the package computed these subspaces and the invariant alternating forms for the irreducible modules W of small dimension on which $Z(E)$ fixes only 0, for $(E, S) = (3_+^{1+2}, \text{SL}(2, 3))$ over $\mathbb{F}_2, \mathbb{F}_5, \mathbb{F}_7$; $(5^{1+2}, \text{SL}(2, 3))$ over $\mathbb{F}_2, \mathbb{F}_3$; $(7^{1+2}, 2.S_4)$ over $\mathbb{F}_2, \mathbb{F}_3$; $(2_-^{1+6}, 3_+^{1+2} \rtimes C_2)$ and $(2_-^{1+6}, 3_+^{1+2} \rtimes \text{SL}(2, 3))$ over $\mathbb{F}_3$; and $(3^{1+4}, Q_8 \wr C_2)$ over $\mathbb{F}_2$. No module satisfies both conditions. This is not evidence that $k = 3$.

6 Computations and reproducibility

Theorems 1.2 and 1.4 and Section 5 use computation only for the modules of Proposition 5.4 and for identifications with library groups (Remark 5.7 reports computations only); Theorem 1.3 is computer-assisted. We used GAP 4.16.1 [GAP26] with SmallGrp 1.7.0 [BEO26, BEO02] and PrimGrp 4.0.3 [HLRDR26, CQRD11]. The reproducibility package has four parts.

The program of this note, check_note.g (about three minutes; 51 checks, all passed). It rebuilds both pairs from their definitions and checks orders, library identifications, derived series, maximality

and cores, the two dimensions of Proposition 5.4 for each of the three modules and the identification with `PrimitiveGroup(64, 37)`; the facts behind Lemmas 2.5 and 2.6; the search of Theorem 1.3 for all orders $144 \leq n \leq 1296$ except 768, 1152, 1280 (130,577 groups, about one minute), with Table 1; the largest difference, 2, over all 28,954 soluble groups of non-prime-power order at most 400, without the filter $d(G) \geq 5$; and the structure of the three groups of Theorem 1.3. *The programs with which the results were first obtained*, with their recorded outputs: the certificates of the two pairs (one second each), the small facts, the complete search (about 1.4 hours of CPU time) and the programs of Remark 5.7. *The programs and outputs of the first two independent verification runs. The programs and outputs of the third verification run (`independent_run_2/`), described in the next paragraph.*

Verification. The results were first obtained, with proofs and with the programs of the package. Three independent verification runs, all AI-assisted, followed; each wrote its own programs. The first two examined the first write-up of the results, not this text. The first run re-derived all proofs, among them those of Lemma 5.5 and Proposition 5.6, re-ran the certificates and the search (except order 1296) with identical outputs, recomputed the six groups of Table 1 of order less than 1296 and the three groups of Theorem 1.3, and found the largest difference 2 for all soluble groups of non-prime-power order at most 500 without the filter $d(G) \geq 5$. The second run built both pairs independently (the first from a complement of the inner automorphisms in `Aut(SmallGroup(27, 3))`), ran its own search over all 2,496,072 library groups of non-prime-power order at most 1296, recomputed the maximal subgroups of the groups of Table 1 by a second algorithm and `DerivedLength` by a second code path on 1615 groups, re-ran four programs of the package, and searched the literature, where it found [MT10]; it read Lemma 5.5 and Proposition 5.6 without proving them independently. Both runs confirmed the results in the scope stated here, and their corrections were made: [MT10] is cited and described, the data of [Doe94, GM24] were corrected, the dependence of Theorem 1.3 on the library and on GAP is stated, the facts used from [DH92, Hup67] are proved in Lemma 2.2, an unused hypothesis was removed from Lemma 5.3, an argument on a further layer below the pair of Proposition 5.4, which had not been written out, was left out, and Table 1 was added. When this note was written, Lemmas 2.5 and 2.6 received proofs which do not use the libraries, Corollary 2.4(iv) was added, the two sufficient conditions in Lemma 5.1 were stated and proved, the hypothesis on W_2 in Proposition 5.6 was weakened to $W_2 \neq 0$, and `check_note.g` was written.

The third run examined the text of this note line by line, and these changes first. It read the sources again (the problem in the Notebook, pp. 63 and 76–78 of the survey [MT10], Satz 1 of [Doe94], the review of [GM24], Theorems 1–4 of [MV25] with the proof of Theorem 4, and Section 1 of [GH92]), re-derived all proofs, and checked the bibliography with Crossref and zbMATH Open. Its own programs, in `independent_run_2/`, rebuild the pair of Theorem 1.2 without GAP in two models, one of them with E the group of unitriangular 3×3 matrices over $\mathbb{F}_3$ and $\text{GL}(2, 3)$ a complement of the inner automorphisms in $\text{Aut}(E)$, found by search (the nine complements are conjugate), and check every subgroup displayed in Section 3; recompute every row of Table 1, with all classes of maximal subgroups obtained as complements of chief factors, without `MaximalSubgroupClassReps`; rebuild the pairs of Proposition 5.4 for the three modules and check Remark 5.7 for $(3_+^{1+2}, \text{SL}(2, 3))$ over $\mathbb{F}_2$; and search all 2,496,072 library groups of non-prime-power order at most 1296, the orders below 144 without the filter $d(G) \geq 5$. All numbers of the note were reproduced; in particular no soluble group of non-prime-power order below 144 has derived length 5, and every such group has difference at most 2. It re-ran every program of the package from the archive, the programs of Remark 5.7 included, except one sampling program of the second run (about half an hour of CPU time); the outputs agree with the recorded ones up to running times, GAP syntax warnings and, in one program of Remark 5.7, the order in which the modules are listed. It found no mathematical error and no earlier answer; its corrections were made: the reference [GH92], the remark on [MT10, Theorem 4.7],

a narrower statement of what is new, the scope of Table 1, the solubility of the layer in Lemmas 5.1 and 5.3, and several sentences.

Scope and priority. The note answers neither part of Problem 1.1. It proves that a constant k , if it exists, satisfies $k \geq 3$ (Theorem 1.2); it determines, by an exhaustive computer search, the least order 1296 of a soluble group with a maximal subgroup of difference 3 and the three groups of this order (Theorem 1.3); and it proves the restrictions of Theorem 1.4 and Section 5. The computations of Remark 5.7 prove nothing about k . The problem is Monakhov’s [KM26, Problem 17.91], [MT10, Question 9]. The related known results are Doerk’s theorem on the nilpotent length [Doe94, Satz 1], its analogues for the π -lengths collected in [MT10, Theorems 4.8 and 4.9] (the $\mathfrak{F}$ -length analogue [MT10, Theorem 4.7] does not hold in general [MV25, Theorem 4]), the theorem of Gol’chuk and Monakhov [GM24, Theorem 1] and the results of Murashka and Vasil’ev [MV25, Theorems 1 and 4]; none of them answers the question. The group of Theorem 1.2 is a term of the extraspecial tower of Glasby and Howlett [GH92]. We use the standard facts on primitive soluble groups [DH92, Hup67], proved in Lemma 2.2, and the libraries [BEO02, CQRD11] in GAP [GAP26, BEO26, HLRDR26]. What was read: Problem 17.91 in version 49 of [KM26] and the record of [ula26]; of [MT10], in the file of the whole issue on the journal’s website, the title page, Section 4 with Theorems 4.6–4.9, Example 15 and Question 9 (pp. 77–78), and all other passages on derived length (the second verification run read it in full, in the copy of Math-Net.Ru); [Doe94] and [MV25] (arXiv version 1) in full; of [GH92], Section 1 of the arXiv version; of [GM24] only the review in zbMATH Open and the Crossref metadata, as the publisher’s page refused automated access. The proofs of the cited theorems were not checked. The searches, on 27 October 2026, used the arXiv, zbMATH Open (derived length and maximal subgroups, Kourovka, Monakhov, Trofimuk, Doerk), OpenAlex (queries in English and Russian, works citing [Doe94, GM24]), Crossref, Math-Net.Ru and five web searches; Semantic Scholar refused automated access. No printed answer to the question was found in these sources; a search that finds nothing is not a proof of novelty, and we claim no priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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