

Elements Inverted by an Automorphism and the Sum of the Character Degrees (Kourovka Notebook Problem 16.60)

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October 18, 2026

Abstract

For a finite group G let $T(G)$ be the sum of the degrees of the irreducible complex characters of G , and for an automorphism α of G let $S_\alpha = \{g \in G : \alpha(g) = g^{-1}\}$. D. MacHale asked in the Kourovka Notebook (Problem 16.60) whether $|S_\alpha| \leq T(G)$ for every $\alpha \in \text{Aut } G$. The inequality is stated as Theorem 3.1 of a 2026 preprint of Yerrapati, Dixit and Shukla; the proof given there for the bound on twisted indicators that it uses (Lemma 2.19 there) is incomplete. We give a complete proof, in two short ways. The first follows the strategy of that preprint: $|S_\alpha| = \sum_\chi \iota_\alpha(\chi) \chi(1)$ with $\iota_\alpha(\chi) = |G|^{-1} \sum_{g \in G} \chi(g\alpha(g))$, and we prove $|\iota_\alpha(\chi)| \leq 1$ for every automorphism from the orthogonality relations for matrix coefficients. The second restricts α to the subgroup of fixed points of α^2 , on which it induces an automorphism of order at most 2, uses the twisted Frobenius–Schur count of Kawanaka and Matsuyama for involutory automorphisms, and then the inequality $T(H) \leq T(G)$ for subgroups H . We also show that equality holds if and only if $\alpha^2 = 1$ and all twisted indicators are equal to 1; in particular the inequality is strict when α has order at least 3. Special cases were known: involutory automorphisms (from the twisted Frobenius–Schur count), inner automorphisms and complete groups (Bors, 2019), and abelian groups, dihedral groups and groups of order p , $2p$, p^2 (Yerrapati, Dixit and Shukla, 2026). The novelty is modest: a complete proof of the general case and the characterisation of equality. This is an unrefereed note.

1 Introduction

1.1 The problem

Let G be a finite group, let $\text{Irr}(G)$ be the set of its irreducible complex characters, and let

$$T(G) = \sum_{\chi \in \text{Irr}(G)} \chi(1)$$

be the sum of their degrees. For $\alpha \in \text{Aut } G$ let

$$S_\alpha = \{g \in G : \alpha(g) = g^{-1}\}$$

be the set of elements inverted by α ; it contains 1, and in general it is not a subgroup. The Kourovka Notebook writes $T(G) = \sum_{i=1}^{k(G)} d_i$, where $k(G)$ is the number of conjugacy classes of G and $d_1, \dots, d_{k(G)}$ are the degrees of the irreducible complex representations, and asks the following question of D. MacHale.

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Problem 1.1 ([KM26, Problem 16.60]). *Is it true that $T(G) \geq |S_\alpha|$ for all $\alpha \in \text{Aut } G$?*

The problem was posed in the 16th issue of the Notebook (2006). We have not consulted the 2006 printing. The wording is the same, word for word, in the 2014 printing (No. 18, [KM14]) and in the current version (No. 21, arXiv version 48 of 6 October 2026, [KM26]), which prints no comment on the problem.

For $\alpha = \text{id}$ the set S_{id} consists of the elements g with $g^2 = 1$, and the inequality $|S_{\text{id}}| \leq T(G)$ is a classical consequence of the theorem of Frobenius and Schur (Section 1.3). The problem asks for all automorphisms; an automorphism may have any order, and inner automorphisms are included.

1.2 Results

For $\chi \in \text{Irr}(G)$ and $\alpha \in \text{Aut } G$ put

$$\iota_\alpha(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g \alpha(g)). \quad (1)$$

For $\alpha = \text{id}$ this is the Frobenius–Schur indicator of χ , and for $\alpha^2 = \text{id}$ it is the twisted Frobenius–Schur indicator of Kawanaka and Matsuyama [KM90]. As in [YDS26a] we use the same notation, and the name twisted indicator, for every automorphism. For $\alpha \in \text{Aut } G$ we write

$$C_G(\alpha^2) = \{g \in G : \alpha^2(g) = g\}$$

for the subgroup of fixed points of α^2 .

Theorem 1.2. *Let G be a finite group and $\alpha \in \text{Aut } G$.*

- (a) $|S_\alpha| \leq T(C_G(\alpha^2)) \leq T(G)$. In particular $|S_\alpha| \leq T(G)$.
- (b) $|S_\alpha| = T(G)$ if and only if $\alpha^2 = \text{id}$ and $\iota_\alpha(\chi) = 1$ for every $\chi \in \text{Irr}(G)$.
- (c) If $\alpha^2 \neq \text{id}$, in particular if α has order at least 3, then $|S_\alpha| < T(G)$.

So the answer to Problem 1.1 is yes. We give two proofs of the inequality $|S_\alpha| \leq T(G)$. The first (Section 4) combines the counting formula $|S_\alpha| = \sum_\chi \iota_\alpha(\chi) \chi(1)$ (Lemma 2.3) and the triangle inequality with the bound $|\iota_\alpha(\chi)| \leq 1$, which we prove for every automorphism (Proposition 3.1): $\iota_\alpha(\chi)$ is real, it vanishes unless $\chi \circ \alpha = \bar{\chi}$, and in that case $\iota_\alpha(\chi) = \text{tr}(U\bar{U})/\chi(1)$ for a unitary matrix U . For $\alpha^2 = \text{id}$ the indicator lies in $\{0, 1, -1\}$, but in general it is not an integer: for the quaternion group Q_8 , an automorphism of order 3 and the character of degree 2 one gets $\iota_\alpha(\chi) = \frac{1}{2}$ (Example 3.3). The second proof (Section 5) gives the sharper form (a): S_α lies in $H = C_G(\alpha^2)$, the restriction of α to H is an involutory automorphism of H with the same inverted elements, the involutory case gives $|S_\alpha| \leq T(H)$, and $T(H) \leq T(G)$ by Frobenius reciprocity, with equality only for $H = G$. Parts (b) and (c) are proved in Section 6, together with the special cases and examples.

1.3 What is known and what is new

The case $\alpha = \text{id}$ is classical: by the theorem of Frobenius and Schur, $|S_{\text{id}}| = \sum_\chi \nu(\chi) \chi(1)$ with $\nu(\chi) = \iota_{\text{id}}(\chi) \in \{0, 1, -1\}$ [Isa06, Chapter 4]. For automorphisms τ with $\tau^2 = 1$, Kawanaka and Matsuyama [KM90] introduced the twisted indicator (1) and showed that some of the standard properties of the Frobenius–Schur indicator generalise; Bump and Ginzburg [BG04] studied the analogous sums for an automorphism τ with $\tau^r = 1$, with $g\tau(g) \cdots \tau^{r-1}(g)$ in place of $g\tau(g)$. Of [KM90] and [BG04] we have read only the zbMATH reviews, not the papers. Vinroot [Vin20, Section 2] states, for an

automorphism τ of G with $\tau^2 = 1$, the twisted count $\sum_{\chi} \varepsilon_{\tau}(\chi) \chi(1) = \#\{g \in G : \tau(g) = g^{-1}\}$ with $\varepsilon_{\tau}(\chi) \in \{0, 1, -1\}$, and the criterion that this number equals $T(G)$ if and only if all $\varepsilon_{\tau}(\chi)$ are 1 [Vin20, Lemma 2.1]; he attributes the basic properties of the twisted indicator to [KM90, BG04]. For involutory α the inequality $|S_{\alpha}| \leq T(G)$ is therefore an immediate consequence of known results.

Bors [Bor19, Section 2.2] showed that the number of elements inverted by an inner automorphism of a finite group G is at most $T(G)$: by his Proposition 2.2.2 this number is the number of square roots of an element, and the Frobenius–Schur count of square roots bounds it by $T(G)$, as in the proof of his Corollary 2.2.5; and Corollary 2.2.5 states the inequality $|S_{\alpha}| \leq T(G)$ for all automorphisms of a complete group. (We use the numbering of the arXiv version of [Bor19]; the journal version was not compared.) Yerrapati, Dixit and Shukla proved the inequality for abelian groups, for dihedral groups and for groups of order p , $2p$ and p^2 [YDS26b, Lemma 2.6, Corollary 2.4, Theorem 2.7], and they state it for all finite groups and all automorphisms as [YDS26a, Theorem 3.1].

Their proof of [YDS26a, Theorem 3.1] combines the counting formula [YDS26a, Lemma 2.20] (our Lemma 2.3), a bound on the twisted indicators, and the triangle inequality. The bound is [YDS26a, Lemma 2.19], whose conclusion reads: “For any automorphism the twisted Frobenius–Schur indicator $\iota_{\alpha}(\chi) \leq 1$.” The proof of Theorem 3.1 uses it in the form $|\iota_{\alpha}(\chi)| \leq 1$. The proof given for Lemma 2.19 is incomplete. It passes from the definition, display (2) on p. 6 of [YDS26a], to the bound (3) by the Cauchy–Schwarz inequality; but the right-hand side of (3) is the Cauchy–Schwarz bound for $|G|^{-1}|\sum_g \chi(g)\chi(\alpha(g))|$, not for $|\iota_{\alpha}(\chi)| = |G|^{-1}|\sum_g \chi(g\alpha(g))|$. The two sums agree term by term when χ is linear, but not when χ is not linear: already at $g = 1$ the two summands are d and d^2 , where $d = \chi(1)$. The statement of the lemma, with the absolute value, is true; it is Proposition 3.1(c) below.

What is new here is the proof of the bound $|\iota_{\alpha}(\chi)| \leq 1$ for an arbitrary automorphism (Proposition 3.1), a second proof of the inequality through the reduction to the involutory case on $C_G(\alpha^2)$ (Section 5), and the characterisation of equality, Theorem 1.2(b) and (c). Our first proof follows the strategy of [YDS26a]. The novelty is modest: the involutory case is classical, the reduction to it takes a few lines, and the statement of the theorem is in [YDS26a].

2 Preliminaries

Throughout, G is a finite group and $\langle \varphi, \psi \rangle = |G|^{-1} \sum_{g \in G} \varphi(g) \overline{\psi(g)}$ is the inner product of class functions. A representation of degree d is a homomorphism $\rho : G \rightarrow \mathrm{GL}_d(\mathbb{C})$; for a matrix A we write $\overline{A}$ for the entrywise complex conjugate, $A^{\top}$ for the transpose and $A^* = \overline{A}^{\top}$. We call an automorphism α *involutory* if $\alpha^2 = \mathrm{id}$ (the identity included). We use the following standard facts, see e.g. [Ser77, Chapter 2] or [Isa06, Chapter 2].

- (F1) Every representation is equivalent to a unitary one (average a Hermitian inner product over G). If ρ is unitary, then $\rho(g^{-1}) = \rho(g)^*$, that is, $\rho(g^{-1})_{ab} = \overline{\rho(g)_{ba}}$.
- (F2) (Schur’s lemma.) Let ρ, ρ' be irreducible representations of degrees d, d' , and let X be a $d \times d'$ matrix with $\rho(g)X = X\rho'(g)$ for all $g \in G$. If ρ and ρ' are not equivalent, then $X = 0$; if $\rho' = \rho$, then X is a scalar matrix.
- (F3) Two representations are equivalent if and only if they have the same character, and the character χ of a representation is irreducible if and only if $\langle \chi, \chi \rangle = 1$.

Let ρ be an irreducible representation with character χ , and let $\alpha \in \mathrm{Aut} G$. Then $g \mapsto \rho(\alpha(g))$ is a representation with character $\chi \circ \alpha$, and it is irreducible by (F3), since $\langle \chi \circ \alpha, \chi \circ \alpha \rangle = \langle \chi, \chi \rangle = 1$ (α permutes G). Entrywise conjugation is a ring automorphism of the matrix algebra and preserves

unitarity, so $g \mapsto \overline{\rho(g)}$ is a representation with character $\bar{\chi}$, again irreducible, and unitary if ρ is. We include the short proofs of the next two lemmas, since the proof of Proposition 3.1 uses the precise form of Lemma 2.1.

Lemma 2.1 (orthogonality of matrix coefficients). *Let ρ, ρ' be irreducible representations of G of degrees d, d' . For all indices $1 \leq a, e \leq d$ and $1 \leq b, c \leq d'$,*

$$\frac{1}{|G|} \sum_{g \in G} \rho(g)_{ab} \rho'(g^{-1})_{ce} = \begin{cases} 0 & \text{if } \rho \text{ and } \rho' \text{ are not equivalent,} \\ \frac{\delta_{ae} \delta_{bc}}{d} & \text{if } \rho' = \rho. \end{cases} \quad (2)$$

Proof. For a $d \times d'$ matrix X put $P(X) = |G|^{-1} \sum_g \rho(g) X \rho'(g)^{-1}$. For $h \in G$, replacing g by hg gives $\rho(h)P(X)\rho'(h)^{-1} = P(X)$, so $\rho(h)P(X) = P(X)\rho'(h)$. By (F2), $P(X) = 0$ if $\rho \not\cong \rho'$, and $P(X) = \mu I$ if $\rho' = \rho$, where $\mu d = \text{tr } P(X) = |G|^{-1} \sum_g \text{tr}(\rho(g) X \rho(g)^{-1}) = \text{tr } X$. For the matrix unit $X = E_{bc}$ the (a, e) entry of $P(E_{bc})$ is the left side of (2), and $(\text{tr } E_{bc}/d) \delta_{ae} = \delta_{bc} \delta_{ae}/d$. $\square$

Lemma 2.2 (regular character). *For $x \in G$, $\sum_{\chi \in \text{Irr}(G)} \chi(1)\chi(x)$ equals $|G|$ if $x = 1$ and 0 otherwise.*

Proof. The left regular representation permutes the basis $\{e_h : h \in G\}$ of the group algebra by $e_h \mapsto e_{xh}$ and fixes no basis vector if $x \neq 1$; so its character ρ_{reg} is $|G|$ at 1 and 0 elsewhere. Hence $\langle \rho_{\text{reg}}, \chi \rangle = \chi(1)$ for every $\chi \in \text{Irr}(G)$, and $\rho_{\text{reg}} = \sum_{\chi} \chi(1)\chi$, since the irreducible characters form an orthonormal basis of the class functions [Isa06, Chapter 2]. $\square$

Lemma 2.3 (counting formula; [YDS26a, Lemma 2.20]). *For every $\alpha \in \text{Aut } G$,*

$$|S_{\alpha}| = \sum_{\chi \in \text{Irr}(G)} \iota_{\alpha}(\chi) \chi(1).$$

Proof. By Lemma 2.2, $\sum_{\chi} \iota_{\alpha}(\chi) \chi(1) = |G|^{-1} \sum_{g \in G} \sum_{\chi} \chi(1) \chi(g\alpha(g)) = \#\{g \in G : g\alpha(g) = 1\}$, and $g\alpha(g) = 1$ means $\alpha(g) = g^{-1}$. $\square$

The proof uses only that α is a map from G to G . For $\alpha = \text{id}$ the formula is the Frobenius–Schur count of the elements g with $g^2 = 1$; for involutory α it is the twisted count of [Vin20, Section 2].

3 Twisted indicators of an arbitrary automorphism

Proposition 3.1. *Let $\alpha \in \text{Aut } G$ and $\chi \in \text{Irr}(G)$, let $d = \chi(1)$, and let ρ be a unitary representation with character χ .*

- (a) $\iota_{\alpha}(\chi)$ is a real number.
- (b) If $\chi \circ \alpha \neq \bar{\chi}$, then $\iota_{\alpha}(\chi) = 0$.
- (c) If $\chi \circ \alpha = \bar{\chi}$, then there is a unitary matrix U such that $\overline{\rho(\alpha(g))} = U\rho(g)U^{-1}$ for all $g \in G$, and

$$\iota_{\alpha}(\chi) = \frac{\text{tr}(U\bar{U})}{d}.$$

In all cases $|\iota_{\alpha}(\chi)| \leq 1$.

- (d) If α is involutory, then $\iota_{\alpha}(\chi) \in \{0, 1, -1\}$; more precisely, in (c) one has $U\bar{U} = \iota_{\alpha}(\chi)I$.

Proof. (a) Since $\chi(x^{-1}) = \overline{\chi(x)}$ and $\chi(xy) = \chi(yx)$,

$$\overline{\iota_\alpha(\chi)} = \frac{1}{|G|} \sum_{g \in G} \chi(\alpha(g^{-1})g^{-1}) = \frac{1}{|G|} \sum_{h \in G} \chi(\alpha(h)h) = \frac{1}{|G|} \sum_{h \in G} \chi(h\alpha(h)) = \iota_\alpha(\chi).$$

(b) and (c). Put $\rho'(g) = \overline{\rho(\alpha(g))}$. By Section 2, ρ' is an irreducible unitary representation with character $\overline{\chi \circ \alpha}$; by (F3), ρ' is equivalent to ρ if and only if $\overline{\chi \circ \alpha} = \chi$, that is, if and only if $\chi \circ \alpha = \overline{\chi}$. By (F1) for ρ' , $\rho(\alpha(g))_{ba} = \rho'(g)_{ba} = \rho'(g^{-1})_{ab}$, hence

$$\chi(g\alpha(g)) = \text{tr}(\rho(g)\rho(\alpha(g))) = \sum_{a,b} \rho(g)_{ab} \rho(\alpha(g))_{ba} = \sum_{a,b} \rho(g)_{ab} \rho'(g^{-1})_{ab},$$

and therefore

$$\iota_\alpha(\chi) = \sum_{a,b=1}^d \frac{1}{|G|} \sum_{g \in G} \rho(g)_{ab} \rho'(g^{-1})_{ab}. \quad (3)$$

If $\chi \circ \alpha \neq \overline{\chi}$, then $\rho' \not\cong \rho$, every inner sum in (3) vanishes by Lemma 2.1 (with $(c, e) = (a, b)$), and $\iota_\alpha(\chi) = 0$. This proves (b).

Now let $\chi \circ \alpha = \overline{\chi}$. Then there is an invertible matrix V with $\rho'(g) = V\rho(g)V^{-1}$ for all g . Taking adjoints in $V\rho(g) = \rho'(g)V$ and using $\rho(g)^* = \rho(g^{-1})$ and $\rho'(g)^* = \rho'(g^{-1})$ gives $\rho(g^{-1})V^* = V^*\rho'(g^{-1})$, hence $\rho(g^{-1})V^*V = V^*\rho'(g^{-1})V = V^*V\rho(g^{-1})$ for all g . By (F2), $V^*V = sI$ for a scalar s , and $s > 0$ since V^*V is positive definite. So $U = s^{-1/2}V$ is unitary and $\rho'(g) = U\rho(g)U^{-1}$ for all g , which is the first claim of (c). Substituting $\rho'(g^{-1})_{ab} = \sum_{c,e} U_{ac} \rho(g^{-1})_{ce} (U^{-1})_{eb}$ into (3) and applying Lemma 2.1 with $\rho' = \rho$, we obtain

$$\iota_\alpha(\chi) = \sum_{a,b,c,e} U_{ac}(U^{-1})_{eb} \frac{\delta_{ae}\delta_{bc}}{d} = \frac{1}{d} \sum_{a,b} U_{ab}(U^{-1})_{ab} = \frac{1}{d} \sum_{a,b} U_{ab} \overline{U_{ba}} = \frac{\text{tr}(U\overline{U})}{d},$$

where $(U^{-1})_{ab} = (U^*)_{ab} = \overline{U_{ba}}$ was used. The matrix $U\overline{U}$ is a product of two unitary matrices, hence unitary, so its d eigenvalues have absolute value 1 and $|\text{tr}(U\overline{U})| \leq d$. Together with (b), $|\iota_\alpha(\chi)| \leq 1$ in all cases.

(d) Let $\alpha^2 = \text{id}$. By (b) we may assume $\chi \circ \alpha = \overline{\chi}$; let U be as in (c). Applying $\overline{\rho(\alpha(h))} = U\rho(h)U^{-1}$ to $h = \alpha(g)$ gives $\overline{\rho(g)} = U\rho(\alpha(g))U^{-1}$, and conjugating entrywise,

$$\rho(g) = \overline{U} \overline{\rho(\alpha(g))} \overline{U}^{-1} = (\overline{U}U) \rho(g) (\overline{U}U)^{-1} \quad (g \in G).$$

By (F2), $\overline{U}U = \lambda I$ for a scalar λ . Then $U\overline{U} = U(\overline{U}U)U^{-1} = \lambda I$, so $\iota_\alpha(\chi) = \text{tr}(U\overline{U})/d = \lambda$. Finally $\overline{U}^{-1} = \overline{U^{-1}} = \overline{U}^* = U^\top$, so $\overline{U}U = \lambda I$ gives $U = \lambda \overline{U}^{-1} = \lambda U^\top$; transposing, $U^\top = \lambda U$, hence $U = \lambda^2 U$ and $\lambda^2 = 1$. $\square$

Remark 3.2. For $\alpha = \text{id}$, part (d) is the theorem of Frobenius and Schur, and for involutory α it is the integrality of the twisted Frobenius–Schur indicator of [KM90]; see [Vin20, Section 2] for the description of the sign through an invariant bilinear form. The proof of (d) is the usual one; parts (b) and (c) extend it to automorphisms of arbitrary order. The matrix U in (c) is unique up to a factor of absolute value 1, by (F2), so $U\overline{U}$ depends only on α and ρ . For α of order at least 3 the indicator need not be an integer, as the next example shows, so an argument that uses only the values $0, \pm 1$ cannot cover these automorphisms; the bound $|\iota_\alpha(\chi)| \leq 1$ of (c) is what the first proof needs.

Example 3.3 (the quaternion group). Let $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ and let α be the automorphism of order 3 with $\alpha(i) = j$, $\alpha(j) = k$, $\alpha(k) = i$ (and $\alpha(-x) = -\alpha(x)$). Then $\alpha(\pm 1) = \pm 1$, while $\alpha(g) \neq$

$-g = g^{-1}$ for $g \in \{\pm i, \pm j, \pm k\}$; so $S_\alpha = \{1, -1\}$ and $|S_\alpha| = 2$, whereas $T(Q_8) = 1 + 1 + 1 + 1 + 2 = 6$. The trivial character has $\iota_\alpha = 1$. The three non-trivial linear characters, with kernels $\langle i \rangle, \langle j \rangle, \langle k \rangle$, are permuted cyclically by α ; they are real, so $\lambda \circ \alpha \neq \bar{\lambda}$ and $\iota_\alpha(\lambda) = 0$ by Proposition 3.1(b). The character χ of degree 2 has $\chi(\pm 1) = \pm 2$ and vanishes elsewhere. For $g = \pm 1$ we have $g\alpha(g) = 1$, and for the other six elements $g\alpha(g) \in \{\pm i, \pm j, \pm k\}$ (for instance $i\alpha(i) = ij = k$). Hence $\iota_\alpha(\chi) = (2 + 2)/8 = \frac{1}{2}$, and Lemma 2.3 reads $2 = 1 \cdot 1 + 3 \cdot (0 \cdot 1) + \frac{1}{2} \cdot 2$. The fixed points of α^2 are ± 1 , so $C_G(\alpha^2) \cong C_2$ and $T(C_G(\alpha^2)) = 2 = |S_\alpha|$: the first inequality of Theorem 1.2(a) can be an equality for an automorphism of order 3.

4 First proof of the inequality

First proof of $|S_\alpha| \leq T(G)$. By Lemma 2.3 and Proposition 3.1(c),

$$|S_\alpha| = \left| \sum_{\chi \in \text{Irr}(G)} \iota_\alpha(\chi) \chi(1) \right| \leq \sum_{\chi \in \text{Irr}(G)} |\iota_\alpha(\chi)| \chi(1) \leq \sum_{\chi \in \text{Irr}(G)} \chi(1) = T(G). \quad \square$$

This is the argument of the proof of [YDS26a, Theorem 3.1], with Proposition 3.1(c) in place of [YDS26a, Lemma 2.19]. It does not give the intermediate bound $T(C_G(\alpha^2))$ of Theorem 1.2(a); for that we use the second proof.

5 Second proof: reduction to an involutory automorphism

Lemma 5.1. *Let $\alpha \in \text{Aut } G$ and $H = C_G(\alpha^2)$. Then $S_\alpha \subseteq H$ and $\alpha(H) = H$. The restriction $\beta = \alpha|_H$ is an involutory automorphism of H , and $S_\beta = S_\alpha$.*

Proof. H is a subgroup, being the set of fixed points of the automorphism α^2 . If $\alpha(g) = g^{-1}$, then $\alpha^2(g) = \alpha(g^{-1}) = \alpha(g)^{-1} = g$; so $S_\alpha \subseteq H$. If $h \in H$, then $\alpha^2(\alpha(h)) = \alpha(\alpha^2(h)) = \alpha(h)$; so $\alpha(H) \subseteq H$, and $\alpha(H) = H$ because α is injective and H is finite. Hence $\beta \in \text{Aut } H$, $\beta^2 = \alpha^2|_H = \text{id}_H$, and $S_\beta = \{h \in H : \alpha(h) = h^{-1}\} = S_\alpha \cap H = S_\alpha$. $\square$

Lemma 5.2. *Let H be a subgroup of G . Then $T(H) \leq T(G)$, and $T(H) < T(G)$ if $H \neq G$.*

Proof. Let $\tau_G = \sum_{\chi \in \text{Irr}(G)} \chi$, so that $T(G) = \tau_G(1)$. Write the restriction of τ_G to H as $\sum_{\psi \in \text{Irr}(H)} m_\psi \psi$. By Frobenius reciprocity [Isa06, Chapter 5],

$$m_\psi = \sum_{\chi \in \text{Irr}(G)} \langle \chi_H, \psi \rangle_H = \sum_{\chi \in \text{Irr}(G)} \langle \chi, \psi^G \rangle_G,$$

which is the number of irreducible constituents of the induced character ψ^G , counted with multiplicity. Since ψ^G is a character of degree $[G : H]\psi(1) > 0$, we have $m_\psi \geq 1$, and $T(G) = \tau_G(1) = \sum_{\psi} m_\psi \psi(1) \geq \sum_{\psi} \psi(1) = T(H)$. If $H \neq G$, consider $\psi = 1_H$. The permutation character $(1_H)^G$ has degree $[G : H] \geq 2$ and contains 1_G exactly once, since $\langle (1_H)^G, 1_G \rangle_G = \langle 1_H, 1_H \rangle_H = 1$; so $(1_H)^G - 1_G$ is a non-zero character, $m_{1_H} \geq 2$, and $T(G) \geq T(H) + 1$. $\square$

The inequality $T(H) \leq T(G)$ is well known; the difference $T(G) - T(H)$ is studied by Berkovich and Mann [BM98] (we have read only the zbMATH review of that paper).

Proof of Theorem 1.2(a). Let $H = C_G(\alpha^2)$ and $\beta = \alpha|_H$. By Lemma 5.1, Lemma 2.3 for (H, β) and Proposition 3.1(d) for (H, β) ,

$$|S_\alpha| = |S_\beta| = \sum_{\psi \in \text{Irr}(H)} \iota_\beta(\psi) \psi(1) \leq \sum_{\psi \in \text{Irr}(H)} \psi(1) = T(H),$$

since each $\iota_\beta(\psi)$ lies in $\{0, 1, -1\}$. By Lemma 5.2, $T(H) \leq T(G)$. $\square$

This proof applies Proposition 3.1 only to the involutory automorphism β , that is, it uses only the classical twisted Frobenius–Schur theory, and not the bound of Proposition 3.1(c) for automorphisms of order at least 3.

6 Equality, special cases and examples

Proof of Theorem 1.2(b) and (c). (b) If $\alpha^2 = \text{id}$ and $\iota_\alpha(\chi) = 1$ for all χ , then Lemma 2.3 gives $|S_\alpha| = \sum_\chi \chi(1) = T(G)$. Conversely, let $|S_\alpha| = T(G)$. By (a), $T(G) = |S_\alpha| \leq T(C_G(\alpha^2)) \leq T(G)$, so $T(C_G(\alpha^2)) = T(G)$, and $C_G(\alpha^2) = G$ by Lemma 5.2; that is, $\alpha^2 = \text{id}$. Now Lemma 2.3 gives $\sum_\chi (1 - \iota_\alpha(\chi))\chi(1) = T(G) - |S_\alpha| = 0$, where every term is non-negative because $\iota_\alpha(\chi) \leq 1$ (Proposition 3.1(d)); hence $\iota_\alpha(\chi) = 1$ for every χ .

(c) If $\alpha^2 \neq \text{id}$, then $C_G(\alpha^2) \neq G$, and (a) together with Lemma 5.2 gives $|S_\alpha| \leq T(C_G(\alpha^2)) < T(G)$. $\square$

Remark 6.1 (equality from the first proof). Part (b) also follows from Proposition 3.1 alone. If $|S_\alpha| = T(G)$, then the inequalities in Section 4 are equalities, so $\iota_\alpha(\chi) = 1$ for every χ (each $\iota_\alpha(\chi)$ is real and at most 1). Let ρ be an irreducible unitary representation, with U as in Proposition 3.1(c). The unitary matrix $U\bar{U}$ has trace d and eigenvalues of absolute value 1, so all its eigenvalues are 1, and $U\bar{U} = I$ since unitary matrices are diagonalisable. Hence $\bar{U} = U^{-1}$ and $\bar{U}U = I$. From $\overline{\rho(\alpha(g))} = U\rho(g)U^{-1}$ we get $\rho(\alpha(g)) = \bar{U}\overline{\rho(g)}\bar{U}^{-1}$, and with $\alpha(g)$ in place of g , $\rho(\alpha^2(g)) = \bar{U}\overline{\rho(\alpha(g))}\bar{U}^{-1} = (\bar{U}U)\rho(g)(\bar{U}U)^{-1} = \rho(g)$. So $\alpha^2(g)g^{-1}$ lies in the kernel of every irreducible representation, hence in the kernel of the regular representation, and $\alpha^2 = \text{id}$.

Remark 6.2 (inner automorphisms; Bors). Let $t \in G$ and $\alpha(x) = txt^{-1}$. If $txt^{-1} = x^{-1}$, then $(xt)^2 = x(txt^{-1})t^2 = t^2$; conversely, if $r^2 = t^2$, then $x = rt^{-1}$ satisfies $txt^{-1} = trt^{-2} = trr^{-2} = tr^{-1} = x^{-1}$. So $x \mapsto xt$ is a bijection from S_α onto the set of square roots of t^2 , and $|S_\alpha| = \theta(t^2)$, where $\theta(y) = \#\{r \in G : r^2 = y\}$. The class function θ satisfies $\langle \theta, \chi \rangle = |G|^{-1} \sum_{r \in G} \overline{\chi(r^2)} = \iota_{\text{id}}(\chi) = \iota_{\text{id}}(\chi)$, so $\theta = \sum_\chi \iota_{\text{id}}(\chi)\chi$ and

$$|S_\alpha| = \sum_{\chi \in \text{Irr}(G)} \iota_{\text{id}}(\chi) \chi(t^2) \leq \sum_{\chi \in \text{Irr}(G)} \chi(1) = T(G),$$

because $|\iota_{\text{id}}(\chi)| \leq 1$ and $|\chi(y)| \leq \chi(1)$. This is the argument of Bors [Bor19, Proposition 2.2.2 and the proof of Corollary 2.2.5], and it covers every automorphism of a complete group [Bor19, Corollary 2.2.5]. If $t^2 = 1$ it gives $|S_\alpha| = |S_{\text{id}}|$; for the symmetric groups this is [YDS26a, Corollary 3.6].

Example 6.3. (1) *Abelian groups.* If G is abelian and $\alpha(g) = g^{-1}$, then $S_\alpha = G$, $\iota_\alpha(\chi) = |G|^{-1} \sum_g \chi(1) = 1$ for every χ , and $|S_\alpha| = |G| = T(G)$. For abelian G and any α , S_α is a subgroup and $|S_\alpha| \leq |G| = T(G)$ trivially [YDS26b, Lemma 2.6].

(2) *The identity.* $|S_{\text{id}}| = T(G)$ holds if and only if $\iota_{\text{id}}(\chi) = 1$ for every χ , that is, if and only if every irreducible representation of G is equivalent to a real one [Isa06, Chapter 4]. For instance $|S_{\text{id}}| = T(G)$ equals 4, 10 and 26 for $G = S_3, S_4, S_5$; for all symmetric groups see [YDS26a, Lemma 2.18] and the references given there.

(3) *An outer automorphism with equality.* Let $G = \text{GL}(3, 2) \cong \text{PSL}(2, 7)$, of order 168, with character degrees 1, 3, 3, 6, 7, 8 and $T(G) = 28$, and let $\alpha(g) = (g^\top)^{-1}$. Then α is an involutory automorphism, and $\alpha(g) = g^{-1}$ if and only if $g^\top = g$; so S_α is the set of invertible symmetric 3×3 matrices over $\mathbb{F}_2$, which has 28 elements. Hence $|S_\alpha| = T(G)$, and by Theorem 1.2(b) all twisted indicators $\iota_\alpha(\chi)$ are equal to 1. This is an instance of [Vin20, Theorem 10.1], which lists, for the finite

classical groups, involutory automorphisms all of whose twisted indicators are 1 (transpose-inverse for the general linear groups), and which collects results of several papers, among them [Vin05, TV18]. By contrast, $|S_{\text{id}}| = 1 + 21 = 22 < 28$, since the two characters of degree 3 are not real-valued. By the computation of Section 7, α is not inner, and among the 336 automorphisms of G exactly the 28 outer involutory ones attain equality.

(4) *Strict inequality for every automorphism.* For the non-abelian group $C_7 \rtimes C_3$ of order 21 ($T = 9$) the largest value of $|S_\alpha|$ over its 42 automorphisms is 7, and for $\text{AGL}(1, 5) \cong C_5 \rtimes C_4$ of order 20 ($T = 8$) the largest value over its 20 automorphisms is 6 (Section 7). Neither group has an automorphism with all twisted indicators equal to 1.

7 Computations and reproducibility

No proof in this note depends on a computation. The program `check_note.py` of the reproducibility package (Python standard library only; a few seconds) checks the statements of the note in exact arithmetic on 42 groups: the cyclic groups of orders 1 to 10 and 12; C_2^2 , C_2^3 , C_3^2 , $C_4 \times C_2$ and C_4^2 ; the dihedral groups of orders 6, 8, 10, 12, 14, 16, 18, 20 and 24; Q_8 , the generalised quaternion group of order 16 and the dicyclic group of order 12; A_4 , S_4 , A_5 and S_5 ; the direct products $C_2 \times S_3$, $C_3 \times S_3$, $S_3 \times S_3$, $C_2 \times Q_8$, $C_2 \times D_8$ and $C_2 \times A_4$; and $C_7 \rtimes C_3$, $\text{AGL}(1, 5)$, $\text{SL}(2, 3)$ and $\text{GL}(3, 2)$. For each group the program computes the character table with values in $\mathbb{Z}[\zeta_e]$, where e is the exponent of the group, from the central characters modulo a prime $p \equiv 1 \pmod{e}$, and certifies it in exact arithmetic: every row satisfies the relations of the class algebra, the value at 1 and the norm relation, and the rows are distinct, so they are the irreducible characters. It enumerates the automorphism group completely (1,778 automorphisms in all; the orders agree with the known values) and checks for every automorphism α : Lemma 2.3; Proposition 3.1(a), (b) and (d); the bound $|\iota_\alpha(\chi)| \leq 1$ of Proposition 3.1(c), exactly when $\iota_\alpha(\chi)$ is rational and otherwise in floating point with a margin of 10^{-9} , far above the rounding error; Lemma 5.1; Theorem 1.2(a), with $T(C_G(\alpha^2))$ computed from the exact character table of $C_G(\alpha^2)$; and Theorem 1.2(b) and (c). It also checks the count of Remark 6.2 for every inner automorphism, the statements of Examples 3.3 and 6.3, and $T(G \times H) = T(G)T(H)$ for eight direct products of the list. All checks pass; 209 of the 1,778 automorphisms attain equality, all of them involutory.

Verification. The results were first obtained with written proofs of Theorem 1.2 (both proofs and the characterisation of equality) and a floating-point program, which tested them on 38 groups (1,315 automorphisms). Three independent verification runs, all AI-assisted, followed; each wrote its own programs. The first two re-derived both proofs and the characterisation of equality step by step before this note was written, and neither read the other's report; the third examined the final text. The first run compared the statement with arXiv versions 1 (2014), 46 and 48 (2026) of the Notebook, read the preprints [YDS26a, YDS26b] and confirmed the gap in the proof of [YDS26a, Lemma 2.19], and tested the statements in floating point on 44 groups of orders 4 to 720 (7,104 automorphisms), among them S_6 , A_6 and $\text{PSL}(2, 7)$ with all their automorphisms; the outer automorphisms of S_6 invert at most $36 < 76 = T(S_6)$ elements, which is the value computed in [YDS26a, Theorem 3.5]. The second run rebuilt the list of all 586 groups of order at most 64 up to isomorphism (for every order the number of groups agrees with the known one), computed their character tables exactly by a modular method, and checked Theorem 1.2, Lemma 2.3 and Proposition 3.1(a), (b), (d) and the bound of (c) for every automorphism of every group of order at most 64, except three abelian groups (C_2^5 , $C_4 \times C_2^4$ and C_2^6), for which the inequality is trivial and 200,000 automorphisms each were checked: 4,319,984 automorphisms in the complete enumerations, without exception. It also checked nine groups of orders 72 to 720 with all their automorphisms, the identity $\iota_\alpha(\chi) = \text{tr}(U\overline{U})/d$

numerically with explicit unitary matrices in 9,391 cases, and the count of Remark 6.2 on all 586 groups; it reproduced the recorded output of the original program exactly and repeated the search of the literature. Neither of these two runs found a mathematical error. The first run required neutral wording on [YDS26a], crediting its strategy and quoting its Lemma 2.19 exactly, a statement of the printings of the Notebook that were compared, and that counts obtained from rate-limited services be marked as unverified; the second required that [Bor19] be credited for inner automorphisms and complete groups, and [Vin20], with [KM90, BG04], for the twisted count. All corrections were made. Remark 6.1, the proof written in Remark 6.2, the description of the example of $GL(3, 2)$ through symmetric matrices, and the program of this note were added when the note was written.

The third run examined the final text, these additions first. It re-derived every lemma, proposition and proof from the text as written, including the indices, the conjugations and the unitarisation of the intertwiner in Proposition 3.1, the strictness in Lemma 5.2 and both directions of Theorem 1.2(b); it compared the quoted statements with the texts of [YDS26a, YDS26b, Bor19, Vin20], with the zbMATH reviews of [KM90, BG04] and with arXiv versions 1, 46 and 48 of the Notebook, and checked the bibliographic data of all references with Crossref and the arXiv API. Its own programs (floating-point character tables and a separate enumeration of the automorphism groups) checked Lemma 2.3, Proposition 3.1(a), (b), (d) and the bound of (c), Lemma 5.1, Lemma 5.2 on the subgroups $C_G(\alpha^2)$ and on random subgroups, Theorem 1.2(a)–(c) and the count of Remark 6.2 for every automorphism of 59 groups of orders 1 to 720 (7,832 automorphisms; among them the 42 groups of this section, for which every figure of the recorded output of `check_note.py` was reproduced, and S_6 , A_6 and $PSL(2, 11)$ with all their automorphisms), the statements of Examples 3.3 and 6.3 as printed, and the identity $\iota_\alpha(\chi) = \text{tr}(U\bar{U})/d$ with explicit unitary matrices in 6,326 cases for 17 groups; it ran the program of this note from the archived source and reproduced its recorded output exactly, and it repeated the search of the literature on 18 October 2026. It found no mathematical error and required three corrections of the text, which were made: this description, the exact number of direct products checked above, and the date of the repeated search. No proof depends on a computation.

Scope and priority. The note answers Problem 1.1 completely (Theorem 1.2(a)) and characterises the cases of equality (Theorem 1.2(b), (c)). The problem is MacHale’s [KM26, Problem 16.60]. The case $\alpha = \text{id}$ is the theorem of Frobenius and Schur; the involutory case follows from the twisted Frobenius–Schur count of Kawanaka and Matsuyama [KM90] (see also [BG04]), which is printed in this form in [Vin20, Section 2]; inner automorphisms and complete groups are due to Bors [Bor19, Section 2.2]; abelian groups, dihedral groups and groups of order p , $2p$, p^2 are due to Yerrapati, Dixit and Shukla [YDS26b]; and the full statement is [YDS26a, Theorem 3.1], where the proof of Lemma 2.19 is incomplete as written (Section 1.3). Our contribution is a complete proof of the general case, in two forms, and the characterisation of equality; the novelty is modest, and no priority is claimed.

What was read: [YDS26a] and [YDS26b] completely (version 1); [Bor19] Section 2.2, in the arXiv version, the journal version not being compared; [Vin20] Section 1, Section 2.2 and Theorem 10.1, in the arXiv version, the journal version not being compared; of [KM90], [BG04] and [BM98] only the zbMATH reviews; of [Vin05, TV18] only what [Vin20] says about them; and Problem 16.60 in arXiv versions 1 and 48 of the Notebook, the 2006 printing not being consulted. The proofs of this note do not depend on any of these texts.

The searches were made on 17 October 2026: queries to the arXiv (twisted Frobenius–Schur indicators, automorphisms inverting many elements, MacHale, sums of character degrees together with automorphisms, twisted involutions, the Kourovka Notebook), to zbMATH Open (all records of D. MacHale, topic queries and reviews), to Crossref and to Semantic Scholar, a look at the website of the Notebook, and six web searches. The third verification run repeated the searches on 18 October

2026 (arXiv, zbMATH Open and Crossref; Semantic Scholar and OpenAlex refused its requests) and found nothing new. Apart from [YDS26a], whose proof is incomplete as written, no printed answer to the question was found in the sources searched; a search that finds nothing is not a proof of novelty. On 17 October 2026 Semantic Scholar listed no work citing [YDS26a] or [YDS26b], and 49, 46 and 18 works citing [KM90], [BG04] and [Vin05], whose titles were scanned; apart from the two preprints [YDS26a, YDS26b], none of these titles concerns the question. This service limited our requests, so that only the count for [YDS26a] could be checked a second time; the other counts are unverified. OpenAlex refused all requests (rate limiting) and was not used. The website of the Notebook showed no post on Problem 16.60 on 17 October 2026.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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