

On Bardakov's Question on the Linearity of Automorphism Groups of Artin Groups of Finite Type: A Reduction to Six Exceptional Types

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Abstract

Problem 17.16 of the Kourovka Notebook, posed by V. G. Bardakov, asks whether the automorphism group $\text{Aut}(A)$ of an Artin group A of finite type is linear, as A is by the theorems of Bigelow, Krammer, Digne and Cohen–Wales; the Notebook records the affirmative answer of Bardakov and Bryukhanov for braid groups. We give a *partial answer*: $\text{Aut}(A)$ is linear for every Artin group A of finite type without components of types $E_6, E_7, E_8, F_4, H_3, H_4$ (type D_5 relying on an unrefereed preprint), and the question reduces to linearity of $\text{Aut}(A/Z(A))$ for these six irreducible types. First, $\text{Aut}(A)$ is linear over $\mathbb{C}$ if and only if $\text{Aut}(A_i/Z(A_i))$ is linear over $\mathbb{C}$ for every irreducible component A_i of A . The proof assembles known facts (linearity of A , a lifting argument of Charney and Crisp, parabolic closures after Cumplido, Gebhardt, González-Meneses and Wiest, a lemma of Cumplido and Paris); what we did not find in print is the reduction itself, for reducible groups and in the converse direction. Second, for the types A_n, B_n, D_n and $I_2(m)$, $\text{Aut}(A)$ has a subgroup of finite index isomorphic to $A/Z(A)$ or to $A/Z(A) \times \mathbb{Z}$. This follows, by the argument of Bardakov and Bryukhanov, from the published computations of $\text{Aut}(A)$ by Dyer–Grossman, Charney–Crisp, Gilbert–Howie–Metaftsis–Raptis, Crisp–Paris, Bell–Margalit, Soroko, Castel–Paris and Paris–Soroko (for D_5 a preprint of 2026), and may be regarded as folklore. For the six exceptional types, finiteness of $\text{Out}(A/Z(A))$ would suffice; we found no computation of it in print. This is an unrefereed note.

1 Introduction

1.1 The question

For a Coxeter matrix $M = (m_{st})_{s,t \in S}$ over a finite set S ($m_{ss} = 1$, $m_{st} = m_{ts} \in \{2, 3, \dots, \infty\}$ for $s \neq t$) and the alternating word $\Pi(s, t; m) = sts \cdots$ of length m , the *Artin group* of M is

$$A = A(M) = \langle S \mid \Pi(s, t; m_{st}) = \Pi(t, s; m_{st}) \text{ for } s \neq t, m_{st} < \infty \rangle,$$

and $W = W(M)$, its quotient by the relations $s^2 = 1$, is the Coxeter group. A is of *finite type* if W is finite. Then A is the direct product $A_1 \times \cdots \times A_k$ of its *irreducible components*, the Artin groups of the connected components of the Coxeter graph, each of one of the types A_n ($n \geq 1$), B_n ($n \geq 2$), D_n ($n \geq 4$), $E_6, E_7, E_8, F_4, H_3, H_4$ and $I_2(m)$ ($m \geq 3$; $I_2(3) = A_2$, $I_2(4) = B_2$, $I_2(6) = G_2$). A group is *linear* if it is isomorphic to a subgroup of $\text{GL}_r(K)$ for some r and some field K .

Problem 17.16 of the Kourovka Notebook [KM26] (17th issue, 2010, proposed by V. G. Bardakov) recalls that an Artin group A of finite type is linear [Big01, Kra02, Dig03, CW02] and asks: “Is it true that the automorphism group $\text{Aut}(A)$ is also linear?” It adds that the answer is affirmative for the

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braid groups Br_n on $n \geq 2$ strands (the groups $A(A_{n-1})$), by a theorem of Bardakov and Bryukhanov [BB07]. In the versions 48 and 49 of [KM26] (6 and 20 October 2026) the problem is not marked as solved and carries no further comment. The question is not restricted to irreducible groups, and neither is this note.

1.2 Results

For a group G we write $Z(G)$ for its centre, $\text{Inn}(G) \cong G/Z(G)$ and $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$; for an Artin group A of finite type we write $\bar{A} = A/Z(A)$.

Theorem A (reduction). *Let A be an Artin group of finite type with irreducible components $A_1, \dots, A_k$. The following are equivalent: (i) $\text{Aut}(A)$ is linear over $\mathbb{C}$; (ii) $\text{Aut}(A_i)$ is linear over $\mathbb{C}$ for every i ; (iii) $\text{Aut}(A_i/Z(A_i))$ is linear over $\mathbb{C}$ for every i . If A is irreducible and $\text{Out}(\bar{A})$ is finite, then $\text{Aut}(A)$ is linear over $\mathbb{C}$.*

Theorem B (the classical types). *Let A be an irreducible Artin group of type A_n ($n \geq 1$), B_n ($n \geq 2$), D_n ($n \geq 4$) or $I_2(m)$ ($m \geq 3$). Then $\text{Aut}(A)$ and $\text{Aut}(\bar{A})$ are linear over $\mathbb{C}$. More precisely, $\text{Aut}(A)$ has order 2 for type A_1 , and otherwise it has a subgroup of finite index isomorphic to $\bar{A}$ (types A_n with $n \geq 2$, D_n , and $I_2(m)$ with m odd) or to $\bar{A} \times \mathbb{Z}$ (types B_n with $n \geq 3$, and $I_2(m)$ with m even, which includes $B_2 = I_2(4)$ and $G_2 = I_2(6)$). For type D_5 the proof uses the preprint [PS26].*

Corollary C. (1) $\text{Aut}(A)$ is linear over $\mathbb{C}$ for every Artin group A of finite type none of whose irreducible components is of type E_6 , E_7 , E_8 , F_4 , H_3 or H_4 ; for components of type D_5 this uses the preprint [PS26]. (2) $\text{Aut}(A)$ is linear over $\mathbb{C}$ for every Artin group A of finite type if and only if $\text{Aut}(A/Z(A))$, equivalently $\text{Aut}(\bar{A})$, is linear over $\mathbb{C}$ for each of the six irreducible Artin groups A of types E_6 , E_7 , E_8 , F_4 , H_3 , H_4 ; for each of them it suffices that $\text{Out}(A/Z(A))$ be finite. The “if” direction uses the preprint [PS26] for components of type D_5 ; without it, the list consists of seven types, the six above and D_5 .

So the note gives a *partial answer*: $\text{Aut}(A)$ is linear for every Artin group A of finite type without components of types E_6 , E_7 , E_8 , F_4 , H_3 , H_4 (type D_5 relying on an unrefereed preprint), and the question reduces to linearity of $\text{Aut}(A/Z(A))$ for these six irreducible types. Our statements concern linearity over $\mathbb{C}$, which for these countable groups means linearity over some field of characteristic 0 (Lemma 3.4(c), (d)); the Notebook allows any field, and positive characteristic is not considered here.

1.3 What is known and what is new

Theorem A assembles known facts, cited where they are used: a linear group modulo its centre is linear [Weh73, Theorem 6.4] (Lemma 3.3); the lift of length-preserving or length-reversing automorphisms [CC05, Proposition 8] (Proposition 3.6); parabolic closures [CGGMW19]; and [CP22, Proposition 6(1) and (4)] (Lemma 4.1 and Corollary 4.2). With [CP22, Theorem 4(1)] one also obtains that $A_i/Z(A_i)$ is directly indecomposable when A_i has rank at least 2 (Remark 4.6); our argument does not need this. What we did not find in print is the reduction itself: the reducible case (Lemma 4.4 and Proposition 4.5, with the standard facts on direct factors collected in Lemma 4.3), the “only if” directions, and its application to the recent computations of $\text{Aut}(A)$. Theorem B is a deduction from the published computations quoted in Section 5. For braid groups it is the theorem of Bardakov and Bryukhanov, and our principle (P1) (a finite $\text{Out}(A)$ gives linearity) is their argument [BB07, Corollaries 1 and 2], which Theorem B extends to the other classical types. The cases B_n , D_n and $I_2(m)$ may be regarded as folklore. For D_4 , linearity of $\text{Aut}(A)$ also follows at once from the linearity of $A/Z(A)$, the pure mapping class group of the torus with three punctures [Sor20, Theorem 1, Corollary 2], which has index 12 in $\text{Aut}(A)$ [Sor21, Theorem 5, Corollary 6]; neither paper states it. No printed answer to

type	N	$w_0 = -1$	ζ	$d = \ell(\zeta)$	c
A_1	1	yes	Δ	1	1
$A_n, n \geq 2$	$n(n+1)/2$	no	Δ^2	$n(n+1)$	1
$B_n, n \geq 2$	n^2	yes	Δ	n^2	2
$D_n, n \geq 4$ even	$n(n-1)$	yes	Δ	$n(n-1)$	1
$D_n, n \geq 5$ odd	$n(n-1)$	no	Δ^2	$2n(n-1)$	1
E_6, E_7, E_8	36, 63, 120	no, yes, yes	Δ^2, Δ, Δ	72, 63, 120	1
F_4	24	yes	Δ	24	2
H_3, H_4	15, 60	yes	Δ	15, 60	1
$I_2(m), m$ odd	m	no	Δ^2	$2m$	1
$I_2(m), m$ even	m	yes	Δ	m	2

Table 1: The irreducible finite types: number N of reflections, whether $w_0 = -1$, the generator ζ of $Z(A)$, $d = \ell(\zeta)$, and the rank c of $H_1(A; \mathbb{Z})$. For $c = 2$ the class of ζ in $H_1(A; \mathbb{Z}) = \mathbb{Z}^2$ is $(n(n-1), n)$ for B_n (coefficients of the class of $s_1, \dots, s_{n-1}$ and of s_n), $(12, 12)$ for F_4 and $(m/2, m/2)$ for $I_2(m)$ [CC05, proof of Proposition 9]. These classical values were recomputed in exact arithmetic (Section 7).

the question beyond braid groups was found in the sources searched (paragraph “Scope and priority” in Section 7).

2 Preliminaries

Throughout, A is an Artin group of finite type on the standard generating set S , $n = |S|$, with irreducible components $A_1, \dots, A_k$, so that $A = A_1 \times \dots \times A_k$ (internal direct product; components of rank one are infinite cyclic, and $S = \emptyset$ gives $A = 1$). Let $\ell: A \rightarrow \mathbb{Z}$ be the length homomorphism ($s \mapsto 1$ for $s \in S$), ℓ_i that of A_i , and $p_i: A \rightarrow A_i$ the projection. For irreducible A , w_0 is the longest element of W , N its length (the number of reflections of W), and $\Delta \in A$ the positive lift of w_0 , a positive word of length N . We use the following standard facts.

(F1) Centre. If A is irreducible, $Z(A)$ is infinite cyclic, generated by $\zeta = \Delta$ if w_0 is central in W , and by $\zeta = \Delta^2$ otherwise [BS72, Satz 7.2] (numbering as cited in [PS26, §2]), [Del72]. For irreducible W , w_0 is central if and only if $w_0 = -1$ in the reflection representation (the eigenspaces of a central involution are invariant, and the representation is irreducible). Hence $d = \ell(\zeta)$ is N or $2N$ (Table 1). If $|S| \geq 2$, then $m_{st} \geq 3$ for some s, t , the dihedral subgroup $\langle s, t \rangle$ of W contains m_{st} reflections, and $d \geq N \geq 3$; $d = 1$ only for $A = \mathbb{Z}$ (type A_1). In general $Z(A) = Z(A_1) \times \dots \times Z(A_k) \cong \mathbb{Z}^k$, with generators $\zeta_1, \dots, \zeta_k$ and $d_i = \ell_i(\zeta_i) \geq 1$.

(F2) Dimension. Let $X_W \subset \mathbb{C}^n$ be the complement of the complexified reflection hyperplanes. The quotient X_W/W has fundamental group A [Bri71], is aspherical [Del72], and is a smooth affine variety of complex dimension n (W acts freely), so it has the homotopy type of a CW complex of dimension at most n [AF59]. Hence $\text{cd}(A) \leq n$: A is torsion-free, and its free abelian subgroups have rank at most n .

(F3) Linearity. A embeds in $\text{GL}_r(K)$ for some r and some field K of characteristic 0 [CW02, Dig03] (for braid groups [Big01, Kra02]; for reducible A take block sums), hence also for $K = \mathbb{C}$ by Lemma 3.4(c).

(F4) Parabolic closures. A parabolic subgroup of A is a conjugate gA_Xg^{-1} of a standard parabolic subgroup $A_X = \langle X \rangle$, $X \subseteq S$. For irreducible A , every $\alpha \in A$ lies in a unique minimal parabolic subgroup P_α [CGGMW19, Theorem 1.1 = Proposition 7.2]; if α is a positive word with support X , then $P_\alpha = A_X$ [CGGMW19, Definition 6.7 and Proposition 6.8]; and $P_{x^{-1}\alpha x} = x^{-1}P_\alpha x$ for $x \in A$ [CGGMW19, Lemma 8.1].

(F5) Selberg's lemma. A finitely generated subgroup of $\mathrm{GL}_r(K)$, $\mathrm{char} K = 0$, has a torsion-free subgroup of finite index [Alp87].

(F6) Abelianization. By the presentation, $H_1(A; \mathbb{Z})$ is free abelian with basis the classes of the generators modulo $s \sim t$ for m_{st} odd. For irreducible A its rank c is 2 for the types B_n , F_4 and $I_2(m)$ with m even, and 1 otherwise; and $H_1(A; \mathbb{Q}) = \bigoplus_i H_1(A_i; \mathbb{Q})$.

3 The central quotient

Lemma 3.1. *The composite $Z(A) \rightarrow A \rightarrow H_1(A; \mathbb{Q})$ is injective. In particular $Z(A) \cap [A, A] = 1$.*

Proof. The homomorphism $\ell_i \circ p_i: A \rightarrow \mathbb{Z}$ factors through $H_1(A; \mathbb{Q})$ and maps ζ_j to d_i if $j = i$ and to 0 otherwise. If $\zeta_1^{a_1} \cdots \zeta_k^{a_k}$ maps to 0, then $a_i d_i = 0$, so $a_i = 0$ for every i , by (F1). Elements of $[A, A]$ map to 0. $\square$

Lemma 3.2. *If $x, y \in A$ and $[x, y] \in Z(A)$, then $[x, y] = 1$. The group $\bar{A}$ has trivial centre.*

Proof. $[x, y] \in Z(A) \cap [A, A] = 1$ by Lemma 3.1. If $xZ(A)$ is central in $\bar{A}$, then $[x, y] \in Z(A)$, hence $[x, y] = 1$, for all $y \in A$, so $x \in Z(A)$. $\square$

Lemma 3.3 (classical). *If $G \leq \mathrm{GL}_r(K)$ for a field K , then $G/Z(G)$ is isomorphic to a subgroup of $\mathrm{GL}_e(K)$ with $e \leq r^2$.*

Proof. Let R be the K -linear span of G in the matrix algebra $M_r(K)$; it is closed under multiplication. For $g \in G$ the map $X \mapsto gXg^{-1}$ preserves R , and $g \mapsto (X \mapsto gXg^{-1})|_R$ is a homomorphism $G \rightarrow \mathrm{GL}(R)$. Its kernel consists of the $g \in G$ that commute with every element of R , that is, with every element of G ; it is $Z(G)$. $\square$

That a linear group modulo its centre is linear is cited as [Weh73, Theorem 6.4] in [PS25, p. 9] and [Sor20, proof of Corollary 2]; the bound $e \leq r^2$ is that of the proof above.

Lemma 3.4. *Let K be a field. (a) If $H \leq G$ has finite index and H embeds in $\mathrm{GL}_r(K)$, then G embeds in $\mathrm{GL}_{r[G:H]}(K)$. (b) Finite direct products of subgroups of $\mathrm{GL}_{r_j}(K)$ embed in $\mathrm{GL}_{\sum r_j}(K)$, and $\mathrm{GL}_r(\mathbb{Q}) \leq \mathrm{GL}_r(K)$ if $\mathrm{char} K = 0$. (c) A countable group that embeds in $\mathrm{GL}_r(K)$ with $\mathrm{char} K = 0$ embeds in $\mathrm{GL}_r(\mathbb{C})$. (d) If G is finitely generated, $\mathrm{Aut}(G)$ is countable.*

Proof. (a) Let G act on the induced module $\bigoplus_{gH \in G/H} g \otimes K^r$. An element acting trivially fixes every coset gH , so it lies in H , and it acts on $1 \otimes K^r$ through the given embedding of H ; so it is 1. (b) is clear. (c) The entries of the matrices of a countable subgroup generate a countable subfield of K of characteristic 0, which embeds in $\mathbb{C}$ ($\mathbb{C}$ is algebraically closed of uncountable transcendence degree over $\mathbb{Q}$). (d) An automorphism is determined by the images of finitely many generators. $\square$

Proposition 3.5. *The map $\Theta: \mathrm{Aut}(A) \rightarrow \mathrm{Aut}(\bar{A}) \times \mathrm{GL}(H_1(A; \mathbb{Q}))$, $\Theta(\varphi) = (\bar{\varphi}, \varphi_*)$, where $\bar{\varphi}$ is the automorphism induced on $\bar{A}$ and φ_* the one induced on $H_1(A; \mathbb{Q})$, is an injective homomorphism. Hence, if $\mathrm{Aut}(\bar{A})$ is linear over $\mathbb{C}$, so is $\mathrm{Aut}(A)$.*

Proof. $Z(A)$ is characteristic, so Θ is a well defined homomorphism. Let $\Theta(\varphi) = (\mathrm{id}, \mathrm{id})$ and put $c(g) = g^{-1}\varphi(g)$ for $g \in A$. Then $c(g) \in Z(A)$, and in $H_1(A; \mathbb{Q})$ we have $[\varphi(g)] = [g] + [c(g)]$, so $[c(g)] = 0$ and $c(g) = 1$ by Lemma 3.1. Thus $\varphi = \mathrm{id}$. The last sentence follows from Lemma 3.4(b), as $H_1(A; \mathbb{Q}) \cong \mathbb{Q}^c$ for some c . $\square$

Proposition 3.6 ([CC05, Proposition 8]). *Let A be irreducible with $|S| \geq 2$, let ζ and d be as in (F1), and let $\bar{\ell}: \bar{A} \rightarrow \mathbb{Z}/d\mathbb{Z}$ be the homomorphism induced by ℓ (well defined, as $\ell(Z(A)) = d\mathbb{Z}$). Let $\text{Aut}^*(\bar{A})$ be the set of $\alpha \in \text{Aut}(\bar{A})$ with $\bar{\ell} \circ \alpha = \pm \bar{\ell}$. (a) $\text{Aut}^*(\bar{A})$ is a subgroup of finite index of $\text{Aut}(\bar{A})$, and for $\alpha \in \text{Aut}^*(\bar{A})$ the sign $\varepsilon(\alpha) \in \{\pm 1\}$ with $\bar{\ell} \circ \alpha = \varepsilon(\alpha)\bar{\ell}$ is unique. (b) For $\alpha \in \text{Aut}^*(\bar{A})$ and $a \in A$ let $\hat{\alpha}(a)$ be the unique $b \in A$ with $bZ(A) = \alpha(aZ(A))$ and $\ell(b) = \varepsilon(\alpha)\ell(a)$. Then $\hat{\alpha} \in \text{Aut}(A)$, and $\alpha \mapsto \hat{\alpha}$ is an injective homomorphism $\text{Aut}^*(\bar{A}) \rightarrow \text{Aut}(A)$.*

Proof. (a) $\text{Hom}(\bar{A}, \mathbb{Z}/d\mathbb{Z})$ is finite, as $\bar{A}$ is finitely generated, and $\text{Aut}(\bar{A})$ acts on it by $f \mapsto f \circ \alpha^{-1}$; the stabilizer of $\bar{\ell}$ has finite index and is contained in $\text{Aut}^*(\bar{A})$, which is clearly closed under products and inverses. If $\bar{\ell} \circ \alpha = \bar{\ell} = -\bar{\ell}$, then $2\bar{\ell} = 0$; as $\bar{\ell}(sZ(A)) = 1$ for $s \in S$, this gives $d \mid 2$, contradicting $d \geq 3$ (F1).

(b) Write $\alpha(aZ(A)) = b_0Z(A)$ and $\varepsilon = \varepsilon(\alpha)$. Then $\ell(b_0) \equiv \varepsilon\ell(a) \pmod{d}$, and the elements $b_0\zeta^j$ of the coset have the lengths $\ell(b_0) + jd$, exactly one of which is $\varepsilon\ell(a)$; so $\hat{\alpha}(a)$ is well defined. The product $\hat{\alpha}(a_1)\hat{\alpha}(a_2)$ lies in $\alpha(a_1a_2Z(A))$ and has length $\varepsilon\ell(a_1a_2)$, so it equals $\hat{\alpha}(a_1a_2)$. For $\alpha, \beta \in \text{Aut}^*(\bar{A})$ we have $\bar{\ell} \circ \alpha\beta = \varepsilon(\alpha)\varepsilon(\beta)\bar{\ell}$, so $\varepsilon(\alpha\beta) = \varepsilon(\alpha)\varepsilon(\beta)$ by (a); the element $\hat{\alpha}(\hat{\beta}(a))$ lies in $\alpha\beta(aZ(A))$ and has length $\varepsilon(\alpha)\varepsilon(\beta)\ell(a)$, so $\widehat{\alpha\beta} = \hat{\alpha}\hat{\beta}$. As $\widehat{\text{id}} = \text{id}$, every $\hat{\alpha}$ is an automorphism with inverse $\widehat{\alpha^{-1}}$, and $\alpha \mapsto \hat{\alpha}$ is a homomorphism; it is injective because $\hat{\alpha}$ induces α on $\bar{A}$. $\square$

The proof is that of [CC05], with the uniqueness of the sign made explicit.

Theorem 3.7. *Let A be irreducible. Then $\text{Aut}(A)$ is linear over $\mathbb{C}$ if and only if $\text{Aut}(\bar{A})$ is linear over $\mathbb{C}$. If $\text{Out}(\bar{A})$ is finite, both are linear over $\mathbb{C}$.*

Proof. For $|S| = 1$ we have $A = \mathbb{Z}$, $\text{Aut}(A) \cong \mathbb{Z}/2$ and $\bar{A} = 1$. Let $|S| \geq 2$. “If” is Proposition 3.5. “Only if”: $\text{Aut}^*(\bar{A})$ embeds in $\text{Aut}(A)$ (Proposition 3.6(b)), so it is linear over $\mathbb{C}$, and it has finite index in $\text{Aut}(\bar{A})$ (Proposition 3.6(a)); apply Lemma 3.4(a). If $\text{Out}(\bar{A})$ is finite, then $\text{Inn}(\bar{A}) \cong \bar{A}$ (Lemma 3.2) has finite index in $\text{Aut}(\bar{A})$, and $\bar{A}$ is linear over $\mathbb{C}$ by Lemma 3.3 and (F3); so $\text{Aut}(\bar{A})$ is linear over $\mathbb{C}$ by Lemma 3.4(a), and so is $\text{Aut}(A)$. $\square$

4 Reducible groups and the proof of Theorem A

Lemma 4.1. *Let $H \leq A$ have finite index, and let $x \in A$ commute with every element of H . Then $x \in Z(A)$.*

For irreducible A this is [CP22, Proposition 6(1)] ($Z(U) = Z(A) \cap U$ for U of finite index), applied to $U = \langle H, x \rangle$; we give the short proof through parabolic closures.

Proof. Write $x = x_1 \cdots x_k$ with $x_i \in A_i$. The subgroup $H \cap A_i$ has finite index in A_i , and x_i commutes with it, as the other factors of x commute with A_i . So we may assume that A is irreducible and $|S| \geq 2$. Let $s \in S$. Two of the cosets s^jH , $0 \leq j \leq [A : H]$, coincide, so $s^q \in H$ for some $q \geq 1$, and $xs^qx^{-1} = s^q$. By (F4), $P_{s^q} = \langle s \rangle$ and $x\langle s \rangle x^{-1} = xP_{s^q}x^{-1} = P_{xs^qx^{-1}} = P_{s^q} = \langle s \rangle$. Hence $xsx^{-1} \in \{s, s^{-1}\}$, and $\ell(xsx^{-1}) = 1$ gives $xsx^{-1} = s$. So x commutes with S , and $x \in Z(A)$. $\square$

Corollary 4.2. *$\bar{A}$ has no nontrivial finite normal subgroup.*

Proof. Let $F \trianglelefteq \bar{A}$ be finite and $\bar{x} \in F$. The conjugacy class of $\bar{x}$ lies in F , so the centralizer of $\bar{x}$ has finite index in $\bar{A}$, and its preimage $H \leq A$ has finite index. For a lift x of $\bar{x}$ and $h \in H$ we have $[x, h] \in Z(A)$, so $[x, h] = 1$ by Lemma 3.2. By Lemma 4.1, $x \in Z(A)$, that is, $\bar{x} = 1$. $\square$

For irreducible A the key step of this proof, that an element of $\overline{A}$ whose centralizer has finite index is trivial, is [CP22, Proposition 6(4)]: the natural map from $\overline{A}$ to its abstract commensurator is injective.

Let G be a group with $Z(G) = 1$. A *direct factor* of G is a subgroup D such that $G = D \times D'$ (internal direct product) for some subgroup D' ; let $\mathfrak{B}(G)$ be the set of direct factors. For $D \leq G$ put $D^1 = D$ and $D^0 = C_G(D)$. The next lemma collects standard facts; we include the short proofs.

Lemma 4.3. *Let $Z(G) = 1$. (a) If $G = D \times D'$, then $D' = C_G(D)$. (b) If $G = H \times H'$ and $G = K_1 \times \cdots \times K_s$, then $H = (H \cap K_1) \times \cdots \times (H \cap K_s)$. (c) If $D_1, \dots, D_t \in \mathfrak{B}(G)$ and $E_\varepsilon = D_1^{\varepsilon_1} \cdots D_t^{\varepsilon_t}$ for $\varepsilon \in \{0, 1\}^t$, then G is the internal direct product of the E_ε , and D_i is the product of the E_ε with $\varepsilon_i = 1$, for every i .*

Proof. (a) Clearly $D' \subseteq C_G(D)$. If $g = dd' \in C_G(D)$ with $d \in D$, $d' \in D'$, then $d = gd'^{-1}$ commutes with D , so $d \in Z(D) \subseteq Z(G) = 1$, and $g \in D'$.

(b) Let $q: G \rightarrow H'$ be the projection with kernel H ; note $Z(H') \subseteq Z(G) = 1$. Let $h \in H$ and $h = x_1 \cdots x_s$ with $x_j \in K_j$. The elements $q(x_j)$ commute pairwise, and their product is $q(h) = 1$. For $l \neq j$, x_j commutes with K_l , so $q(x_j)$ commutes with $q(K_l)$; and $q(x_j) = \prod_{l \neq j} q(x_l)^{-1}$ commutes with $q(K_j)$, because every x_l with $l \neq j$ commutes with K_j . Hence $q(x_j)$ commutes with $q(K_1) \cdots q(K_s) = q(G) = H'$, so $q(x_j) = 1$ and $x_j \in H \cap K_j$. Thus H is generated by the subgroups $H \cap K_j$, which are normal in G , commute pairwise and lie in the independent factors K_j ; so H is their internal direct product.

(c) Induction on t ; for $t = 1$ this is (a). If the E_ε with $\varepsilon \in \{0, 1\}^{t-1}$ form a direct decomposition of G , then each of them is a direct factor of G , and (b), applied to it and to $G = D_t \times C_G(D_t)$ (by (a)), gives $E_\varepsilon = (E_\varepsilon \cap D_t) \times (E_\varepsilon \cap C_G(D_t))$; this is the decomposition for t . Next, the product P of the E_ε with $\varepsilon_i = 1$ lies in D_i , the product Q of the others lies in $C_G(D_i)$, and $G = P \times Q$. For $d \in D_i$ write $d = pq$ with $p \in P$, $q \in Q$; then $q = p^{-1}d \in D_i \cap C_G(D_i) = Z(D_i) \subseteq Z(G) = 1$. So $D_i = P$. $\square$

Lemma 4.4. *If $\overline{A} = F_1 \times \cdots \times F_r$ (internal) with $F_j \neq 1$ for all j , then $r \leq n - k$. Consequently $\mathfrak{B}(\overline{A})$ is finite, with at most 2^{n-k} elements.*

Proof. Each F_j is a nontrivial normal subgroup of $\overline{A}$, hence infinite by Corollary 4.2. It is finitely generated (a quotient of $\overline{A}$) and linear over $\mathbb{C}$ (a subgroup of $\overline{A}$, which is linear by Lemma 3.3 and (F3)). By (F5) it has a torsion-free subgroup of finite index, which is infinite; so F_j contains an element $\bar{x}_j$ of infinite order. Choose lifts $x_j \in A$. For $j \neq l$ we have $[\bar{x}_j, \bar{x}_l] = 1$, so $[x_j, x_l] = 1$ by Lemma 3.2. Hence $X = \langle Z(A), x_1, \dots, x_r \rangle$ is a finitely generated abelian group, torsion-free by (F2), containing $Z(A) \cong \mathbb{Z}^k$, and $X/Z(A) \cong \langle \bar{x}_1 \rangle \times \cdots \times \langle \bar{x}_r \rangle \cong \mathbb{Z}^r$. So $X \cong \mathbb{Z}^{k+r}$, and $k + r \leq n$ by (F2).

Now let $D_1, \dots, D_t \in \mathfrak{B}(\overline{A})$ be distinct; $\overline{A}$ is centreless by Lemma 3.2. By Lemma 4.3(c) the nontrivial E_ε form a direct decomposition of $\overline{A}$, so there are at most $n - k$ of them, and each D_i is the product of the nontrivial E_ε with $\varepsilon_i = 1$. Distinct D_i give distinct sets of such factors, so $t \leq 2^{n-k}$. $\square$

Proposition 4.5. *Let $\overline{A}_i$ be the image of A_i in $\overline{A}$, so that $\overline{A} = \overline{A}_1 \times \cdots \times \overline{A}_k$ and $\overline{A}_i \cong A_i/Z(A_i)$. The subgroup $\text{Aut}_{\mathfrak{B}}(\overline{A})$ of the automorphisms that fix every element of $\mathfrak{B}(\overline{A})$ has finite index in $\text{Aut}(\overline{A})$, and $\varphi \mapsto (\varphi|_{\overline{A}_1}, \dots, \varphi|_{\overline{A}_k})$ is an injective homomorphism $\text{Aut}_{\mathfrak{B}}(\overline{A}) \rightarrow \text{Aut}(\overline{A}_1) \times \cdots \times \text{Aut}(\overline{A}_k)$.*

Proof. As $Z(A) = \prod_i Z(A_i)$, $\overline{A}$ is the internal direct product of the $\overline{A}_i \cong A_i/Z(A_i)$. Automorphisms map direct factors to direct factors, so $\text{Aut}(\overline{A})$ acts on the finite set $\mathfrak{B}(\overline{A})$ (Lemma 4.4), and $\text{Aut}_{\mathfrak{B}}(\overline{A})$ is the kernel of this action. Each $\overline{A}_i$ is a direct factor, so every $\varphi \in \text{Aut}_{\mathfrak{B}}(\overline{A})$ restricts to an automorphism of $\overline{A}_i$. The restriction map is a homomorphism, and it is injective because the $\overline{A}_i$ generate $\overline{A}$. (Components of rank one give $\overline{A}_i = 1$.) $\square$

Remark 4.6. For irreducible A with $|S| \geq 2$, $\bar{A}$ is directly indecomposable. Indeed, let $P = \ker(A \rightarrow W)$; its image $\bar{P}$ in $\bar{A}$ has finite index and is isomorphic to $P/Z(P)$, as $Z(P) = Z(A) \cap P$ [CP22, Proposition 6(1)]; every subgroup of finite index of $P/Z(P)$ is directly indecomposable [CP22, Theorem 4(1)]. If $\bar{A} = H \times K$ with $H, K \neq 1$, then H, K are infinite (Corollary 4.2), and $(H \cap \bar{P}) \times (K \cap \bar{P})$ is a decomposable subgroup of finite index of $\bar{P}$. With Lemma 4.3(b), every direct factor of $\bar{A}$ for reducible A is then a product of some $\bar{A}_i$; this gives a second proof of Proposition 4.5.

Proof of Theorem A. (i) $\Rightarrow$ (ii): $\alpha \mapsto \alpha \times \text{id}$ embeds $\text{Aut}(A_i)$ in $\text{Aut}(A)$. (ii) $\Leftrightarrow$ (iii): Theorem 3.7 for each A_i . (iii) $\Rightarrow$ (i): $\prod_i \text{Aut}(\bar{A}_i)$ is linear over $\mathbb{C}$ (Lemma 3.4(b)); hence so is $\text{Aut}_{\mathfrak{B}}(\bar{A})$ (Proposition 4.5), hence $\text{Aut}(\bar{A})$ (Lemma 3.4(a)), hence $\text{Aut}(A)$ (Proposition 3.5). The last sentence of Theorem A is part of Theorem 3.7. $\square$

5 The classical types: proof of Theorem B and Corollary C

Let A be irreducible, with ζ and d as in (F1). We use two principles.

(P1) If $\text{Out}(A)$ is finite, then $\text{Inn}(A) \cong \bar{A}$ has finite index in $\text{Aut}(A)$, and $\text{Aut}(A)$ is linear over $\mathbb{C}$ (Lemma 3.3, (F3), Lemma 3.4(a)). For braid groups this is the argument of [BB07, Corollaries 1 and 2], with $|\text{Aut}(\text{Br}_n) : \text{Inn}(\text{Br}_n)| = 2$ from [DG81].

(P2) Transvections [CC05, §3]. For a homomorphism $\lambda: A \rightarrow \mathbb{Z}$ the map $T_\lambda(x) = x\zeta^{\lambda(x)}$ is an endomorphism of A .

Lemma 5.1. (a) $T_\lambda T_\mu = T_\nu$ with $\nu = \lambda + (1 + \lambda(\zeta))\mu$. (b) T_λ is an automorphism if and only if $\lambda(\zeta) \in \{0, -2\}$. (c) If $\lambda \neq 0$ and $\lambda(\zeta) = 0$, then T_λ is an automorphism of infinite order that commutes with every inner automorphism, no nontrivial power of T_λ is inner, and $\text{Inn}(A)\langle T_\lambda \rangle \cong \bar{A} \times \mathbb{Z}$. If this subgroup has finite index in $\text{Aut}(A)$, then $\text{Aut}(A)$ is linear over $\mathbb{C}$.

Proof. (a) $T_\lambda(T_\mu(x)) = T_\lambda(x)T_\lambda(\zeta)^{\mu(x)} = x\zeta^{\lambda(x)+(1+\lambda(\zeta))\mu(x)}$. (b) $T_\lambda(\zeta) = \zeta^{1+\lambda(\zeta)}$. If $T_\lambda(x) \in Z(A)$, then $x \in Z(A)$; so T_λ is injective if and only if $1 + \lambda(\zeta) \neq 0$, and $T_\lambda(A) \cap Z(A) = T_\lambda(Z(A))$. As $T_\lambda(A)Z(A) = A$, the map T_λ is onto if and only if $T_\lambda(Z(A)) = Z(A)$, that is, $|1 + \lambda(\zeta)| = 1$. (c) By (a), $T_\lambda^j = T_{j\lambda} \neq \text{id}$ for $j \neq 0$. For $g \in A$ we have $T_\lambda \iota_g T_\lambda^{-1} = \iota_{T_\lambda(g)} = \iota_g$, where ι_g is conjugation by g . Since $\ell \circ T_{j\lambda} = \ell + jd\lambda \neq \ell = \ell \circ \iota_g$ for $j \neq 0$, no T_λ^j with $j \neq 0$ is inner. So $\text{Inn}(A)\langle T_\lambda \rangle$ is the internal direct product $\text{Inn}(A) \times \langle T_\lambda \rangle \cong \bar{A} \times \mathbb{Z}$, which is linear over $\mathbb{C}$ by Lemma 3.3, (F3) and Lemma 3.4(b); the last claim follows from Lemma 3.4(a). $\square$

Lemma 5.2. Let $m = 2k \geq 4$ be even, $A = A(I_2(m)) = \langle a, b \mid (ab)^k = (ba)^k \rangle$ and $\Delta = (ab)^k$. (a) With $y = ab$ we have $A = \langle a, y \mid ay^k = y^k a \rangle$, $Z(A) = \langle \Delta \rangle = \langle y^k \rangle$, and $\bar{A} \cong \mathbb{Z} * \mathbb{Z}/k$. (b) $\eta(a) = aba$, $\eta(b) = a^{-1}$ defines an automorphism of A , and $\eta^k = T_\lambda$ with $\lambda(a) = 1$, $\lambda(b) = -1$, so that $\lambda(\Delta) = 0$.

Proof. (a) In the free group $F(a, b)$ we have $a^{-1}(ab)^k a = (ba)^k$, so the relation reads $a^{-1}y^k a = y^k$, and $(a, b) \mapsto (a, y)$ is a Nielsen transformation. By (F1) and Table 1, $\Delta = y^k$ generates $Z(A)$; hence $\bar{A} = \langle a, y \mid [a, y^k], y^k \rangle = \langle a, y \mid y^k \rangle \cong \mathbb{Z} * \mathbb{Z}/k$. (The presentation with the generators ab and b , and the isomorphism $\bar{A} \cong \mathbb{Z}/k * \mathbb{Z}$, are in [CP02, proof of Theorem 5.1].)

(b) In $F(a, b)$, $\eta(ab) = ab$ and $\eta(ba) = ba$; the endomorphism ψ with $\psi(a) = b^{-1}$, $\psi(b) = bab$ satisfies $\psi\eta = \eta\psi = \text{id}$ on a and b . So η is an automorphism of $F(a, b)$ fixing the relator $(ab)^k(ba)^{-k}$, and it induces an automorphism of A with inverse induced by ψ . By induction on j , $\eta^j(a) = (ab)^j a$ and $\eta^j(b) = a^{-1}(ab)^{-(j-1)}$ in $F(a, b)$: indeed $\eta((ab)^j a) = (ab)^j aba = (ab)^{j+1} a$ and $\eta(a^{-1}(ab)^{-(j-1)}) = a^{-1}b^{-1}a^{-1}(ab)^{-(j-1)} = a^{-1}(ab)^{-j}$. For $j = k$ this gives $\eta^k(a) = \Delta a = a\Delta$ (Δ is central) and $\eta^k(b) = a^{-1}(ab)(ab)^{-k} = b\Delta^{-1}$. The relation is balanced in a and in b , so λ is a homomorphism, and $\eta^k = T_\lambda$. $\square$

type	published description	source
$A_n, n \geq 3$	$\text{Aut}(A) = \text{Inn}(A) \rtimes \langle \varepsilon \rangle$	[CC05, Theorem 1(ii)(a), p. 2], after [DG81]
$B_n, n \geq 3$	$\text{Aut}(A) \cong (\text{Inn}(A) \times \text{Tv}(A)) \rtimes (\mathbb{Z}/2 \times \mathbb{Z}/2),$ $\text{Tv}(A) \cong \mathbb{Z}$	[CC05, Propositions 9, 10]; [BM07, Theorem 6]
D_4	$\text{Aut}(A) \cong \bar{A} \rtimes (S_3 \times \mathbb{Z}/2)$	[Sor21, Theorem 5, Corollary 6]
D_5	$\text{Aut}(A) = \text{Inn}(A) \rtimes \langle \chi \rangle$	[PS26, Theorems 1, 2] (preprint)
$D_n, n \geq 6$	$\text{Aut}(A) = \text{Inn}(A) \rtimes \langle \tau, \chi \rangle$ (n even), $\text{Inn}(A) \rtimes \langle \chi \rangle$ (n odd)	[CP25, Corollaries 2.6, 2.10]
$I_2(m), m$ odd	$\text{Aut}(A) = \text{Inn}(A) \rtimes \langle \varepsilon \rangle$	[CC05, §3], after [GHMR00]; [CP02, Theorem 5.1]
$I_2(m), m$ even	$\text{Out}(A) = (\langle [\eta] \rangle \rtimes \langle [\gamma] \rangle) \times \langle [\varepsilon] \rangle$	[CC05, §3], after [GHMR00]; [CP02, Theorem 5.1]

Table 2: The descriptions of $\text{Aut}(A)$ used for Theorem B. Here ε and χ denote the inversion $s \mapsto s^{-1}$, τ the graph automorphism of D_n exchanging the generators at the ends of the two short branches, γ the graph automorphism $a \leftrightarrow b$ of $I_2(m)$, and η is as in Lemma 5.2; $[\gamma]$ inverts $[\eta]$. [CC05, Theorem 1] is stated for $n \geq 3$; $A_2 = I_2(3)$. The theorem of [BM07] is numbered as in its arXiv version (Paris and Soroko cite it as Theorem 7), and [PS25, Corollaries 2.5 and 2.10] recover the case B_n for $n \geq 5$. The second references for D_5 and D_n describe $\text{Aut}(A/Z(A))$ in the same way.

Proof of Theorem B. By Theorem 3.7 it suffices to treat $\text{Aut}(A)$. For A_1 , $\text{Aut}(\mathbb{Z}) \cong \mathbb{Z}/2$. For A_n ($n \geq 3$), D_n ($n \geq 4$) and $I_2(m)$ with m odd, which includes A_2 , Table 2 shows that $\text{Out}(A)$ is finite, and (P1) applies.¹ For D_5 this uses the preprint [PS26].

$B_n, n \geq 3$. Here $\text{Tv}(A)$ is the group of automorphisms of A that are transvections. By Table 1 the class of ζ in $H_1(A; \mathbb{Z}) = \mathbb{Z}^2$ is $(n(n-1), n)$, so $\lambda(\zeta) \in n\mathbb{Z}$ for every λ , and by Lemma 5.1(b) the infinite cyclic group $\text{Tv}(A)$ is generated by some T_λ with $\lambda \neq 0$ and $\lambda(\zeta) = 0$. So $\text{Inn}(A) \times \text{Tv}(A)$ has index 4 in $\text{Aut}(A)$ (Table 2), and Lemma 5.1(c) applies.

$I_2(m), m = 2k$ even, which includes B_2 and G_2 . The subgroup $\langle [\eta]^k \rangle$ is normal in $\text{Out}(A)$ (Table 2), with quotient $(\mathbb{Z}/k \rtimes \mathbb{Z}/2) \times \mathbb{Z}/2$ of order $2m$. Its preimage $\text{Inn}(A)\langle \eta^k \rangle$ has index $2m$ in $\text{Aut}(A)$, and η^k is a transvection with $\lambda \neq 0$ and $\lambda(\zeta) = 0$ (Lemma 5.2(b)); so Lemma 5.1(c) applies. (The automorphism $\alpha \mapsto \beta^{-1}$, $\beta \mapsto \beta\alpha\beta$ of [CP02, Theorem 5.1(ii)] is our η^{-1} ; for $m \geq 6$, Charney and Crisp note that the transvections have index $m/2$ in $\langle \eta \rangle$ [CC05, p. 9].) $\square$

Remark 5.3. For B_n ($n \geq 3$) and $I_2(m)$ (m even) linearity alone also follows from Theorem 3.7, as $\text{Out}(\bar{A})$ is finite: $\text{Out}(\bar{A}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ for B_n [CC05, Theorem 1(ii)(b), Proposition 5], and $\text{Out}(\mathbb{Z} * \mathbb{Z}/k)$ is finite [CP02, Lemma 5.3], where $\bar{A} \cong \mathbb{Z} * \mathbb{Z}/k$ for $I_2(2k)$ (Lemma 5.2(a)).

Proof of Corollary C. The irreducible types other than $E_6, E_7, E_8, F_4, H_3, H_4$ are A_n, B_n, D_n and $I_2(m)$. (1) By Theorem B and Theorem 3.7, $\text{Aut}(A_i/Z(A_i))$ is linear over $\mathbb{C}$ for every component A_i of such an A ; apply Theorem A. (2) “Only if” is the case of irreducible A , with Theorem 3.7 for the equivalence of the two conditions. “If”: under the hypothesis, $\text{Aut}(A_i/Z(A_i))$ is linear over $\mathbb{C}$ for every component of every A , by the hypothesis and by Theorem B and Theorem 3.7; apply Theorem A. Finiteness of $\text{Out}(A/Z(A))$ suffices by Theorem 3.7. Theorem B uses [PS26] only for D_5 . $\square$

¹Castel and Paris [CP25] write ζ for the graph automorphism τ ; in this note ζ denotes the generator of $Z(A)$.

6 The six exceptional types

For the types $E_6, E_7, E_8, F_4, H_3, H_4$ we found no computation of $\text{Aut}(A)$ or of $\text{Out}(\bar{A})$ in print; [CP25, §1] lists $A_n, B_n, I_2(m)$ and D_4 as the spherical types with known $\text{Aut}(A)$ and treats D_n for $n \geq 6$, and [PS26, §1] lists these, D_n ($n \geq 6$) included, and adds D_5 . For F_4 , the transvection automorphisms and the automorphisms preserving ℓ up to sign generate a subgroup of finite index of $\text{Aut}(A)$ [CC05, Proposition 9]; the index is not determined there. By Theorem 3.7, the question for these six groups is the linearity of $\text{Aut}(\bar{A})$, and finiteness of $\text{Out}(\bar{A})$ would suffice. The computations of [CC05, BM07, Sor21, CP25, PS25, PS26] realize $\bar{A}$, or a subgroup of finite index, in a mapping class group and use rigidity theorems [CC05, BM07, Sor21] or classifications of homomorphisms from braid groups to mapping class groups [CP25, PS25, PS26]; in rank two, [CP02] uses the free-product structure of $\bar{A}$. We know of no such model for the six exceptional types, and we could not bound $\text{Out}(\bar{A})$ without one.

Conclusion. The note gives a *partial answer*: $\text{Aut}(A)$ is linear for every Artin group A of finite type without components of types $E_6, E_7, E_8, F_4, H_3, H_4$ (type D_5 relying on an unrefereed preprint), and the question reduces to linearity of $\text{Aut}(A/Z(A))$ for these six irreducible types. Without the preprint [PS26] the same holds with D_5 added to the six types.

7 Computations and reproducibility

No theorem, proposition or lemma of this note depends on a computation; the programs check the numbers and identities that are quoted. The package contains (1) `check_note.py`, written with this note (standard library, under one second): for the 45 types A_1 – A_8, B_2 – B_8, D_4 – D_8, E_6 – E_8, F_4, G_2, H_3, H_4 and $I_2(m)$, $3 \leq m \leq 20$, it builds the root system in exact arithmetic (integer Cartan matrices, the ring $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$, and $2m$ roots as a permutation model for $I_2(m)$) and recomputes N , a reduced word of w_0 , whether $w_0 = -1$ and, separately, whether w_0 is central, the classes of generators, the class of ζ in $H_1(A; \mathbb{Z})$, d , and the transvection data of Section 5, all as in Table 1; and it checks the identities of Lemma 5.2, with $\gamma\eta\gamma = \eta^{-1}$ and $\varepsilon\eta\varepsilon = \eta$, in the free group for all even $m \leq 20$; all checks passed. (2) `coxeter_facts.py`, with which these values were first computed, and its output. (3) The programs and outputs of the three verification runs described below. All programs were run again from the package, the last time from the archive, and reproduced their recorded outputs exactly.

Verification. The results were first obtained, with proofs and with `coxeter_facts.py`. Two independent verification runs, both AI-assisted, followed; each wrote its own programs. The first re-derived every statement of Sections 3 and 4 and (P1), (P2) line by line, compared each statement used for Theorem B with the source at the place cited, and found the English version of [BB07]; its programs checked Lemma 5.2 for all even $m \leq 40$ and recomputed N , $w_0 = -1$ and d for 40 types in floating point. The second recomputed Table 1 for 44 types in exact arithmetic and wrote an exact solver for the word problem (Garside normal forms), with which it checked 55 statements inside the Artin groups: among them the centrality of Δ or Δ^2 , Lemma 5.2 for $m \leq 16$, the automorphism of $A(B_n)$ described after [CC05, Proposition 10] for $n = 3, 4, 5$, and non-inner automorphisms for D_4, D_6 and $I_2(m)$, m even; eight deliberately wrong inputs were rejected. It read [CP02, BM07, Sor20, CP22, BB07] as further sources and repeated the literature search. Neither run found a mathematical error. Their required corrections, all made, are: the uniqueness of the sign in Proposition 3.6; the English version of [BB07], with the attribution of (P1); the derivation in Lemma 5.2(a) instead of a second-hand citation; the footnote on the notation of [CP25]; the attribution of Lemma 4.1 and Remark 4.6 to

[CP22]; the caveat on D_5 in Corollary C(2); and precise citations. Remark 5.3 follows a suggestion of both runs. The program `check_note.py`, the exact program of this note, was written afterwards.

A third independent verification run, also AI-assisted, then examined the text of this note as if nothing had been checked. It re-derived every proof line by line; read again Problem 17.16 in the versions 48 and 49 of [KM26], [CC05] completely and the cited places of [CP25, PS25, PS26, Sor21, Sor20, CP22, CP02, BM07, CGGMW19, BB07]; checked every DOI with Crossref; and repeated the literature search. Its programs, written independently of the earlier ones, recompute Table 1 for 48 types from explicit root systems in standard coordinates rather than from Cartan matrices (exact arithmetic in $\mathbb{Q}$, $\mathbb{Q}(\sqrt{5})$ and cyclotomic fields), in agreement with `check_note.py` on all 45 common types; and they check Lemma 5.2 in the free group for all even $m \leq 20$ and inside $A(I_2(m))$, $3 \leq m \leq 20$, with a solver for the word problem based on the central quotient $(\mathbb{Z} * \mathbb{Z}/k$ or $\mathbb{Z}/2 * \mathbb{Z}/m$) and the length, together with $\gamma\eta\gamma = \eta^{-1}$, $\varepsilon\eta\varepsilon = \eta$ and the order $2m$ of the quotient used for $I_2(m)$ in the proof of Theorem B. It found no mathematical error. Its required corrections, all made, concern the credit to [CP22, Proposition 6(4)], [CP02] and [Weh73], the list of known cases in [CP25, §1], the description of the methods in Section 6, the standard character of Lemma 4.3, and the number of types of the first run.

Scope and priority. The note gives a partial answer to Problem 17.16 (Corollary C); the question stays open for the types $E_6, E_7, E_8, F_4, H_3, H_4$, the case D_5 rests on the unrefereed preprint [PS26], and the equivalences concern linearity over $\mathbb{C}$. The question is due to V. G. Bardakov [KM26]; the braid group case and the principle (P1) to Bardakov and Bryukhanov [BB07]; the linearity of A to [Big01, Kra02, Dig03, CW02]; the descriptions of $\text{Aut}(A)$ in Table 2 to [DG81, CC05, GHMR00, BM07, Sor21, CP25, PS25, PS26, CP02]; Proposition 3.6 to [CC05]; Lemma 3.3 is classical [Weh73]; parabolic closures are due to [CGGMW19], and Lemma 4.1, the key step of Corollary 4.2 and Remark 4.6 rest on [CP22]. We read, in the arXiv versions (whose numbering we use; journal versions were not compared), [CC05] completely and the cited sections of [CP25, PS25, PS26, Sor21, Sor20, CP22, CP02, BM07, CGGMW19], the introduction, Theorem 1 and Corollaries 1–3 of [BB07], and Problem 17.16 in the versions 48 and 49 of [KM26]; the third verification run read them again. We did not read [DG81], [GHMR00] or [Weh73]; what we use from them is quoted from [CC05], [PS25] and [Sor20], and the statements of [GHMR00] that we use are also proved in [CP02]. The classical results [BS72, Bri71, Del72, AF59, Alp87, Big01, Kra02, Dig03, CW02] are used as stated there. The bibliographic data were checked with Crossref or the arXiv API, and with zbMATH for [Alp87, BB07]. The searches (21 and 22 October 2026) used the arXiv API (automorphisms and endomorphisms of Artin groups of spherical, finite or exceptional type, holomorphs of braid groups, linearity of automorphism groups, the word Kurovka, author searches, and the newest listings, whose latest relevant entry on 22 October was arXiv:2610.13904 of 9 October 2026), the zbMATH Open API, OpenAlex (the works citing [CC05, CP25, Sor21, BB07, PS26]; titles read), Crossref, Semantic Scholar (the works citing [BB07]; a keyword search there was refused with HTTP status 429) and five web searches. The closest results found are [BB07], the computations in Table 2, and, for D_4 , [Sor20] with [Sor21]. No printed answer to the question beyond braid groups was found in the sources searched; a search that finds nothing is not a proof of novelty, and we claim no priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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