

Lower Bounds for the Finite Quotients of Conder's Group on the Cubic Tree (Kourovka Notebook Problem 16.46)

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October 24, 2026

Abstract

Problem 16.46 of the Kourovka Notebook, posed by M. Conder and going back to his paper of 1992, concerns the free product $G = G_3 *_{G_1} G_4$ of two of the Djoković–Miller groups that act arc-transitively on the cubic tree, amalgamated along the modular group G_1 , and the normal subgroup K of index 8 of G , which has a presentation with six generators and 22 relators: does K have a non-trivial finite quotient? Conder found no proper subgroup of finite index. We do not decide the question. We prove, without computation, that K is perfect, that in every non-trivial image of K the generators keep their orders and the symmetric group S_4 embeds, and that K is the amalgamated free product of four copies of $L = S_4 *_{D_8} S_4$ over a common subgroup $C_3 * C_3$, glued along its four sign automorphisms; a non-trivial finite quotient of K gives one of the subgroup J generated by two of these copies. By exhaustive computer searches we show that J , and hence K , has no non-trivial action on at most 36 points, so K has no subgroup of index between 2 and 36, and, with GAP, that no non-abelian finite simple group of order less than 10^8 receives a non-trivial homomorphism from J . Hence every non-trivial finite quotient of K has order at least 10^8 . Jambor (2015) proved, by a machine computation, that K has no quotient $\mathrm{PSL}_2(q)$ or $\mathrm{PGL}(2, q)$; together with a search of the twelve other simple groups of order between 10^8 and 10^9 , every non-trivial finite quotient of K has order at least 10^9 . In contrast, the amalgam of two copies differing by the twist $(A, B) \mapsto (A^{-1}, B^{-1})$ maps onto A_6 and $\mathrm{PSL}(2, 7)$. The bounds are computer-assisted; the question itself remains open. This is an unrefereed note.

1 Introduction

1.1 The problem

Djoković and Miller [DM80] proved that a group which acts transitively on the arcs (ordered edges) of the cubic tree T_3 , with finite vertex stabilisers, is isomorphic to one of seven finitely presented groups. Conder [Con92, §2] lists them with the presentations of [CL89]. Two of them are

$$\begin{aligned} G_3 &= \langle h, a, P, Q \mid h^3, a^2, P^2, Q^2, [P, Q], [h, P], (hQ)^2, a^{-1}PaQ \rangle, \\ G_4 &= \langle h, a, p, q, r \mid h^3, a^2, p^2, q^2, r^2, [p, q], [p, r], p(qr)^2, \\ &\quad h^{-1}phq, h^{-1}qhpq, (hr)^2, [a, p], a^{-1}qar \rangle, \end{aligned}$$

which act regularly on the paths of length 3 and 4 of T_3 , with vertex stabilisers $S_3 \times C_2$ and S_4 ([Con92, §2], where G_4 is called G_4^1). Both contain the modular group $G_1 = \langle h, a \mid h^3, a^2 \rangle \cong \mathrm{PSL}_2(\mathbb{Z})$, with index 4 and 8. Let $G = G_3 *_{G_1} G_4$ be the free product with $G_1 = \langle h, a \rangle$ amalgamated. By

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Conder's theorem [Con92, §4], two arc-transitive subgroups of $\text{Aut}(T_3)$ with finite vertex stabilisers which meet in an arc-regular subgroup generate their amalgamated free product unless one of them acts transitively on the paths of length 5; so G acts faithfully on T_3 . Killing h, p, q, r defines a homomorphism of G onto the dihedral group $\langle a, P, Q \rangle$ of order 8. Its kernel K is generated by

$$A = h, \quad B = aha, \quad C = p, \quad D = PpP, \quad E = QpQ, \quad F = PQpPQ,$$

and Conder obtained by the Reidemeister–Schreier process, with the Schreier transversal $\{1, a, P, Q, PQ, aP, aQ, aPQ\}$, the presentation

$$\begin{aligned} K = \langle A, B, C, D, E, F \mid & A^3, B^3, C^2, D^2, E^2, F^2, (AC)^3, (AD)^3, (AE)^3, \\ & (AF)^3, (BC)^3, (BD)^3, (BE)^3, (BF)^3, (ABA^{-1}C)^2, \\ & (ABA^{-1}D)^2, (A^{-1}BAE)^2, (A^{-1}BAF)^2, (BAB^{-1}C)^2, \\ & (B^{-1}ABD)^2, (BAB^{-1}E)^2, (B^{-1}ABF)^2 \rangle \end{aligned} \quad (1)$$

[Con92, §5], [KM26, Problem 16.46]. (In [Con92] the group is called S .)

Problem 1.1 (Conder [Con92, §5], [KM26, Problem 16.46]). *Does the group K have a non-trivial finite quotient?*

Conder also asked whether K can be simple. If K had no non-trivial finite quotient, it would provide an explicit finite amalgam which embeds in no finite group [Con92, §5]. His only evidence, he wrote, was that he had not found a proper subgroup of finite index in K ; no bound on the index is stated. In the same section he recalls that for two copies of G_4^1 (Problem 6 of [DM80]) the group generated by the vertex stabilisers is residually finite, by work of Djoković, and he notes that it maps onto $\text{PSL}_2(7)$. The Notebook prints Problem 16.46 without comment and without a mark (neither the $*$ of a solved problem nor the $\dagger$ of an unconfirmed AI solution) in its versions 48 and 49 (October 2026) [KM26], and it is record KOU-16.46 of the UnsolvedMath collection [ula26], where it is listed as open.

Conventions. $[x, y] = x^{-1}y^{-1}xy$; a relator w stands for $w = 1$; permutations act on the right (xy means first x , then y); a non-trivial finite quotient is one of order greater than 1. The group K is infinite (it contains $C_3 * C_3$, Lemmas 2.3 and 2.4), so the question is whether K has a proper normal subgroup of finite index.

1.2 Results

Let $H = \langle A, B \mid A^3, B^3 \rangle \cong C_3 * C_3$ and

$$L = \langle A, B, C \mid A^3, B^3, C^2, (AC)^3, (BC)^3, (ABA^{-1}C)^2, (BAB^{-1}C)^2 \rangle,$$

and let J be the group with generators A, B, C, D and the 12 relators of (1) that do not involve E or F .

Theorem 1.2 (structure; no computation). (a) K, J and L are perfect.

(b) Let f be a non-trivial homomorphism of K , or of J , into a group. Then $f(A)$ and $f(B)$ have order 3, the images of C, D, E, F (respectively of C, D) have order 2, and f is injective on a subgroup isomorphic to S_4 .

(c) $L \cong S_4 *_{D_8} S_4$; the subgroup $\langle A, B \rangle$ of L is isomorphic to H and has index 8.

- (d) K is the amalgamated free product of four copies L_C, L_D, L_E, L_F of L over H , where L_X is generated by A^e, B^d and X , playing the roles of A, B, C , with $(e, d) = (1, 1), (1, -1), (-1, 1), (-1, -1)$ for $X = C, D, E, F$; and $J \cong L_C *_H L_D$.
- (e) If K has a non-trivial finite quotient, then so has J .

Theorem 1.3 (computer-assisted, exhaustive). (a) Every homomorphism from J to a symmetric group of degree at most 36 is trivial. Hence neither J nor K has a subgroup of index m with $2 \leq m \leq 36$.

- (b) There is no non-trivial homomorphism from J to a non-abelian finite simple group of order less than 10^8 .
- (c) There is no non-trivial homomorphism from J to one of the twelve non-abelian finite simple groups T with $10^8 \leq |T| < 10^9$ that are not isomorphic to a group $\mathrm{PSL}_2(q)$.

Corollary 1.4. Every non-trivial finite quotient of J or of K has order at least 10^8 . Together with Jambor's theorem that K has no quotient isomorphic to $\mathrm{PSL}_2(q)$ or $\mathrm{PGL}(2, q)$ [Jam15, Proposition 12.2], every non-trivial finite quotient of K has order at least 10^9 .

Theorems 1.2 and 1.3 and Corollary 1.4 are proved in Sections 2 to 4. The amalgam $L_C *_H L_F$, whose factors differ by the twist $(A, B) \mapsto (A^{-1}, B^{-1})$, maps onto A_6 and onto $\mathrm{PSL}(2, 7)$ (Proposition 5.1); the obstruction found by the searches lies in the twist $(A, B) \mapsto (A, B^{-1})$.

1.3 What is known, what is new, and what stays open

The one partial answer we found in print is Jambor's: K , given by exactly the presentation (1), has no quotient isomorphic to $\mathrm{PSL}_2(q)$ or $\mathrm{PGL}(2, q)$ for any prime power q [Jam15, Proposition 12.2], a result obtained with his Magma implementation of the L_2 -quotient algorithm. Theorem 1.3(b) contains this statement again, by a different method, for the 123 groups $\mathrm{PSL}_2(q)$ of order less than 10^8 ($q \leq 577$, $q \neq 512$); a quotient $\mathrm{PGL}(2, q)$ with q odd is excluded anyway, as K is perfect. Jambor's theorem is used only for the step from 10^8 to 10^9 . Theorem 1.2 is elementary; part (d) is what Bass–Serre theory [Ser80] gives for the action of K on the tree of G (Remark 2.6), and it may be known to experts. Theorem 1.3 and the first bound of Corollary 1.4 are, to our knowledge, not in print; [Con92] gives no bound. They rest on exhaustive computer searches, parts (b) and (c) on computations in GAP [GAP26]; the dependencies are listed in Section 4.1. The question itself remains open, as do the existence of a non-trivial finite quotient of J and the simplicity of K .

2 The structure of K

For an involution x and an element w of a group we say that x *inverts* w if $xwx = w^{-1}$; then $(wx)^2 = 1$ if and only if x inverts w , if and only if x inverts w^{-1} . For elements a, b of a group S let

$$\mathrm{Sol}_S(a, b) = \{x \in S : x^2 = 1, (ax)^3 = (bx)^3 = 1, x \text{ inverts } aba^{-1} \text{ and } bab^{-1}\}.$$

If $a^3 = b^3 = 1$, then $\mathrm{Sol}_S(a, b)$ is the set of images of C under the homomorphisms $L \rightarrow S$ with $A \mapsto a, B \mapsto b$.

Lemma 2.1. K, J and L are perfect.

Proof. Write the abelianisation of K additively. Then $3A = 3B = 0$ and $2X = 0$ for $X \in \{C, D, E, F\}$; the relator $(AX)^3$ gives $3X = 0$, hence $X = 0$. Now $(ABA^{-1}C)^2$ gives $2B = 0$, hence $B = 0$, and $(BAB^{-1}C)^2$ gives $A = 0$. For J and L the same relators are available. $\square$

Lemma 2.2. *Let f be a homomorphism of K into a group. If one of $f(A), \dots, f(F)$ is trivial, then f is trivial. The same holds for homomorphisms of J (with the generators A, B, C, D) and of L (with A, B, C). In particular, if $f \neq 1$, then $f(A)$ and $f(B)$ have order 3 and the images of the other generators have order 2.*

Proof. Let $f(B) = 1$. The relators $(BAB^{-1}C)^2$ and $(AC)^3$ show that the order of $f(AC)$ divides 2 and 3, so $f(C) = f(A)^{-1}$; since $f(C)^2 = f(A)^3 = 1$, this gives $f(A) = f(C) = 1$. For $X \in \{D, E, F\}$ the relators $(AX)^3$ and X^2 then give $f(X) = 1$, so f is trivial. If $f(A) = 1$, the relators $(ABA^{-1}C)^2$ and $(BC)^3$ give in the same way $f(B) = f(C) = 1$, and f is trivial by the first case. Finally, let $f(X) = 1$ for some $X \in \{C, D, E, F\}$. The relator $(wX)^2$ of (1) with $w = ABA^{-1}$ (for $X = C, D$) or $w = A^{-1}BA$ (for $X = E, F$) gives $f(w)^2 = 1$, so $f(B)^2 = 1$; with $f(B)^3 = 1$ we get $f(B) = 1$. The arguments use, for J and L , only the relators of J and L . $\square$

Lemma 2.3. (a) *In any group, if $b^3 = c^2 = (bc)^3 = 1$, then $c(b^{-1}cb) = (b^{-1}cb)c = bcb^{-1}$. The group $\langle b, c \mid b^3, c^2, (bc)^3 \rangle$ is isomorphic to A_4 , by $b \mapsto (123)$, $c \mapsto (12)(34)$.*

(b) *The group $\Sigma = \langle h, p, r \mid h^3, p^2, (hp)^3, r^2, (hr)^2, [p, r] \rangle$ is isomorphic to S_4 , by $h \mapsto (123)$, $p \mapsto (12)(34)$, $r \mapsto (12)$; then $h^{-1}ph \mapsto (14)(23)$.*

(c) *In L put $q = A^{-1}CA$ and $r = B^{-1}CB$. Then L is the amalgamated free product $\Sigma_1 *_\Delta \Sigma_2$ of the subgroups $\Sigma_1 = \langle A, C, r \rangle \cong S_4$ and $\Sigma_2 = \langle B, C, q \rangle \cong S_4$ over $\Delta = \langle C, q, r \rangle \cong D_8$. The centre of Δ is $\langle C \rangle$; q lies in the normal Klein four-group of Σ_1 and is a transposition of Σ_2 , and r lies in the normal Klein four-group of Σ_2 and is a transposition of Σ_1 .*

(d) *The subgroup $\langle A, B \rangle$ of L is isomorphic to $H = C_3 * C_3$, and $[L : \langle A, B \rangle] = 8$.*

Proof. (a) From $(bc)^3 = 1$ we get $cbc = b^{-1}cb^{-1}$. Since $(cb)^3 = c(bc)^3c^{-1} = 1$, also $(b^{-1}c)^3 = ((cb)^{-1})^3 = 1$, and the same computation with b^{-1} in place of b gives $cb^{-1}c = bcb$. Hence $c(b^{-1}cb) = (cb^{-1}c)b = bcb^2 = bcb^{-1}$ and $(b^{-1}cb)c = b^{-1}(cbc) = b^{-2}cb^{-1} = bcb^{-1}$. So $c, c' = b^{-1}cb$ and $c'' = bcb^{-1} = cc'$ commute, $V = \{1, c, c', c''\}$ is a subgroup, conjugation by b permutes c, c', c'' cyclically, and $\langle b, c \rangle = V\langle b \rangle$ has order at most 12. The permutations (123) and $(12)(34)$ satisfy the relations (their product is (243)) and generate A_4 .

(b) By (a), $\langle h, p \rangle$ has order at most 12. The relations $r^2 = (hr)^2 = [p, r] = 1$ say that $rhr = h^{-1}$ and $rpr = p$, so r normalises $\langle h, p \rangle$ and $|\Sigma| \leq 24$. The three permutations satisfy the relations $(hp = (243), hr = (23))$ and generate S_4 . A direct computation gives $h^{-1}ph \mapsto (14)(23)$.

(c) The relations of Σ for $(h, p, r) = (A, C, r)$ and for $(h, p, r) = (B, C, q)$ hold in L : the relations $h^3, p^2, (hp)^3$ are relators of L ; $r^2 = B^{-1}C^2B = 1$ and $q^2 = 1$; $(Ar)^2 = B^{-1}(BAB^{-1}C)^2B$ and $(Bq)^2 = A^{-1}(ABA^{-1}C)^2A$; and $[C, r] = [C, q] = 1$ by (a), applied to $(b, c) = (B, C)$ and (A, C) . Conversely, these relations contain all relators of L . So L is presented by two copies $\Sigma_1 = \langle h_1, p_1, r_1 \rangle$ and $\Sigma_2 = \langle h_2, p_2, r_2 \rangle$ of Σ together with the identifications

$$p_1 = p_2 (= C), \quad r_1 = h_2^{-1}p_2h_2 (= r), \quad h_1^{-1}p_1h_1 = r_2 (= q),$$

with $h_1 = A$ and $h_2 = B$. By (b), in $\Sigma_1 \cong S_4$ the elements $p_1 = (12)(34)$, $h_1^{-1}p_1h_1 = (14)(23)$ and $r_1 = (12)$ generate a Sylow 2-subgroup Δ_1 with centre $\langle p_1 \rangle$, and in Σ_2 the elements $p_2, r_2 = (12)$ and $h_2^{-1}p_2h_2 = (14)(23)$ generate Δ_2 . In both, q and r are involutions which generate a group of order 8 with $(qr)^2 = C$. As $\langle q, r \mid q^2, r^2, (qr)^4 \rangle$ has order 8, the map $q \mapsto q, r \mapsto r$ is an isomorphism $\Delta_1 \rightarrow \Delta_2$, and the identifications define the amalgamated free product $\Sigma_1 *_\Delta \Sigma_2$. The factors embed in it [Ser80, Ch. I, §1]. The remaining statements follow from (b).

(d) Since Δ is a 2-group, $A^{\pm 1} \in \Sigma_1 \setminus \Delta$ and $B^{\pm 1} \in \Sigma_2 \setminus \Delta$. A non-empty reduced word in $A^{\pm 1}, B^{\pm 1}$ is therefore an alternating product of elements of $\Sigma_1 \setminus \Delta$ and $\Sigma_2 \setminus \Delta$, and is not 1 by the normal

form theorem [Ser80, Ch. I, §1]. So $\langle A, B \rangle \cong H$. For the index, let X be the tree of the amalgam [Ser80, Ch. I, §4]: its vertices are the cosets of Σ_1 and of Σ_2 , its edges the cosets $g\Delta$, and $g\Delta$ joins $g\Sigma_1$ and $g\Sigma_2$. The three edges at the vertex Σ_1 correspond to Σ_1/Δ , on which $\Sigma_1 \cong S_4$ acts through S_3 , and A acts on them as a 3-cycle; likewise B at Σ_2 . Hence, if an edge $h\Delta$ with $h \in H$ ends at $h\Sigma_i$, the element hAh^{-1} or hBh^{-1} of H permutes the three edges at this vertex transitively. As X is connected, H is transitive on the edges. The stabiliser $H \cap \Delta$ of the edge Δ is trivial, because every element of finite order of $C_3 * C_3$ has order 1 or 3. So every element of L is uniquely $h\delta$ with $h \in H$, $\delta \in \Delta$, and $[L : H] = |\Delta| = 8$. $\square$

For $X \in \{C, D, E, F\}$ let $(e_X, d_X) = (1, 1), (1, -1), (-1, 1), (-1, -1)$, respectively, and let L_X be a copy of L with generators a_X, b_X, x_X . The map $\iota_X: H \rightarrow L_X$, $A \mapsto a_X^{e_X}$, $B \mapsto b_X^{d_X}$, is injective by Lemma 2.3(d), because $(A, B) \mapsto (A^{e_X}, B^{d_X})$ is an automorphism of H .

Lemma 2.4. (a) *Apart from A^3 and B^3 , the relators of (1) fall into four sets of five: those that involve X , for $X = C, D, E, F$; no relator involves two of these letters. As cyclic words up to inversion in the free product of $\langle A \rangle$, $\langle B \rangle$ and $\langle X \rangle$, of orders 3, 3 and 2, the five relators that involve X are the relators x^2 , $(ax)^3$, $(bx)^3$, $(aba^{-1}x)^2$, $(bab^{-1}x)^2$ of L evaluated at $(a, b, x) = (A^{e_X}, B^{d_X}, X)$.*

(b) *Hence K is the amalgamated free product of L_C, L_D, L_E, L_F along the embeddings ι_X of H , where A, B, X correspond to $a_X^{e_X}, b_X^{d_X}, x_X$; and J is the amalgamated free product of L_C and L_D in the same way.*

(c) *Let S be a group. Sending $(A, \dots, F)$ to $(a, \dots, f)$ is a bijection between the homomorphisms $K \rightarrow S$ and the tuples $(a, b, c, d, e, f) \in S^6$ with $a^3 = b^3 = 1$, $c \in \text{Sol}_S(a, b)$, $d \in \text{Sol}_S(a, b^{-1})$, $e \in \text{Sol}_S(a^{-1}, b)$ and $f \in \text{Sol}_S(a^{-1}, b^{-1})$; likewise for J and the tuples (a, b, c, d) .*

Proof. (a) Let x be an involution. Then $(w^{-1}x)^2 = 1$ if and only if $(wx)^2 = 1$, and $(b^{-1}x)^3 = 1$ if and only if $(xb)^3 = 1$, if and only if $(bx)^3 = 1$. For $X = C$ the statement is literal. For $X = D$ the relators of L at (A, B^{-1}, D) are D^2 , $(AD)^3$, $(B^{-1}D)^3$, $(AB^{-1}A^{-1}D)^2$ and $(B^{-1}ABD)^2$, equivalent to D^2 , $(AD)^3$, $(BD)^3$, $(ABA^{-1}D)^2$ and $(B^{-1}ABD)^2$. For $X = E$, at (A^{-1}, B, E) we get $(A^{-1}E)^3$, $(BE)^3$, $(A^{-1}BAE)^2$ and $(BA^{-1}B^{-1}E)^2$, equivalent to $(AE)^3$, $(BE)^3$, $(A^{-1}BAE)^2$ and $(BAB^{-1}E)^2$. For $X = F$, at (A^{-1}, B^{-1}, F) we get $(A^{-1}F)^3$, $(B^{-1}F)^3$, $(A^{-1}B^{-1}AF)^2$ and $(B^{-1}A^{-1}BF)^2$, equivalent to $(AF)^3$, $(BF)^3$, $(A^{-1}BAF)^2$ and $(B^{-1}ABF)^2$. These are the relators of (1).

(b) By (a) and Tietze transformations, (1) is the presentation of the quotient of $L_C * L_D * L_E * L_F * H$ by the identifications $\iota_X(g) = g$, $g \in H$, that is, of the amalgamated free product of the four copies of L along the four embeddings of H (formed step by step, $L_C * _H (L_D * _H (L_E * _H L_F))$, with H embedded in each step by [Ser80, Ch. I, §1]). The same holds for J .

(c) By (a), the relators of (1) that involve X hold for an image x of X exactly when $x \in \text{Sol}_S(a^{e_X}, b^{d_X})$. $\square$

Corollary 2.5. (a) *Every non-trivial homomorphism f from K to a group restricts to a non-trivial homomorphism of J ; more precisely, the composite of f with the homomorphism $J \rightarrow K$, $A, B, C, D \mapsto A, B, C, D$, is non-trivial.*

(b) *If f is a non-trivial homomorphism of K or of J into a group, then f is injective on $\langle A, C, B^{-1}CB \rangle$, and this subgroup is isomorphic to S_4 .*

Proof. The relators of J are relators of K , so $J \rightarrow K$ exists. (a) By Lemma 2.2, $f(A) \neq 1$. (b) By Lemma 2.3(c), $\Sigma_1 = \langle A, C, r \rangle \leq L$ is isomorphic to S_4 , with C in its normal Klein four-group, and f composed with $L \rightarrow K$ (or $L \rightarrow J$) maps C to $f(C) \neq 1$. As every non-trivial normal subgroup of S_4 contains the Klein four-group, the composite is injective on Σ_1 , whose image is $\langle A, C, B^{-1}CB \rangle$. $\square$

Proof of Theorem 1.2. (a)–(d) are Lemmas 2.1 to 2.4 with Corollary 2.5(b); for (e), the image of J in a non-trivial finite quotient of K is non-trivial by Corollary 2.5(a). $\square$

Remark 2.6 (the presentation and the tree of G). Lemma 2.4(b) can also be read off from the action of K on the tree Y of the amalgam G [Ser80, Ch. I, §4]; this confirms that (1) presents the subgroup of index 8 defined in the introduction, independently of the Reidemeister–Schreier computation. (In the scanned copy of [Con92] that was read, the presentation, on p. 217, is not in the text layer; read from the rendered page, it has the same 22 relators as (1), which is also the text of [KM26, Problem 16.46] and of [Jam15, Proposition 12.2]; the derivation below confirms them.) The vertices of Y are the cosets of G_3 and of G_4 , and the images of G_3 , G_4 , G_1 in $G/K = \langle a, P, Q \rangle$ are D_8 , $\langle a \rangle$, $\langle a \rangle$. So K has one orbit on the vertices of type G_3 and four on those of type G_4 and on the edges, with representatives $1, P, Q, PQ$. The stabiliser $K \cap G_3$ contains $\langle h, aha \rangle$, of index 8 in G_3 , so it equals $\langle A, B \rangle \cong H$, and so does every edge stabiliser. The group $K \cap G_4$ has index 2 in G_4 and preserves the bipartition of T_3 ; it is transitive on the edges of T_3 , with vertex stabilisers $\langle h, p, q, r \rangle = \langle A, C, r \rangle$ and $a\langle h, p, q, r \rangle a = \langle B, C, q \rangle$ and edge stabiliser $\langle p, q, r \rangle$, where $q = h^{-1}ph = A^{-1}CA$ and, by $a^{-1}qar = [a, p] = 1$, $r = aqa = B^{-1}CB$. By Lemma 2.3(c) and [Ser80, Ch. I, §4], $K \cap G_4 = \langle A, B, C \rangle \cong L$. As $K \backslash Y$ is a star with four edges, K is the amalgamated free product over H of the four groups $g(K \cap G_4)g^{-1}$, $g = 1, P, Q, PQ$ [Ser80, Ch. I, §§4–5]. Finally $[h, P] = 1$, $Pa = aQ$ and $QhQ = h^{-1}$ give $PAP = A$, $PBP = aQhQa = B^{-1}$, $QAQ = A^{-1}$ and $QBQ = aPhPa = B$, so conjugation by P, Q, PQ maps (A, B, C) to (A, B^{-1}, D) , (A^{-1}, B, E) , (A^{-1}, B^{-1}, F) .

3 Actions on at most 36 points

An L -set is a finite set with an action of L ; the transitive L -sets of degree n , up to isomorphism, correspond to the conjugacy classes of subgroups of index n in L .

Theorem 3.1. *For $n \leq 36$, every homomorphism from J to the symmetric group S_n is trivial. Consequently neither J nor K has a subgroup of index m with $2 \leq m \leq 36$.*

Proof (computer-assisted). Let J act on a set Ω of $n \leq 36$ points by a, b, c, d . By Lemma 2.4(c), (a, b, c) and (a, b^{-1}, d) are actions of L on Ω ; we call c and d solutions of the C -role and of the D -role. If the action is non-trivial, then $a \neq 1$ (Lemma 2.2). Decompose the L -set $(\Omega; a, b, c)$ into transitive L -sets. A transitive L -set on which a acts trivially is a single point (Lemma 2.2), so at least one component is non-trivial. Since an isomorphism of L -sets carries solutions of the D -role to solutions, the theorem follows from two computations.

Step 1: the transitive L -sets. The transitive L -sets of degree at most 36 were enumerated up to isomorphism by the low-index subgroups algorithm, a depth-first search over standardised coset tables, with a canonicity test that keeps one table in each conjugacy class of subgroups [Sim94]. There are 1156 classes; Table 1 gives their degrees.

Step 2: the D -role. For every *configuration*, that is, a multiset of non-trivial transitive L -sets together with a number of fixed points, of total degree at most 36, a backtracking search over the images of the points under d , which deduces all consequences of the relators of L for (a, b^{-1}, d) after each choice, decided whether an involution d of Ω solving the D -role exists. There are 55862 configurations, and in none of them does such a d exist.

So J has no action on at most 36 points with $a \neq 1$. A subgroup of index m , $2 \leq m \leq 36$, of J or of K would give a non-trivial homomorphism into S_m , also for K by Corollary 2.5(a). $\square$

Step 1 ran in three deterministic parts of 712, 764 and 947 seconds, Step 2 in 11 seconds. The programs reproduce the subgroup numbers of A_5 , $\text{PSL}(2, 7)$ and (by a plain search) of L up to index

n	cl.	a_n	n	cl.	a_n	n	cl.	a_n	n	cl.	a_n	n	cl.	a_n
1	1	1	13	4	52	21	32	672	27	16	432	32	64	2048
6	2	12	14	6	56	22	8	176	28	104	2912	33	64	2112
7	4	28	15	8	120	24	24	448	29	16	464	34	128	4352
8	2	16	18	16	264	25	16	400	30	20	480	35	96	3360
10	2	20	20	7	120	26	30	624	31	64	1984	36	422	14688

Table 1: The numbers of conjugacy classes (cl.) of subgroups of index $n \leq 36$ in L and the numbers $a_n = a_n(L)$ of subgroups of index n (0 for the other $n \leq 36$); 1156 classes in all.

10, the solver agrees with a search over all involutions on 336 test pairs, and the programs find the actions of $L_C *_H L_F$ of Proposition 5.1.

Four further computations confirm Theorem 3.1. (i) A plain low-index search on the presentation of J finds only the subgroup of index 1 up to index 12, and up to index 16 with the programs of both verification runs (one of them GAP's `LowIndexSubgroupsFpGroup` [GAP26]). (ii) The first verification run re-implemented Steps 1 and 2, with the same numbers of classes and configurations in every degree and no solution. (iii) The second verification run obtained the 1156 classes from `LowIndexSubgroupsFpGroup` and certified that the list is complete: from Lemma 2.3(c) it counted the homomorphisms $L \rightarrow S_n$ as pairs of actions of S_4 that agree on D_8 , obtained $a_n(L)$ by Hall's recursion [Hal49], and checked $\sum_X n/|\text{Aut}(X)| = a_n(L)$ for $n \leq 36$, the sum over the listed X of degree n . It then replaced Step 2 by another argument. An action of J with $a \neq 1$ amounts to L -sets $X = (\Omega; a, b, c)$ and $Y = (\Omega'; a', b', d')$ with an isomorphism of H -sets from $(\Omega; a, b)$ onto $(\Omega'; a', b'^{-1})$, and finite H -sets are determined by the multiset of the isomorphism types of their orbits. Discard every non-trivial transitive L -set one of whose non-trivial H -orbit types does not occur among the twisted types of the remaining ones, and conversely, and repeat; the components of a solution are never discarded. For J a single round empties both sides: only six non-trivial H -orbit types occur on both sides, and every L -set with one of them also has an H -orbit of another type (the programs count a second, confirming pass). For $L_C *_H L_F$, 26 L -sets survive on each side. `check_note.py` repeats this in a few seconds. (iv) The third verification run certified the same list by its own count of the $a_n(L)$, decided the D -role in all 55862 configurations with its own solver (no solution), and repeated the elimination with its own programs.

4 Simple groups of order less than 10^9

Lemma 4.1. *Let T be a finite group, let R be a set of representatives of the conjugacy classes of elements of order 3 of T , and for $a, b_0 \in R$ let $\mathcal{D}(a, b_0)$ be a set of representatives of the double cosets $C_T(a)gC_T(b_0)$ in T . Then the elements $b_0^{g^{-1}}$, $g \in \mathcal{D}(a, b_0)$, represent the orbits of $C_T(a)$, acting by conjugation, on the class of b_0 . There is a non-trivial homomorphism $J \rightarrow T$ if and only if there are $a, b_0 \in R$ and $g \in \mathcal{D}(a, b_0)$ such that, for $b = b_0^{g^{-1}}$, both $\text{Sol}_T(a, b)$ and $\text{Sol}_T(a, b^{-1})$ are non-empty. For K the condition is that also $\text{Sol}_T(a^{-1}, b)$ and $\text{Sol}_T(a^{-1}, b^{-1})$ are non-empty.*

Here $x^g = g^{-1}xg$. All four sets lie in $X_a = \{x \in T : x^2 = 1, (ax)^3 = 1\}$, which is also the set for a^{-1} , since $(a^{-1}x)^3 = 1$ if and only if $(xa)^3 = 1$, if and only if $(ax)^3 = 1$. We call the four conditions the C -, D -, E - and F -role.

Proof. The map $C_T(b_0)x \mapsto b_0^x$ is a bijection from the right cosets of $C_T(b_0)$ onto the class of b_0 , and conjugation by $c \in C_T(a)$ corresponds to $C_T(b_0)x \mapsto C_T(b_0)xc$. So the $C_T(a)$ -orbits on the class correspond to the double cosets $C_T(b_0)xC_T(a)$, that is, under $x \mapsto x^{-1}$, to the double cosets $C_T(a)gC_T(b_0)$ with $g = x^{-1}$. Let $f: J \rightarrow T$ be non-trivial. By Lemma 2.2, $f(A)$ has order 3; after

conjugating f we may assume $f(A) = a \in R$, and after conjugating with an element of $C_T(a)$, that $f(B) = b_0^{g^{-1}}$ with $b_0 \in R$ and $g \in \mathcal{D}(a, b_0)$ (by Lemma 2.2, $f(B)$ has order 3). Then $f(C) \in \text{Sol}_T(a, b)$ and $f(D) \in \text{Sol}_T(a, b^{-1})$. The converse is Lemma 2.4(c), and so is the statement for K . $\square$

Theorem 4.2. *There is no non-trivial homomorphism from J , and hence none from K , to a non-abelian finite simple group of order less than 10^8 .*

Proof (computer-assisted). Let T be such a group. If there is a non-trivial homomorphism $J \rightarrow T$, then S_4 embeds in T (Corollary 2.5(b)), so 24 divides $|T|$. By the classification of the finite simple groups and the order formulas [Wil09], the non-abelian simple groups of order less than 10^8 are the following 161 groups: the 123 groups $\text{PSL}_2(q)$ for the 124 prime powers q with $4 \leq q \leq 577$ and $q \neq 512$ (as $\text{PSL}_2(4) \cong \text{PSL}_2(5)$); the Suzuki groups $\text{Sz}(8)$ and $\text{Sz}(32)$; and the 36 groups of Table 2 before $S_4(7)$. The order $q^2(q^2 + 1)(q - 1)$ of $\text{Sz}(q)$, $q = 2^{2m+1}$, is not divisible by 3, so the Suzuki groups are excluded. (Neither is $|\text{PSL}_2(q)|$ divisible by 8 when $q \equiv \pm 3 \pmod{8}$.) For each of the other 159 groups the search of Lemma 4.1 was carried out in GAP 4.16.1 [GAP26], for $\text{PSL}_2(q)$ on the $q + 1$ points of the projective line and for the others as given by `SimpleGroupsIterator` (in a representation of smaller degree if the given one had more than 1000 points), with `ConjugacyClasses`, `Centralizer` and `DoubleCosetRepsAndSizes`, testing the roles over all of X_a . For no group and no pair (a, b) are the C - and the D -role solvable together (Table 2; for $\text{PSL}_2(q)$ the C -role is solvable for some pair exactly when q is odd and $q \equiv \pm 1 \pmod{8}$, that is, when S_4 embeds in $\text{PSL}_2(q)$). $\square$

The groups listed by `SimpleGroupsIterator` agree, order by order, with the order formulas (checked by `check_note.py`). The longest search, for $U_3(11)$, took 622 seconds; the 124 groups $\text{PSL}_2(q)$ took 94 minutes.

Controls. The GAP program first searched A_6 , $\text{PSL}(2, 7)$, $\text{PSU}_3(5)$ and $\text{PSp}_4(4)$, where a Python program examining all pairs (a, b) with $a \in R$ had found solutions of the C - and of the C - and F -roles but not of the C - and D -roles. GAP counts $C_T(a)$ -orbits of pairs; weighted by their sizes they give the Python counts (for A_6 , 32 orbits and 160 pairs, 4 and 36 with a solvable C -role). The orbits are not always regular (in HS, 4500 pairs with a solvable C -role form 13 orbits, one of size 180).

A second algorithm. GAP's `GQuotients` found no epimorphism of J onto 74 groups: all non-abelian simple T with $|T| < 10^8$ and $3 \mid |T|$, except $\text{PSL}_2(q)$ with $q > 125$ and $U_3(11)$ (memory limit of 16 GB reached). As J is perfect, the image of a non-trivial homomorphism $J \rightarrow T$ is perfect and maps onto a simple group of order at most $|T|$; so for the bound 10^8 it suffices to exclude epimorphisms, and for these 74 groups `GQuotients` does so by a second algorithm. As controls, it finds one class of epimorphisms of L onto A_6 , two onto $\text{PSL}(2, 7)$, and one of $L_C *_H L_F$ onto each of them.

Theorem 4.3. *There is no non-trivial homomorphism from J to any of the twelve non-abelian simple groups T with $10^8 \leq |T| < 10^9$ that are not isomorphic to a group $\text{PSL}_2(q)$.*

Proof (computer-assisted). By the order formulas, the non-abelian simple groups T with $10^8 \leq |T| < 10^9$ are the 104 groups $\text{PSL}_2(q)$ with $q = 512$ or q a prime power with $587 \leq q \leq 1259$, and the twelve groups of Table 2 from $S_4(7)$ on; there is no Suzuki or Ree group of such order. The search of Lemma 4.1 was carried out for the twelve groups as in Theorem 4.2. $\square$

Proof of Corollary 1.4. Let Q be a non-trivial finite quotient of J or of K . By Lemma 2.1, Q is perfect, so it maps onto a non-abelian simple group T with $|T| \leq |Q|$. Composing (and, for K , restricting to J by Corollary 2.5(a)) we get a non-trivial homomorphism $J \rightarrow T$, so $|T| \geq 10^8$ by Theorem 4.2. Now let Q be a quotient of K and suppose $|Q| < 10^9$. Then $10^8 \leq |T| < 10^9$, and K maps onto T . By [Jam15, Proposition 12.2], T is not isomorphic to $\text{PSL}_2(q)$, and by Theorem 4.3 it is none of the other twelve groups; this is a contradiction. $\square$

T	pairs	C	C, F	T	pairs	C	C, F	T	pairs	C	C, F
$A_7^{\dagger}$	75	10	8	$S_6(2)^*$	302	8	7	$U_3(9)$	6800	10	2
$L_3(3)^{\dagger}$	116	2	0	A_{10}^*	743	10	7	$HS^{\dagger}$	399	13	12
$U_3(3)^{\dagger}$	104	4	2	$L_3(7)^{\dagger}$	1472	18	3	J_3	1346	8	6
$M_{11}^{\dagger}$	30	2	2	$U_4(3)^{\dagger}$	1018	50	48	$U_3(11)$	3494	36	12
$A_8^{\dagger}$	108	10	8	$G_2(3)^{\dagger}$	1460	5	2	$S_4(7)$	876	16	8
$L_3(4)^{\dagger}$	256	20	18	$S_4(5)$	234	6	6	$O_8^+(2)$	1326	20	19
$U_4(2)^{\dagger}$	114	2	2	$U_3(8)$	1136	0	0	$O_8^-(2)^{\dagger}$	1632	10	7
$U_3(4)^{\dagger}$	284	0	0	$U_3(7)$	2468	8	1	${}^3D_4(2)$	1170	2	1
$M_{12}^{\dagger}$	243	2	2	$L_4(3)^{\dagger}$	1618	6	4	$L_3(11)$	14762	10	2
$U_3(5)^{\dagger}$	118	14	12	$L_5(2)^*$	700	14	12	A_{12}^*	3396	14	12
$J_1^{\dagger}$	200	0	0	M_{23}^*	356	16	14	$M_{24}^{\dagger}$	2242	25	15
$A_9^{\dagger}$	284	7	6	$U_5(2)$	782	2	2	$G_2(4)$	8618	12	3
$L_3(5)^{\dagger}$	650	4	0	$L_3(8)$	4160	16	2	$L_3(13)$	13108	36	0
$M_{22}^{\dagger}$	364	24	22	${}^2F_4(2)'$	1564	8	6	$U_3(13)$	28760	14	2
$J_2^{\dagger}$	581	4	1	A_{11}^*	1331	10	7	McL	1186	28	26
$S_4(4)^{\dagger}$	224	2	2	$L_3(9)$	6836	8	4	$L_4(4)$	5154	22	8

Table 2: The search of Lemma 4.1 in the 48 non-abelian simple groups of order less than 10^9 other than $\text{PSL}_2(q)$ and $\text{Sz}(q)$, in the order of `SimpleGroupsIterator`; from $S_4(7)$ on, $|T| \geq 10^8$. $L_n = \text{PSL}_n$, $U_n = \text{PSU}_n$, $S_{2m} = \text{PSp}_{2m}$, $O_{2m}^{\pm} = \text{P}\Omega_{2m}^{\pm}$. “pairs”: pairs (a, b) examined, one for each $C_T(a)$ -orbit; “ C ”, “ C, F ”: pairs for which the C -role, resp. the C - and F -roles, are solvable. The C - and D -roles are never solvable together. *: a faithful permutation representation of degree at most 36 exists (Theorem 3.1). †: searched again, outside GAP, in a verification run.

4.1 What the bounds depend on

Theorem 3.1 rests on the programs of Steps 1 and 2, their re-implementation in the first verification run, the elimination of the second, and the certified list, solver and elimination of the third. In Theorem 4.2, the groups $\text{PSL}_2(q)$, $q \leq 577$, $q \neq 512$, are covered by three independent computations (the GAP search, the first run’s arithmetic in $\text{SL}_2(q)$ for all these q , the second run’s program for the primes $p \leq 199$) and by Jambor’s computation for all q . Of the 36 other groups, 16 are also covered by Theorem 3.1 (marked * in Table 2). The first run searched $\text{PSp}_4(4)$, $\text{PSU}_3(4)$ and $\text{PSU}_3(5)$, and the third $\text{PSL}_3(7)$ and $\text{PSL}_4(3)$, built from matrices, with their own programs (the third with all pairs (a, b) , reproducing the orbit counts of Table 2); the second ran its own Python search on permutation generators of 24 simple groups taken from GAP’s library of primitive groups [HLRDR26]; none found a solution, and comparable counts agree. For the 15 groups J_1 , J_2 , J_3 , HS , $L_3(8)$, $L_3(9)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_3(11)$, $U_4(3)$, $U_5(2)$, $S_4(5)$, $G_2(3)$, ${}^2F_4(2)'$, the bound 10^8 rests on GAP 4.16.1 alone: on the search above and, except for $U_3(11)$, on `GQuotients`; five of them (J_1 , J_2 , HS , $U_4(3)$, $G_2(3)$) were searched again by the second run’s Python program, but on generators exported from GAP. The bound 10^9 rests in addition on the GAP search for the twelve groups of Theorem 4.3 (with M_{24} and $O_8^-(2)$ searched again in Python by the second run, and A_{12} , M_{24} also covered by Theorem 3.1) and on Jambor’s Magma computation for the 104 groups $\text{PSL}_2(q)$.

5 The twist, and remarks

Proposition 5.1. *Let $J' = L_C *_H L_F$ be the group with generators A, B, C, F and the 12 relators of (1) that do not involve D or E . The permutations*

$$A = (3\,5\,4), \quad B = (1\,2\,3)(4\,5\,6), \quad C = (2\,4)(3\,5), \quad F = (1\,5)(3\,4)$$

satisfy these relators and generate A_6 , and

$$A = (245)(367), \quad B = (123)(576), \quad C = (46)(57), \quad F = (16)(37)$$

satisfy them and generate the simple group $\text{PSL}(2,7)$ of order 168, acting on 7 points. So J' has non-trivial finite quotients.

Proof. A direct check of the 12 relators, with right actions; the groups generated have orders 360 and 168 (`check_note.py`, part [5], which also shows by a search over all involutions of S_6 and S_7 that for these a, b neither the D - nor the E -role is solvable). $\square$

So J and J' differ only in the twist of the second factor; on the 161 non-trivial transitive L -sets of degree ≤ 26 the D -role is never solvable, the F -role on 18.

Remark 5.2 (stated without proof). As G acts faithfully on T_3 and acts on the tree Y of Remark 2.6, the stabiliser of a vertex of $T_3 \times Y$ being the stabiliser of a vertex of T_3 in a conjugate of G_3 or G_4 , hence finite, one expects G, K, J to be cocompact lattices in products of two trees (for J the second tree is $(8,8)$ -biregular, as $[L : H] = 8$), to which normal subgroup theorems [Cap19, §4.7] might apply; we have not checked their hypotheses.

6 Computations and reproducibility

The reproducibility package contains `check_note.py`, the exact program of this note (Python, standard library, about 30 seconds), which checks the finite and exact parts of Section 2 and Proposition 5.1, small cases of Theorems 3.1 and 4.2, the lists of simple groups against the recorded GAP outputs, and the elimination of Section 3; the programs of the first computation with their recorded outputs; and those of the three verification runs (details in `README.md`).

Verification. The results were first obtained, with proofs and with the programs of the package. Two independent verification runs, both AI-assisted, followed; each re-derived the proofs of Section 2 and wrote its own programs (both examined the first write-up of the result, not this text). The first run read Problem 16.46 in both versions of [KM26], [Con92] and Section 12 of [Jam15], re-derived Remark 2.6, re-implemented Steps 1 and 2 of Theorem 3.1 (same counts in every degree, no solution), searched all $\text{PSL}_2(q)$ with $q \leq 577$ in its own arithmetic in $\text{SL}_2(q)$ and $\text{P}\text{Sp}_4(4)$, $\text{PSU}_3(4)$, $\text{PSU}_3(5)$ built from matrices, and checked from the order formulas that the lists of simple groups are complete. The second run checked Lemma 2.4(a) at the level of words, proved Theorem 3.1 a second time as described in Section 3, searched 24 simple groups and $\text{PSL}(2,p)$ for the primes $p \leq 199$ with its own Python program, repeated the weighting of orbits for thirteen groups, re-ran part of the original programs with identical outputs, and repeated the literature search. Both runs confirmed the results in the scope stated here. Their corrections concerned the presentation and have all been made (for instance, Theorem 4.2 is stated for all homomorphisms, as the search gives and Corollary 1.4 needs). When this note was written, `check_note.py` was added.

A third run, also AI-assisted, examined the final text of this note. It re-derived every proof in the wording of the note, among them Remark 2.6 and the reduction in the proof of Corollary 1.4; read the sources again (Problem 16.46 in versions 48 and 49 of [KM26], [Con92], whose display of the presentation of K it read from the rendered scan, and Section 12 of [Jam15]); checked the references with Crossref; and repeated the literature search. Its programs (folder `independent_run_2/` of the package), written before the programs of the package were run again, certify the list of the 1156 transitive L -sets by their own count of the $a_n(L)$ (Table 1), decide the D -role in all 55862 configurations with their own solver and repeat the elimination of Section 3; recompute the lists of simple groups

from the order formulas and compare every entry and mark of Table 2 with the recorded GAP outputs; check Proposition 5.1 and the controls; and search $\mathrm{PSL}_3(7)$ and $\mathrm{PSL}_4(3)$, built from matrices over $\mathbb{F}_7$ and $\mathbb{F}_3$, with all pairs (a, b) : no solution, and the orbit counts of Table 2 are reproduced. It also ran the package again from the source archive (outputs identical apart from running times). It found no mathematical error and required eleven corrections of wording, precision and bookkeeping, which were made, among them the account of Section 4.1, the strength of the **GQuotients** check, the range $q \neq 512$ of the re-check of [Jam15], the number of rounds of the elimination and the first sentence of Remark 5.2.

Scope and priority. The note does not answer Problem 16.46. It proves Theorem 1.2 and, by exhaustive computer searches, Theorem 1.3 and Corollary 1.4; the bound 10^9 also depends on Jambor’s machine computation. The problem, the group K and its presentation are Conder’s [Con92, KM26]; G_1 , G_3 , G_4 are Djoković–Miller groups [DM80], with the presentations of [CL89]; the exclusion of all $\mathrm{PSL}_2(q)$ and $\mathrm{PGL}(2, q)$ is Jambor’s [Jam15, Proposition 12.2], re-checked here for the $\mathrm{PSL}_2(q)$ of order less than 10^8 . We use, without re-reading them for this note, amalgams and Bass–Serre theory [Ser80], the low-index subgroups algorithm [Sim94], Hall’s recursion [Hal49], the classification of the finite simple groups with the order formulas [Wil09], and GAP [GAP26, HLRDR26]. What was read: Problem 16.46 in versions 48 and 49 of [KM26]; [Con92] in full, in an open scanned copy, whose display of the presentation of K (p. 217) was read from the rendered page (Remark 2.6); Section 12 of [Jam15] in arXiv version 1, the only arXiv version (the journal version was not compared); Jambor’s doctoral thesis (RWTH Aachen, 2012), which treats two-generator groups and does not mention K ; and [Cap19], which does not mention K (arXiv version 3). The proofs of the cited theorems were not checked. The searches, on 22 and 24 October 2026, used the arXiv (Conder’s papers, the cubic tree, Kourovka 16.46, amalgams of arc-transitive groups, finite quotients of lattices in products of trees), zbMATH Open, Crossref, the works citing [Con92] and [Jam15] in OpenAlex and Semantic Scholar, and four web searches; the third verification run repeated the searches in the arXiv, Crossref, OpenAlex and zbMATH Open, with one of the web searches, on 24 October. None of the citing works read addresses the question, and a thesis of 2010 citing [Con92] could not be retrieved. No printed answer to the question was found in these sources; a search that finds nothing is not a proof of novelty, and we claim no priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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