

Minimal Non- $\mathfrak{F}$ -Groups with Sylow Residual and the Unitary Groups $\text{PSU}(3, p^{2n})$: A Negative Answer to Kourovka Notebook Problem 15.98

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Abstract

In Problem 15.98 of the Kourovka Notebook (2002), L. A. Shemetkov asked whether, for a saturated formation $\mathfrak{F}$ and a finite soluble minimal non- $\mathfrak{F}$ -group G whose $\mathfrak{F}$ -residual is a Sylow p -subgroup, this residual is always isomorphic to a factor group of the Sylow p -subgroup of $\text{PSU}(3, p^{2n})$. For the formation of nilpotent groups this is a theorem of Mazurov and Syskin ($p = 2$) and of Zhurtov and Syskin on Schmidt groups. We show that the answer is no: there is a saturated formation $\mathfrak{F}$ and a finite soluble minimal non- $\mathfrak{F}$ -group G of order 2304 whose $\mathfrak{F}$ -residual is a Sylow 2-subgroup isomorphic to no factor group of a Sylow 2-subgroup of $\text{PSU}(3, 2^{2n})$; 2304 is the least order of such a counterexample. The obstruction is elementary: in a nonabelian factor group Q of the Sylow p -subgroup of $\text{PSU}(3, p^{2n})$ the commutators $[y, x]$, $x \in Q$, fill Q' for every non-central y , while our residual, a special group of order 2^8 with centre of order 4, has a non-central element with only two commutators. The example comes from an elementary construction, a cyclic group acting on a special p -group, which gives a second counterexample of order $3^8 \cdot 2^4$ and shows that the answer stays no for subgroup-closed formations containing all supersoluble groups, for formations contained in $\mathfrak{N}_p\mathfrak{N}_r$, and for the formations of metanilpotent and of nilpotent-by-abelian groups. For the formation of supersoluble groups the answer is yes; we include a proof, not claimed as new, which uses the theorem of Zhurtov and Syskin as stated in its review. The least-order statement does not use that theorem. No printed answer to the question was found in the sources we searched; a search that finds nothing is not a proof of novelty, and the construction may be known to specialists. This is an unrefereed note.

1 Introduction

1.1 The question

All groups are finite. For a formation $\mathfrak{F}$, a group G is a *minimal non- $\mathfrak{F}$ -group* (or $\mathfrak{F}$ -critical) if $G \notin \mathfrak{F}$ while every proper subgroup of G lies in $\mathfrak{F}$, and the $\mathfrak{F}$ -residual $G^{\mathfrak{F}}$ is the smallest normal subgroup of G with quotient in $\mathfrak{F}$. Problem 15.98 of the Kourovka Notebook [KM26], posed by L. A. Shemetkov in the 15th issue (2002), concerns a saturated formation $\mathfrak{F}$ and a soluble minimal non- $\mathfrak{F}$ -group G such that $G^{\mathfrak{F}}$ is a Sylow p -subgroup of G . It asks whether $G^{\mathfrak{F}}$ must then be “isomorphic to a factor group of the Sylow p -subgroup of $\text{PSU}(3, p^{2n})$ ”. The Notebook adds that this holds for the formation $\mathfrak{N}$ of nilpotent groups, citing Mazurov and Syskin [MS73] and Zhurtov and Syskin [ZS87]. We read the problem in versions 48 and 49 of the Notebook (6 and 20 October 2026, printed page 99); the text is

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the same in both, no comment follows it, and it is not marked as solved. It is record KOU-15.98 of the UnsolvedMath collection [ula26], listed there as open, with the Notebook’s text.

Conventions. $\text{PSU}(3, p^{2n})$ is the projective special unitary group of degree 3 over the field with p^{2n} elements, the group $U_3(p^n)$ of order $p^{3n}(p^{2n} - 1)(p^{3n} + 1)/\gcd(3, p^n + 1)$; the zbMATH review of [ZS87] states the known case in the same notation. Its Sylow p -subgroup is denoted $U(p^n)$ (Section 3). The exponent $n \geq 1$ is not fixed, so the question is whether $G^{\mathfrak{F}}$ is isomorphic to a factor group of $U(p^n)$ for some n ; factor groups may be trivial or the whole group. Under the other conceivable reading, over the field with p^{4n} elements, the Sylow subgroup would be $U(p^{2n})$, and the statement would fail already for $\mathfrak{F} = \mathfrak{N}$: $\text{SL}(2, 3) = Q_8 \rtimes C_3$ is a minimal non-nilpotent group with nilpotent residual $Q_8 \cong U(2)$, and $|Q_8 : Q'_8| = 4$ is not of the form p^{4n} (Lemma 3.1). So the first reading is the intended one; our counterexamples work under both, while the least-order statement refers to the first (under the second, $\text{SL}(2, 3)$, of order 24, would already be a counterexample).

1.2 Results

Theorem 1.1. *For Problem 15.98 as posed, the answer is no: there is a saturated formation $\mathfrak{F}$ and a finite soluble minimal non- $\mathfrak{F}$ -group G of order 2304 whose $\mathfrak{F}$ -residual is a Sylow 2-subgroup isomorphic to no factor group of a Sylow 2-subgroup of $\text{PSU}(3, 2^{2n})$; 2304 is the least order of such a counterexample.*

Here “no factor group” means: for no $n \geq 1$. Explicitly (Theorem 4.2), let $\mathfrak{F} = \text{LF}(f)$ be the local formation with $f(2) = \mathfrak{A}(3)$, the class of abelian groups of exponent dividing 3, and $f(l)$ the class of all groups for odd primes l . Let $P = \mathbb{F}_{64} \times \mathbb{F}_4$ with the product $(v, w)(v', w') = (v + v', w + w' + \text{Tr}_{\mathbb{F}_{64}/\mathbb{F}_4}(vv'^2))$, and let h act on P by $(v, w) \mapsto (\zeta v, \zeta^3 w)$, where $\zeta \in \mathbb{F}_{64}$ has order 9. Then $G = P \rtimes \langle h \rangle$ is a minimal non- $\mathfrak{F}$ -group of order $2^8 \cdot 3^2$ with $G^{\mathfrak{F}} = P$. In every nonabelian factor group Q of $U(2^n)$ the set $[y, Q]$ of commutators equals Q' for each $y \notin Z(Q)$ (Lemma 3.1), while P' has order 4 and $[(1, 0), P]$ has two elements. The least order is Theorem 5.4. The other readings that we examine are summarised in Table 1; in each “no” case the witness is a soluble group, so nothing changes if formations are taken in the universe of soluble groups.

Condition on the saturated formation $\mathfrak{F}$	Answer	Where
none (the question as posed)	no; least order 2304	Theorems 4.2 and 5.4
subgroup-closed, $\mathfrak{U} \subseteq \mathfrak{F}$, $f(p)$ abelian	no	Proposition 6.1
$\mathfrak{F} \subseteq \mathfrak{N}_p \mathfrak{N}_r$ for primes $p \neq r$	no	Proposition 6.1
$\mathfrak{F} = \mathfrak{N}^2$ or $\mathfrak{F} = \mathfrak{N}\mathfrak{A}$	no	Proposition 6.3
$\mathfrak{F} = \mathfrak{U}$	yes (via [ZS87]; not new)	Theorem 6.5
$\mathfrak{F} = \mathfrak{N}$	yes [MS73, ZS87]	Remark 6.4

Table 1: Answers to Problem 15.98 under the readings examined here; $\mathfrak{N}^2$ is the class of metanilpotent groups, $\mathfrak{N}\mathfrak{A}$ that of nilpotent-by-abelian groups, $\mathfrak{U}$ that of supersoluble groups.

1.3 Context

For $\mathfrak{F} = \mathfrak{N}$ the groups in question are the Schmidt groups (minimal non-nilpotent groups), and Theorem 1 of [ZS87] says, according to its zbMATH review (Zbl 0622.20010), that the normal Sylow p -subgroup of a Schmidt group is abelian or isomorphic to U/Z with U a Sylow p -subgroup of $\text{PSU}_3(p^{2n})$ and $Z \leq Z(U)$; for $p = 2$ this is due to [MS73]. We know both papers only from zbMATH: the review of [ZS87], which also attributes the case $p = 2$ to [MS73], and the record of [MS73]. V. N. Semenchuk

studied minimal non- $\mathfrak{F}$ -groups in [Sem79] and in later papers; we have not read them, and the question was posed after [Sem79] and is still listed as open. Ballester-Bolínches, Esteban-Romero and Robinson [BBERR05] determine the critical groups for several classes and classify the Schmidt groups, and Ballester-Bolínches and Esteban-Romero [BBER07] classify the minimal non- p -supersoluble groups in the p -soluble universe; according to their abstracts they concern specific classes, and they do not answer Problem 15.98. In the terminology of Beisiegel [Bei77] (see the survey [Lew17]), Lemma 3.1 says that every nonabelian factor group of $U(q)$ is semi-extraspecial (for $U(q)$ itself this is due to Beisiegel), and the bound $|Q'|^2 \leq |Q : Q'|$ is Beisiegel's inequality for such groups; Brenner, Camina and Lewis [BCL25] characterise the Sylow p -subgroups of $\mathrm{SU}_3(p^{2a})$ among semi-extraspecial groups by an automorphism of large order. Our residual P is special but not semi-extraspecial.

The counterexamples, the least order and the answers for the restricted readings are, to our knowledge, not in print; the construction (Lemma 4.1) is elementary and may be known to specialists. Theorem 6.5 ($\mathfrak{F} = \mathfrak{U}$) is not claimed as new. The searches and their limits are described in the paragraph “Scope and priority” (Section 8).

2 Preliminaries

For a group X we write X' , $Z(X)$, $\Phi(X)$ for the derived subgroup, the centre and the Frattini subgroup, $[x, y] = x^{-1}y^{-1}xy$, and, for $x \in X$, $[x, X] = \{[x, y] : y \in X\}$, the *set* of commutators of x . If X has nilpotency class at most 2, then $y \mapsto [x, y]$ is a homomorphism $X \rightarrow X'$, so $[x, X]$ is a subgroup; this applies to all groups $[x, U]$, $[y, Q]$ and $[x, P]$ below, and $|[x, X]|$ is the length of the conjugacy class of x . A p -group P is *special* if $Z(P) = P' = \Phi(P)$ is elementary abelian, and *extraspecial* if moreover $|Z(P)| = p$. We write $\mathfrak{E}$ for the class of all groups, $\mathfrak{A}$ for the abelian groups, $\mathfrak{A}(e)$ for the abelian groups of exponent dividing e , $\mathfrak{N}$, $\mathfrak{U}$ for the nilpotent and the supersoluble groups, $\mathfrak{N}^2$ and $\mathfrak{N}\mathfrak{A}$ for the groups with a nilpotent normal subgroup with nilpotent, respectively abelian, quotient, and $\mathfrak{N}_p\mathfrak{N}_r$ for the groups with a normal p -subgroup whose quotient is an r -group.

A *formation* is a class of groups closed under homomorphic images and finite subdirect products ($X/K_1, X/K_2 \in \mathfrak{F}$ implies $X/(K_1 \cap K_2) \in \mathfrak{F}$); it is *saturated* if $X/\Phi(X) \in \mathfrak{F}$ implies $X \in \mathfrak{F}$. For a nonempty formation $\mathfrak{F}$ the $\mathfrak{F}$ -residual exists. A *formation function* f assigns to each prime l a formation $f(l)$, possibly empty, and $\mathrm{LF}(f)$ is the class of groups X such that $X/C_X(A/B) \in f(l)$ for every chief factor A/B of X and every prime l dividing $|A/B|$. Every $\mathrm{LF}(f)$ is a saturated formation, and conversely every saturated formation is of this form (the theorem of Gaschütz, Lubeseder and Schmid); see [DH92, Chapter IV, §§3–4] and [BBE06, Chapter 3]. The negative answer uses only the first, easy statement; the converse is used only in Theorem 5.4, where $\mathfrak{F}$ is an arbitrary saturated formation (Remark 6.4 and Theorem 6.5 use the explicit local definitions below). For example $\mathfrak{N} = \mathrm{LF}(f)$ with $f(l) = \{1\}$ for all l , and $\mathfrak{U} = \mathrm{LF}(u)$ with $u(l) = \mathfrak{A}(l-1)$: a chief factor of order l has centralizer quotient inside $\mathrm{Aut}(C_l) \cong C_{l-1}$; conversely, a nonabelian chief factor embeds in its centralizer quotient, and an irreducible $\mathbb{F}_l X$ -module on which X acts through an abelian group of exponent dividing $l-1$ has dimension 1. If $f(l) \subseteq f'(l)$ for all l , then $\mathrm{LF}(f) \subseteq \mathrm{LF}(f')$.

We use the following standard facts [DH92, Hup67].

- (F1) If K is a normal p -subgroup of X and A/B is a chief factor of X with $A \leq K$, then $K \leq C_X(A/B)$: the fixed points of K on the p -group A/B form a nontrivial normal subgroup of X/B .
- (F2) X -isomorphic chief factors have the same centralizer, and any two chief series of X have pairwise X -isomorphic factors (Jordan–Hölder); so membership in $\mathrm{LF}(f)$ can be tested on one chief series.
- (F3) Let a p' -group H act on a p -group P . If H acts trivially on $P/\Phi(P)$, or on all factors of an H -invariant normal series of P , then it acts trivially on P ; for an H -invariant normal subgroup

K of P , $C_{P/K}(H) = C_P(H)K/K$; and $P = [P, H]C_P(H)$. Modules of H over $\mathbb{F}_p$ are semisimple (Maschke).

(F4) (Schur–Zassenhaus) A normal Sylow p -subgroup of X has a complement, and all complements are conjugate.

3 Factor groups of the Sylow p -subgroup of $\text{PSU}(3, q^2)$

Let $q = p^n$, $K = \mathbb{F}_{q^2}$ and $\bar{a} = a^q$. On K^3 take the Hermitian form $(u, v) = u_1\bar{v}_3 + u_2\bar{v}_2 + u_3\bar{v}_1$, with Gram matrix $J = \text{antidiag}(1, 1, 1)$. For an upper unitriangular matrix g with entries a, b, c in positions $(1, 2), (1, 3), (2, 3)$, the entries of $g^T J \bar{g}$ in positions $(2, 3), (3, 2), (3, 3)$ are $a + \bar{c}, c + \bar{a}, b + \bar{b} + c\bar{c}$, and the others are those of J . So the unitriangular elements of $\text{SU}(3, q^2)$ are the matrices

$$u(a, b) = \begin{pmatrix} 1 & a & b \\ 0 & 1 & -\bar{a} \\ 0 & 0 & 1 \end{pmatrix}, \quad b + \bar{b} + a\bar{a} = 0,$$

and they form a group $U(q)$ with

$$u(a, b)u(a', b') = u(a + a', b + b' - a\bar{a}'). \quad (1)$$

For each a the equation $b + \bar{b} = -a\bar{a}$ has q solutions, since the trace $K \rightarrow \mathbb{F}_q$ is onto; so $|U(q)| = q^3$, the full p -part of $|\text{SU}(3, q^2)| = q^3(q^2 - 1)(q^3 + 1)$. The centre of $\text{SU}(3, q^2)$ has order $\gcd(3, q + 1)$, prime to p , so $U(q)$ is also isomorphic to a Sylow p -subgroup of $\text{PSU}(3, q^2)$. From (1), $u(a, b)^k = u(ka, kb - \binom{k}{2}a\bar{a})$, the elements $u(0, e)$ with $e + \bar{e} = 0$ are central, and

$$[u(a, b), u(a', b')] = u(0, a'\bar{a} - a\bar{a}'), \quad (2)$$

because $xy = u(a + a', b + b' - a\bar{a}')$, $yx = u(a + a', b + b' - a'\bar{a})$ and $u(c, d)u(0, e) = u(c, d + e)$.

Lemma 3.1. *Let $q = p^n$, $U = U(q)$ and $Z = \{u(0, e) : e + \bar{e} = 0\}$.*

- (a) $Z = Z(U) = U' = \Phi(U)$ has order q , $U/Z \cong (K, +)$, and $[x, U] = Z$ for every $x \in U \setminus Z$.
- (b) Let $N \trianglelefteq U$ and $Q = U/N$. If Q is abelian, then Q is elementary abelian. If Q is nonabelian, then $N < Z$, $Z(Q) = Q' = \Phi(Q) = Z/N$, $|Q : Q'| = q^2$ and $1 < |Q'| \leq q$, so that $|Q'|^2 \leq |Q : Q'|$; and $[y, Q] = Q'$ for every $y \in Q \setminus Z(Q)$.

Proof. (a) Let $a \neq 0$. The map $a' \mapsto a'\bar{a} - a\bar{a}'$ is $\mathbb{F}_p$ -linear on K with kernel $a\mathbb{F}_q$ (for $a' \neq 0$, $a'a^q = aa'^q$ means $(a'/a)^q = a'/a$), so its image has q elements; the image lies in $\{e : e + \bar{e} = 0\}$, which also has q elements. By (2), $[x, U] = Z$ for $x = u(a, b)$, $a \neq 0$. The elements of Z are central, so $Z(U) = Z = U'$. By (1), $u(a, b) \mapsto a$ induces $U/Z \cong (K, +)$, which is elementary abelian; hence $\Phi(U) = U'$.

(b) If Q is abelian, then $Z = U' \leq N$. If $N \not\leq Z$, take $x \in N \setminus Z$; then $Z = [x, U] \subseteq N$. In both cases Q is a quotient of the elementary abelian group U/Z . Now let Q be nonabelian. Then $N \leq Z$, $N \neq Z$, and $Q' = Z/N \neq 1$. For $x \in U \setminus Z$ we get $[xN, Q] = [x, U]N/N = Z/N = Q' \neq 1$, so xN is not central; hence $Z(Q) = Z/N = Q'$. As $Q/Q' \cong U/Z$ is elementary abelian, $\Phi(Q) = Q'$. Finally $|Q : Q'| = |U : Z| = q^2$ and $|Q'| = |Z : N| \leq q$. $\square$

Corollary 3.2. *If a p -group P is isomorphic to a factor group of $U(p^n)$, then P is elementary abelian, or P is nonabelian with $Z(P) = P' = \Phi(P)$, $|P : P'| = p^{2n}$, $|P'|^2 \leq |P : P'|$ and $[y, P] = P'$ for all $y \in P \setminus Z(P)$. Conversely, an elementary abelian group of order p^m is a factor group of $U(p^n)$ whenever $2n \geq m$. The same holds for $U(p^{2n})$, with p^{4n} in place of p^{2n} .*

Remark 3.3. A p -group X is *semi-extraspecial* if X/M is extraspecial for every maximal subgroup M of $Z(X)$ [Bei77]. By a theorem of Verardi, as stated in [Lew17, Theorem 5.2], these are the groups of class 2 in which the conjugacy class of each $x \notin X'$ is xX' , that is, $[x, X] = X'$. Beisiegel showed that the Sylow p -subgroups of $SU_3(p^{2a})$ are semi-extraspecial [Bei77, Lemmas 4 and 5], as reported in [Lew17, §4]; Lemma 3.1(a) is a direct proof of this, and Lemma 3.1(b) says that every nonabelian factor group of $U(q)$ is semi-extraspecial. The inequality $|Q'|^2 \leq |Q : Q'|$ is an instance of Beisiegel's bound for such groups [Bei77, Satz 1], as reported in [Lew17, §4]. We do not use these results.

4 A construction and two counterexamples

Lemma 4.1. *Let p be a prime, $\mathfrak{X}$ a formation, and H a soluble p' -group acting faithfully on a p -group $P \neq 1$ such that*

- (i) $V = P/\Phi(P)$ is an irreducible $\mathbb{F}_p H$ -module;
- (ii) $H \notin \mathfrak{X}$, and every proper subgroup of H lies in $\mathfrak{X}$;
- (iii) $H/C_H(\Phi(P)) \in \mathfrak{X}$.

Let $f(p) = \mathfrak{X}$, $f(l) = \mathfrak{E}$ for primes $l \neq p$, and $\mathfrak{F} = \text{LF}(f)$. Then $\mathfrak{F}$ is a saturated formation, $G = P \rtimes H$ is a soluble minimal non- $\mathfrak{F}$ -group, and $G^\mathfrak{F} = P$ is the normal Sylow p -subgroup of G .

The formation $\mathfrak{X}$ need not be subgroup-closed.

Proof. $\mathfrak{F}$ is saturated (Section 2), and G is soluble. For $L \leq G$, $L \cap P$ is the normal Sylow p -subgroup of L , and the chief factors of L above $L \cap P$ have order prime to p , with no condition attached ($f(l) = \mathfrak{E}$). By (F2), therefore,

$$(*) \quad L \in \mathfrak{F} \iff L/C_L(A/B) \in \mathfrak{X} \text{ for every chief factor } A/B \text{ of } L \text{ with } A \leq L \cap P.$$

$G \notin \mathfrak{F}$: V is a chief factor of G by (i), and $C_H(V) = C_H(P) = 1$ by (F3), so $C_G(V) = P$ and $G/C_G(V) \cong H \notin \mathfrak{X}$.

Let $L < G$ and $K = L \cap P$; by (F4), $L = KL_0$ with $L_0 \cong L/K \cong LP/P$. If $LP \neq G$, then L_0 is isomorphic to a proper subgroup of H , so $L_0 \in \mathfrak{X}$ by (ii), and by (F1) every quotient $L/C_L(A/B)$ in $(*)$ is a quotient of $L/K \cong L_0$, hence lies in $\mathfrak{X}$. If $LP = G$, then L_0 is a complement of P , conjugate to H by (F4); after conjugation $H \leq L$ and $L = KH$ with $K < P$ invariant under H . Then $K\Phi(P)/\Phi(P)$ is a proper submodule of V , as $K\Phi(P) = P$ would give $K = P$; so $K \leq \Phi(P)$ by (i), and $C := C_H(\Phi(P))$ centralizes K . Now $KC = K \times C$ is normal in L with $L/KC \cong H/C$, and both K (by (F1)) and C centralize every A/B in $(*)$; so $L/C_L(A/B)$ is a quotient of $H/C \in \mathfrak{X}$. Hence $L \in \mathfrak{F}$.

Finally $G/P \cong H$ is a p' -group, so $G/P \in \mathfrak{F}$ and $G^\mathfrak{F} \leq P$. If $G^\mathfrak{F} < P$, then $G^\mathfrak{F} \leq \Phi(P)$ as above, and V would be a chief factor of $G/G^\mathfrak{F} \in \mathfrak{F}$ with centralizer quotient $H \notin \mathfrak{X}$. So $G^\mathfrak{F} = P$. $\square$

If $H = \langle h \rangle \cong C_{r^a}$ for a prime $r \neq p$ and $a \geq 1$, and $\mathfrak{X} = \mathfrak{A}(r^{a-1})$, then (ii) holds, and (iii) says that $h^{r^{a-1}}$ centralizes $\Phi(P)$.

Theorem 4.2 (Witness A). *Let $\zeta \in \mathbb{F}_{64}$ have order 9, so that $\omega = \zeta^3$ generates $\mathbb{F}_4^* \subset \mathbb{F}_{64}^*$. Let $T(y) = y + y^4 + y^{16}$ be the trace $\mathbb{F}_{64} \rightarrow \mathbb{F}_4$, $\gamma(v, v') = T(vv'^2)$, and let $P = \mathbb{F}_{64} \times \mathbb{F}_4$ with*

$$(v, w)(v', w') = (v + v', w + w' + \gamma(v, v')), \quad h(v, w) = (\zeta v, \zeta^3 w).$$

Let $\mathfrak{F} = \text{LF}(f)$ with $f(2) = \mathfrak{A}(3)$ and $f(l) = \mathfrak{E}$ for odd l .

- (a) P is a special group of order 2^8 with $Z(P) = P' = \Phi(P) = \{0\} \times \mathbb{F}_4$, and $[(1,0), P] = \{(0,0), (0,1)\}$.
- (b) h is an automorphism of P of order 9, h^3 centralizes $\Phi(P)$, and $P/\Phi(P)$ is a faithful irreducible $\mathbb{F}_2\langle h \rangle$ -module.
- (c) $G = P \rtimes \langle h \rangle$, of order 2304, is a soluble minimal non- $\mathfrak{F}$ -group, and $G^\mathfrak{F} = P$ is a Sylow 2-subgroup of G .
- (d) P is isomorphic to no factor group of $U(2^n)$, and to no factor group of $U(4^n)$, $n \geq 1$.

Proof. (a) γ is biadditive ($v' \mapsto v'^2$ is additive), so the product is associative, $(0,0)$ is the identity, $(v,w)^{-1} = (v, w + \gamma(v,v))$, and the elements $(0,w)$ are central. One finds $[(v,w), (v',w')] = (0, \beta(v,v'))$ with $\beta(v,v') = T(vv'^2 + v^2v')$, and $(v,w)^2 = (0, T(v^3))$. For $d \in \mathbb{F}_{64}^*$ let $B_d(v,v') = \text{Tr}_{\mathbb{F}_{64}/\mathbb{F}_2}(d(vv'^2 + v^2v'))$. Since $\text{Tr}(uy^2) = \text{Tr}(u^3y^2)$, we have $B_d(v,v') = \text{Tr}(((dv)^{32} + dv^2)v')$, so v lies in the radical of B_d if and only if $(dv)^{32} = dv^2$, that is (squaring) $dv = d^2v^4$, that is $v = 0$ or $v^3 = d^{-1}$. The linear functionals on $\mathbb{F}_4$ are $y \mapsto \text{Tr}_{\mathbb{F}_4/\mathbb{F}_2}(\mu y)$, $\mu \in \mathbb{F}_4$, and $\text{Tr}_{\mathbb{F}_4/\mathbb{F}_2}(\mu\beta(v,v')) = B_\mu(v,v')$. For a central element (v,w) with $v \neq 0$, v would lie in the radicals of B_1 and B_ω , so that $v^3 = 1$ and $v^3 = \omega^{-1} \neq 1$; hence $Z(P) = \{0\} \times \mathbb{F}_4$. The values of β span a nonzero $\mathbb{F}_2$ -subspace of $\mathbb{F}_4$, which is invariant under multiplication by ω because $\beta(\zeta v, \zeta v') = \omega\beta(v, v')$; so it is $\mathbb{F}_4$. Hence $P' = \{0\} \times \mathbb{F}_4 = Z(P)$, and as squares lie in P' , $\Phi(P) = P'$. Finally $\beta(1, v') = T(v'^2 + v') = t^2 + t$ with $t = T(v') \in \mathbb{F}_4$, since $T(y^2) = T(y)^2$; and $t^2 + t \in \mathbb{F}_2$ for $t \in \mathbb{F}_4$. So $[(1,0), P] \subseteq \{(0,0), (0,1)\}$, with equality as $(1,0) \notin Z(P)$.

(b) $\gamma(\zeta v, \zeta v') = T(\zeta^3 vv'^2) = \zeta^3 \gamma(v, v')$, since T is $\mathbb{F}_4$ -linear; so h is an automorphism. It has order 9, and $h^3(v, w) = (\zeta^3 v, w)$. On $V = P/\Phi(P) \cong (\mathbb{F}_{64}, +)$, h acts as multiplication by ζ , whose minimal polynomial over $\mathbb{F}_2$ has degree $\text{ord}_9(2) = 6$; so $\mathbb{F}_2[\zeta] = \mathbb{F}_{64}$, and the nonzero h -invariant subspaces of V are $\mathbb{F}_{64}$ -subspaces, that is, V .

(c) By (b) and the remark after Lemma 4.1 (with $p = 2$, $r = 3$, $a = 2$), Lemma 4.1 applies; $|G| = 2^8 \cdot 9$.

(d) P is nonabelian, and $(1,0) \notin Z(P)$ has $[(1,0), P] \neq P'$. So Corollary 3.2 excludes both cases. $\square$

Corollary 3.2 also shows that only $U(8)$ could have a factor group of the order and shape of P ($|P : P'| = 2^6$). A second invariant separates P from these quotients. A normal subgroup N of $U(8)$ of order 2 is central, $N = \{1, u(0, e_0)\}$ with $e_0 \in \mathbb{F}_8^*$, and by (1) $u(a, b)^2 = u(0, a^9)$. So the involutions of $U(8)/N$ are the three nontrivial elements of $Z(U(8))/N$ and, for each of the 9 elements a with $a^9 = e_0$, four cosets $u(a, b)N$: there are 39. In P , (v, w) is an involution if and only if $(v, w) \neq 1$ and $T(v^3) = 0$, and a direct count (Section 8) gives 28 such v and 111 involutions.

Proposition 4.3 (Witness B). *Let $\zeta \in \mathbb{F}_{81}$ have order 16, $\beta(x, y) = xy^3 - x^3y$, and let $P = \mathbb{F}_{81} \times \mathbb{F}_{81}$ with $(v, w)(v', w') = (v + v', w + w' + \beta(v, v'))$ and $h(v, w) = (\zeta v, \zeta^4 w)$. Let $\mathfrak{F} = \text{LF}(f)$ with $f(3) = \mathfrak{A}(8)$ and $f(l) = \mathfrak{E}$ for $l \neq 3$. Then $G = P \rtimes \langle h \rangle$, of order $3^8 \cdot 2^4 = 104976$, is a soluble minimal non- $\mathfrak{F}$ -group with $G^\mathfrak{F} = P$, and $|P'| = |P : P'| = 3^4$. So P is isomorphic to no factor group of $U(3^n)$ or of $U(9^n)$.*

Proof. β is $\mathbb{F}_3$ -bilinear and alternating, with $\beta(\zeta x, \zeta y) = \zeta^4 \beta(x, y)$. So P is a group, $(v, w)^{-1} = (-v, -w)$, h is an automorphism of order 16, $[(v, w), (v', w')] = (0, 2\beta(v, v'))$ and $(v, w)^3 = (0, 2\beta(v, v)) = (0, 0)$. For $x \neq 0$, $\beta(x, y) = x^4(t^3 - t)$ with $t = y/x$, and $t \mapsto t^3 - t$ has kernel $\mathbb{F}_3$ and image $\ker \text{Tr}_{\mathbb{F}_{81}/\mathbb{F}_3}$; so $\beta(x, \cdot) \neq 0$ and $Z(P) = \{0\} \times \mathbb{F}_{81}$. If the span of the values of β were not $\mathbb{F}_{81}$, it would be $\ker \text{Tr}$, which would then be invariant under multiplication by the fourth powers; these form the group of order 20, whose generator has degree $\text{ord}_{20}(3) = 4$ over $\mathbb{F}_3$, and $\ker \text{Tr}$ would be a 3-dimensional $\mathbb{F}_{81}$ -subspace. So $P' = \Phi(P) = Z(P) = \{0\} \times \mathbb{F}_{81}$. On $P/\Phi(P)$, h acts as ζ ,

irreducibly since $\text{ord}_{16}(3) = 4$, and h^8 acts on $\Phi(P)$ as $\zeta^{32} = 1$. Lemma 4.1 with $r = 2$, $a = 4$, $\mathfrak{X} = \mathfrak{A}(8)$ gives the claims on G , and $|P'|^2 = 3^8 > 3^4 = |P : P'|$ contradicts Corollary 3.2. $\square$

5 The least order of a counterexample

We first record two elementary lemmas.

Lemma 5.1. *Let $\zeta \in \mathbb{F}_{p^m}$ have multiplicative order k and $\mathbb{F}_p[\zeta] = \mathbb{F}_{p^m}$, and let $C = \langle \zeta \rangle$ act on $V = \mathbb{F}_{p^m}$ by multiplication.*

- (a) *The C -invariant bilinear forms on V form a space of dimension $\#\{(i, j) : 0 \leq i, j < m, k \mid p^i + p^j\}$.*
- (b) *Let $p = 2$, $m = 2n$, $q = 2^n$ and $k = q + 1$, and let W be an $\mathbb{F}_2$ -space. Every C -invariant quadratic map $Q : V \rightarrow W$ is $\lambda \circ N$ for an $\mathbb{F}_2$ -linear $\lambda : \mathbb{F}_q \rightarrow W$, where $N(a) = a^{q+1}$.*

Proof. (a) The minimal polynomial of ζ has the m distinct roots ζ^{p^i} , so $V \otimes \overline{\mathbb{F}}_p = \bigoplus_{i < m} L_i$ with ζ acting on the line L_i as ζ^{p^i} . An invariant form vanishes on $L_i \times L_j$ unless $\zeta^{p^i + p^j} = 1$, and on each such pair it may be chosen freely. The invariance conditions are linear over $\mathbb{F}_p$, so the dimension over $\mathbb{F}_p$ is the same.

(b) It suffices to take $W = \mathbb{F}_2$. As $2^n \equiv -1 \pmod{k}$, the order of 2 modulo k is $2n$, and for $0 \leq i \leq j < 2n$ we have $k \mid 2^i + 2^j$ only if $j = i + n$. Extend Q to a quadratic form on $V \otimes \overline{\mathbb{F}}_2$. For $v \in L_i$ and $\mu = \zeta^{2^i}$ we get $Q(v) = Q(\mu v) = \mu^2 Q(v)$ with $\mu^2 \neq 1$, so Q vanishes on every L_i and is determined by its polarization, an invariant symmetric form. By (a) the invariant symmetric forms live on the n pairs $L_i \times L_{i+n}$, $L_{i+n} \times L_i$, so the invariant quadratic forms form a space of dimension at most n . The 2^n forms $\pi \circ N$, $\pi \in \text{Hom}(\mathbb{F}_q, \mathbb{F}_2)$, are invariant quadratic forms (the polarization of N is $aa'^q + a^q a'$, and $N(\zeta a) = N(a)$), and they are distinct because N is onto $\mathbb{F}_q$. $\square$

Lemma 5.2. *Let P_1, P_2 be p -groups with $\Phi(P_i) = P'_i \leq Z(P_i)$ elementary abelian, and of exponent p if p is odd. Let $V_i = P_i/\Phi(P_i)$, $W_i = \Phi(P_i)$, and let κ_i be the squaring map $V_i \rightarrow W_i$, $x\Phi(P_i) \mapsto x^2$, if $p = 2$, and the commutator map $V_i \times V_i \rightarrow W_i$ if p is odd. If there are linear isomorphisms $\sigma : V_1 \rightarrow V_2$ and $\tau : W_1 \rightarrow W_2$ with $\kappa_2 \circ \sigma = \tau \circ \kappa_1$, then $P_1 \cong P_2$.*

Proof. For $p = 2$ the commutator map is the polarization of the squaring map, as $(xy)^2 = x^2 y^2 [y, x]$; so σ, τ also match the commutator maps. Let $e_1, \dots, e_m$ be a basis of V_1 , and let $x_i \in P_1$, $y_i \in P_2$ map to e_i and $\sigma(e_i)$. Every element of P_1 is uniquely $x_1^{a_1} \cdots x_m^{a_m} w$ with $0 \leq a_i < p$ and $w \in W_1$, and since W_1 is central and commutators are bilinear,

$$\left(\prod_i x_i^{a_i} w \right) \left(\prod_i x_i^{b_i} w' \right) = \prod_i x_i^{c_i} \cdot ww' \cdot \prod_{i < j} [x_j, x_i]^{a_j b_i} \cdot \prod_{a_i + b_i \geq p} x_i^p,$$

where $c_i \equiv a_i + b_i \pmod{p}$, $0 \leq c_i < p$. Here $x_i^p = 1$ if p is odd and $x_i^2 = \kappa_1(e_i)$ if $p = 2$. The same formula holds in P_2 , so $\prod x_i^{a_i} w \mapsto \prod y_i^{a_i} \tau(w)$ is an isomorphism. $\square$

Setting. Let $\mathfrak{F}$ be a saturated formation, so $\mathfrak{F} = \text{LF}(f)$ for some f (Section 2), and G a soluble minimal non- $\mathfrak{F}$ -group with $P = G^{\mathfrak{F}} \in \text{Syl}_p(G)$. Let H be a complement of P (F4), $V = P/\Phi(P)$, $m = \dim V$, $C = C_H(P)$, $\tilde{H} = H/C$, $N = C_H(\Phi(P))$, $\tilde{N} = N/C$, $W = P'$ and $d = \dim W$ when P' is elementary abelian.

Proposition 5.3. *In this setting:*

- (a) *$P \neq 1$, and V is a faithful irreducible $\mathbb{F}_p \tilde{H}$ -module.*

- (b) If $f(p) = \emptyset$, then $G \cong C_p$. Otherwise $\tilde{H} \notin f(p)$, every proper subgroup of $\tilde{H}$ lies in $f(p)$, $H/N \in f(p)$, $\tilde{N} \neq 1$ and $C_V(\tilde{N}) = 0$.
- (c) $\Phi(P) = P' \leq Z(P)$, P' is elementary abelian, and P has exponent p if p is odd. If P is nonabelian, then P is special, the commutator map $\beta: V \times V \rightarrow W$ has radical 0 (so $d \geq 2$ if m is odd), and for every nonzero linear functional λ on W , $\lambda \circ \beta$ is a nonzero $\tilde{N}$ -invariant alternating form on V .
- (d) If P is abelian, it is elementary abelian, hence a factor group of $U(p^n)$ for $2n \geq m$.
- (e) If $m \leq 2$, then P is elementary abelian or $P \cong U(p)$.
- (f) If $|\tilde{H}|$ is prime, then P is abelian or isomorphic to U/Z_0 , where U is a Sylow p -subgroup of $\text{PSU}(3, p^{2n})$ for some n and $Z_0 \leq Z(U)$. This uses Theorem 1 of [ZS87] as stated in its zbMATH review (for $p = 2$, [MS73]).

Proof. (a) $P \neq 1$ as $G \notin \mathfrak{F}$. If V were reducible, then by (F3) $V = R/\Phi(P) \oplus S/\Phi(P)$ with proper H -invariant subgroups R, S , normal in G as they contain P' . The proper subgroups RH and SH lie in $\mathfrak{F}$; since $RH \cap S = R \cap S = \Phi(P)$ and $(RH)S = G$, we get $G/S \cong RH/\Phi(P) \in \mathfrak{F}$, likewise $G/R \in \mathfrak{F}$, so $G/\Phi(P) \in \mathfrak{F}$. As $\Phi(P) \leq \Phi(G)$, saturation gives $G \in \mathfrak{F}$, a contradiction. By (F3), $C_H(V) = C$.

(b) If $f(p) = \emptyset$, the members of $\mathfrak{F}$ are p' -groups, all proper subgroups of G are p' -groups, and $G \cong C_p$. Let $f(p) \neq \emptyset$ and take a chief series of G through $\Phi(P)$ and P . The factors above P are chief factors of $G/P \in \mathfrak{F}$. A chief factor A/B with $A \leq \Phi(P)$ is centralized by P (F1), so it is a chief factor of the proper subgroup $K = \Phi(P)H \in \mathfrak{F}$, and $G/C_G(A/B) \cong H/C_H(A/B) \cong K/C_K(A/B) \in f(p)$. If $G/C_G(V) \cong \tilde{H}$ were in $f(p)$, then $G \in \mathfrak{F}$; so $\tilde{H} \notin f(p)$. By (F3) the intersection of the $C_H(A/B)$ for the factors inside $\Phi(P)$ is N , so $H/N \in f(p)$, and $N \neq C$. For $C \leq H_0 < H$, the proper subgroup PH_0 lies in $\mathfrak{F}$; V is a sum of simple H_0 -modules X_i , which are chief factors of PH_0 for a suitable chief series, and $\bigcap_i C_{H_0}(X_i) = C$; so $H_0/C \in f(p)$. Finally $C_V(\tilde{N})$ is a submodule different from V , hence 0.

(c) By (b) and (F3), $C_P(N)\Phi(P)/\Phi(P) = C_V(N) = 0$, so $P = [P, N]C_P(N) = [P, N]\Phi(P)$ and $P = [P, N]$. As N centralizes $\Phi(P)$, $[[N, \Phi(P)], P] = 1 = [[\Phi(P), P], N]$, and the three subgroups lemma gives $[[P, N], \Phi(P)] = 1$, that is, $\Phi(P) \leq Z(P)$. So P has class at most 2, $[x, y]^p = [x^p, y] = 1$, and P' is elementary abelian. Let $\pi(x) = x^p$ if p is odd and $\pi(x) = x^2P'$ if $p = 2$. Since $(xy)^p = x^p y^p [y, x]^{p(p-1)/2}$, π is an H -equivariant homomorphism into $\Phi(P)$, respectively $\Phi(P)/P'$, on which N acts trivially; so $\pi([x, n]) = \pi(x)^{-1}\pi(x)^n = 1$ for $n \in N$, and $P = [P, N] \leq \ker \pi$. Hence P has exponent p for odd p , and $\Phi(P) = P'$ in all cases. The image of $Z(P)$ in V is a submodule; if it is V , then $P = Z(P)$. So for nonabelian P , $Z(P) \leq \Phi(P) \leq Z(P)$, and the radical of β is $Z(P)/\Phi(P) = 0$; if $d = 1$, β is a nondegenerate alternating form, so m is even. The map β is bilinear, alternating and H -equivariant, its values generate W , and N centralizes W ; this gives the last claim.

(d) By (c), $\Phi(P) = P' = 1$; then use Corollary 3.2.

(e) Let P be nonabelian; then $m = 2$, $|W| = p$ and $|P| = p^3$. For odd p , P and $U(p)$ have exponent p ($u(a, b)^p = u(0, -(\frac{p}{2})a\bar{a}) = 1$) and nondegenerate commutator forms on 2-dimensional spaces, so $P \cong U(p)$ by Lemma 5.2. For $p = 2$, $P \cong D_8$ or Q_8 ; as $\tilde{H}$ has odd order, is nontrivial (V is irreducible of dimension 2) and acts faithfully on P , while $\text{Aut}(D_8)$ is a 2-group, $P \cong Q_8 \cong U(2)$ (the six elements $u(a, b)$, $a \neq 0$, of $U(2)$ have order 4).

(f) By (b), $\tilde{N} = \tilde{H}$, so $N = H$. Let $h \in H$ be an r -element, $r = |\tilde{H}|$, whose image generates $\tilde{H}$; then $\langle h \rangle \cap C = \langle h^r \rangle$, and $C \trianglelefteq G$. In $X = P\langle h \rangle$ every proper subgroup L is nilpotent: a Sylow r -subgroup L_r of L lies in some $\langle h \rangle^x$, $x \in P$. If $L_r \leq \langle h^r \rangle^x \leq C$, it centralizes P . Otherwise $L_r = \langle h \rangle^x$, and $L \cap P$ is a proper h^x -invariant subgroup of P , hence lies in $\Phi(P)$ as in the proof of Lemma 4.1, and h^x centralizes $\Phi(P)$. In both cases L_r centralizes $L \cap P$, so L is nilpotent. Hence X is a non-nilpotent

group all of whose proper subgroups are nilpotent, a Schmidt group with normal Sylow p -subgroup P , and the cited theorem applies. $\square$

Theorem 5.4. *Let $\mathfrak{F}$ be a saturated formation and G a soluble minimal non- $\mathfrak{F}$ -group such that $P = G^{\mathfrak{F}}$ is a Sylow p -subgroup of G isomorphic to no factor group of $U(p^n)$, $n \geq 1$. Then $|G| \geq 2304$. So 2304 is the least order of a counterexample (Theorem 4.2). The proof does not use Proposition 5.3(f), and hence does not depend on [MS73, ZS87].*

Proof. By Proposition 5.3(d),(e), P is nonabelian and $m \geq 3$, and $d \geq 1$. $\tilde{H} \neq 1$ is a p' -subgroup of $\text{GL}(V)$ acting irreducibly; if it is abelian, it is cyclic of some order k with $m = \text{ord}_k(p)$ (Schur's lemma). Also $|G| \geq p^{m+d}|\tilde{H}|$. Suppose $|G| < 2304 = 2^8 \cdot 3^2$.

$p \geq 5$. If $m = 3$, then $d \geq 2$ by Proposition 5.3(c); so $m + d \geq 5$ and $|G| \geq 2 \cdot 5^5$. Impossible.

$p = 3$, $m \geq 5$. $|\tilde{H}| < 2304/3^6 < 4$, so $\tilde{H} \cong C_2$, which acts diagonalizably over $\mathbb{F}_3$, hence reducibly.

$p = 3$, $m = 4$. If $d \geq 2$, then again $\tilde{H} \cong C_2$. So $d = 1$: P is extraspecial of order 3^5 and exponent 3. So is $U(9)/N_0$ for any $N_0 \leq Z(U(9))$ of order 3 (Lemma 3.1; $U(9)$ has exponent 3). Nondegenerate alternating forms on $\mathbb{F}_3^4$ are equivalent, so $P \cong U(9)/N_0$ by Lemma 5.2.

$p = 3$, $m = 3$. Here $d \geq 2$, so $|\tilde{H}| \leq 9$, and as $5, 7 \nmid |\text{GL}_3(3)| = 2^5 \cdot 3^3 \cdot 13$, $\tilde{H}$ is a 2-group of order at most 8. The cyclic ones act irreducibly only in dimension $\text{ord}_k(3) \leq 2$. D_8 and Q_8 contain an element x of order 4 whose square is the central involution, which acts as -1 on V (its eigenspaces are submodules); so V is a vector space over $\mathbb{F}_3[x] \cong \mathbb{F}_9$, and m is even. Impossible.

$p = 2$, $m \geq 7$. $|\tilde{H}| < 9$ is odd, so $\tilde{H} \cong C_k$ with $k \in \{3, 5, 7\}$, and $\text{ord}_k(2) \leq 4 < m$.

$p = 2$, $m = 6$. $|\tilde{H}| < 18$ is odd; nonabelian groups of odd order have order at least 21; so $\tilde{H} \cong C_k$ with $\text{ord}_k(2) = 6$, that is, $k = 9$. Then $d = 1$, so H centralizes W , and the squaring map $q_P: V \rightarrow W$ is invariant under $\tilde{H}$, which acts on $V \cong \mathbb{F}_{64}$ as the group of 9th roots of unity. By Lemma 5.1(b), $q_P = \lambda \circ N$ with $\lambda: \mathbb{F}_8 \rightarrow W$ linear and onto (the values of the polarization span W). In $Q = U(8)/N_0$ with $N_0 = \{u(0, b) : b \in \ker \lambda\}$ the squaring map is the map induced by $a \mapsto a^9$, as $u(a, b)^2 = u(0, a^9)$ by (1); so $P \cong U(8)/N_0$ by Lemmas 3.1 and 5.2.

$p = 2$, $m = 5$. $d \geq 2$, so $|\tilde{H}| < 18$ is odd, $\tilde{H} \cong C_k$ with $\text{ord}_k(2) = 5$, so $k = 31$. Impossible.

$p = 2$, $m = 4$. $|\tilde{H}| < 72$ is odd and divides $|\text{GL}_4(2)| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$. If 7 divides $|\tilde{H}|$, a Sylow 7-subgroup is normal (the number of them divides $|\tilde{H}|/7 \leq 9$ and is 1 modulo 7) and has a nonzero fixed vector, as the simple $\mathbb{F}_2 C_7$ -modules have dimensions 1 and 3; its fixed space is a proper submodule. If $|\tilde{H}|$ divides 9, then $\tilde{H}$ is abelian, so cyclic, of order 3 or 9; but $\text{ord}_3(2) = 2$, and $\text{GL}_4(2)$ has no element of order 9 as $\text{ord}_9(2) = 6$. So a Sylow 5-subgroup $S \cong C_5$ exists; it is normal (the number of Sylow 5-subgroups divides $|\tilde{H}|/5 \in \{1, 3, 9\}$ and is 1 modulo 5), and acts irreducibly (otherwise $C_V(S) \neq 0$). So $C_{\text{GL}(V)}(S) \cong \mathbb{F}_{16}^*$, and $N_{\text{GL}(V)}(S)/C_{\text{GL}(V)}(S)$ embeds in $\text{Gal}(\mathbb{F}_{16}/\mathbb{F}_2) \cong C_4$; hence $\tilde{H} \in \{C_5, C_{15}\}$. If $\tilde{H} \cong C_{15} = C_3 \times C_5$, its proper subgroups C_3, C_5 lie in $f(2)$, hence so does $\tilde{H}$, a subdirect product of them, contradicting Proposition 5.3(b). So $\tilde{H} \cong C_5$, $\tilde{N} = \tilde{H}$ by (b), H centralizes W , and as in the case $m = 6$, with $\mathbb{F}_{16}$, $N(a) = a^5$ and $U(4)$, we get $P \cong U(4)/N_0$.

$p = 2$, $m = 3$. $\tilde{H} \leq \text{GL}_3(2)$ has odd order and acts irreducibly, so $\tilde{H} \cong C_7$ or $C_7 \rtimes C_3$ (C_3 has no simple module of dimension 3, and $\text{GL}_3(2)$ has no element of order 21). In both cases $\tilde{N}$ contains the normal Sylow 7-subgroup, so by Proposition 5.3(c) $V \cong \mathbb{F}_8$ carries a nonzero C_7 -invariant alternating form; but $7 \nmid 2^i + 2^j$ for $i, j \in \{0, 1, 2\}$, contradicting Lemma 5.1(a). $\square$

6 Restricted readings

6.1 Restrictions satisfied by the witnesses

Proposition 6.1. *Let $\mathfrak{F} = \text{LF}(f)$ with $f(p) = \mathfrak{A}(e)$ and $f(l) = \mathfrak{C}$ for $l \neq p$, as for witnesses A ($p = 2$, $e = 3$) and B ($p = 3$, $e = 8$).*

- (a) $\mathfrak{F}$ is subgroup-closed, $f(p)$ is abelian, and $\mathfrak{U} \subseteq \mathfrak{F}$ in both cases.
- (b) Let $H = C_{r^a}$, P and G be as in Lemma 4.1 with $\mathfrak{X} = \mathfrak{A}(r^{a-1})$, and let $\mathfrak{F}' = \text{LF}(f')$ with $f'(p) = \mathfrak{A}(r^{a-1})$, $f'(r) = \{1\}$ and $f'(l) = \emptyset$ for $l \notin \{p, r\}$. Then $\mathfrak{F}' \subseteq \mathfrak{N}_p \mathfrak{N}_r$, and G is a minimal non- $\mathfrak{F}'$ -group with $G^{\mathfrak{F}'} = P$.

So the answer to Problem 15.98 stays no if $\mathfrak{F}$ is required in addition to be subgroup-closed, to contain all supersoluble groups, to have an abelian value $f(p)$, or to lie in $\mathfrak{N}_p \mathfrak{N}_r$.

Proof. (a) Let $X \in \mathfrak{F}$ and $Y \leq X$. For a chief series (X_i) of X , the groups $X_i \cap Y$ form a normal series of Y ; refine it to a chief series. A chief factor A/B of Y of order divisible by p is a section of some $(X_i \cap Y)/(X_{i-1} \cap Y)$, which embeds Y -equivariantly in X_i/X_{i-1} ; so p divides $|X_i/X_{i-1}|$, the group $Y \cap C_X(X_i/X_{i-1})$ centralizes A/B , and $Y/C_Y(A/B)$ is a quotient of a subgroup of $X/C_X(X_i/X_{i-1}) \in \mathfrak{A}(e)$. So $Y \in \mathfrak{F}$. Next, $\mathfrak{U} = \text{LF}(u)$ with $u(2) = \{1\} \subseteq \mathfrak{A}(3)$, $u(3) = \mathfrak{A}(2) \subseteq \mathfrak{A}(8)$ and $u(l) \subseteq \mathfrak{E}$.

(b) The chief factors of a group $X \in \mathfrak{F}'$ have p -power or r -power order, and those of r -power order are central; so X is soluble. Put $\bar{X} = X/O_{r'}(X)$, where $O_{r'}(X)$ is a p -group. An r' -element of $\bar{X}$ acts trivially on the central chief factors of $\bar{X}$ inside $O_r(\bar{X})$, hence on $O_r(\bar{X})$ by (F3), and so lies in $C_{\bar{X}}(O_r(\bar{X})) \leq O_r(\bar{X})$ (the Hall–Higman lemma, as $O_{r'}(\bar{X}) = 1$; see [Hup67, Chapter VI]); so $\bar{X}$ is an r -group and $X \in \mathfrak{N}_p \mathfrak{N}_r$. For the second claim, repeat the proof of Lemma 4.1: for $L \leq G$ the chief factors of L above $L \cap P$ are chief factors of a cyclic r -group, hence central, and satisfy the condition at r ; all other steps are unchanged. $\square$

6.2 Metanilpotent and nilpotent-by-abelian groups

Lemma 6.2. Let $\mathfrak{Y} \in \{\mathfrak{N}, \mathfrak{A}\}$ and $\mathfrak{F} = \text{LF}(f)$ with $f(l) = \mathfrak{Y}$ for every prime l ; this is $\mathfrak{N}^2$, respectively $\mathfrak{N}\mathfrak{A}$: a nonabelian chief factor embeds in its centralizer quotient, which therefore is not nilpotent, so the members of $\mathfrak{F}$ are soluble, and for a soluble group the intersection of the centralizers of its chief factors is its Fitting subgroup [DH92, Chapter A]. Let H be a soluble p' -group acting faithfully on a p -group $P \neq 1$ with $P/\Phi(P)$ irreducible, $H \in \mathfrak{F}$, $H \notin \mathfrak{Y}$, every proper subgroup of H in $\mathfrak{Y}$, and $H/C_H(\Phi(P)) \in \mathfrak{Y}$. Then $G = P \rtimes H$ is a minimal non- $\mathfrak{F}$ -group with $G^{\mathfrak{F}} = P$.

Proof. As for Lemma 4.1 with $\mathfrak{X} = \mathfrak{Y}$, with one more step: for $L \leq G$ the chief factors of L above $L \cap P$ are chief factors of $L/(L \cap P)$, which is isomorphic to H or to a proper subgroup of H , so lies in $\mathfrak{F}$; hence they satisfy the local condition. In particular $G/P \in \mathfrak{F}$. $\square$

Proposition 6.3. In the following two cases $G = P \rtimes H$ is a minimal non- $\mathfrak{F}$ -group with $G^{\mathfrak{F}} = P$ and $|P'|^2 > |P : P'|$, so P is isomorphic to no factor group of $U(p^n)$ or $U(p^{2n})$. In both, $P = V \times W$ with $(v, w)(v', w') = (v + v', w + w' + \beta(v, v'))$ and H acting diagonally.

- (i) $\mathfrak{F} = \mathfrak{N}^2$, $p = 5$: $H = \text{SL}(2, 3) = Q_8 \rtimes \langle c \rangle \leq \text{SL}(2, 5)$ acts on $V = \mathbb{F}_{25}^2$, viewed over $\mathbb{F}_5$, by $g \mapsto \omega(g)g$, where $\omega: H \rightarrow \mathbb{F}_{25}^*$ has kernel Q_8 and $\omega(c)$ has order 3; W is the module of Q_8 -coinvariants of $\Lambda_{\mathbb{F}_5}^2 V$, and $\beta(v, v')$ is the image of $v \wedge v'$. Here $|P| = 5^7$ and $|P'| = 5^3$.
- (ii) $\mathfrak{F} = \mathfrak{N}\mathfrak{A}$, $p = 3$: $H = M_{16} = \langle a, b \mid a^8 = b^2 = 1, a^b = a^5 \rangle$ acts on $V = \mathbb{F}_9^2$, viewed over $\mathbb{F}_3$, by $a \mapsto \text{diag}(z, z^5)$, $b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with $z \in \mathbb{F}_9$ of order 8; $W = \Lambda_{\mathbb{F}_3}^2 V$ and $\beta(v, v') = v \wedge v'$. Here $|P| = 3^{10}$ and $|P'| = 3^6$.

Proof. In both cases β is bilinear, alternating and H -equivariant, and its values span W ; P has exponent p and $[(v, w), (v', w')] = (0, 2\beta(v, v'))$, so $P' = \Phi(P) = \{0\} \times W$, and $Z(P) = P'$ once the radical of β is 0. We check the hypotheses of Lemma 6.2.

(i) $\text{SL}(2, 3)$ is a Schmidt group, so it lies in $\mathfrak{N}^2$ but not in $\mathfrak{N}$, and its proper subgroups are nilpotent. Over $\mathbb{F}_{25}$, $V \otimes \mathbb{F}_{25} = U \oplus U^\sigma$ with $U = \omega \otimes V_0$ and $U^\sigma = \omega^{-1} \otimes V_0$, V_0 the natural module and σ the

Frobenius. U and U^σ are irreducible (Q_8 acts irreducibly on V_0) and not isomorphic (the traces of c are $-\omega(c)$ and $-\omega(c)^{-1}$), so V is irreducible over $\mathbb{F}_5$; it is faithful. Restricted to Q_8 , $\Lambda^2 U$ and $\Lambda^2 U^\sigma$ are trivial and $U \otimes U^\sigma \cong V_0 \otimes V_0$ is the sum of the four linear characters of Q_8 ; so $W \otimes \mathbb{F}_{25}$ is the sum of $\Lambda^2 U$, $\Lambda^2 U^\sigma$ and a line in $U \otimes U^\sigma$, and $\dim W = 3$. If $v = u + u' \neq 0$ with $u \in U$, $u' \in U^\sigma$, say $u \neq 0$, choose $u_1 \in U$ independent of u ; the $\Lambda^2 U$ -component of $\beta(v, u_1)$ is $u \wedge u_1 \neq 0$. So the radical is 0. Finally $Q_8 \leq C_H(W)$, so $H/C_H(\Phi(P))$ is a quotient of C_3 .

(ii) M_{16} is nilpotent and nonabelian, and its maximal subgroups contain $Z(M_{16}) = \langle a^2 \rangle$ with index 2, so they are abelian. The matrices satisfy the relations and give a faithful action. Over $\mathbb{F}_9$, $V \otimes \mathbb{F}_9 = U \oplus U^\sigma$, where a has eigenvalues z, z^5 on U and z^3, z^7 on U^σ , and U is irreducible as b swaps the two eigenlines of a ; so V is irreducible over $\mathbb{F}_3$. $\beta(v, \cdot) \neq 0$ for $v \neq 0$. As $a^4 = -1$ on V , a^4 centralizes W , and $H/\langle a^4 \rangle$ is abelian since $a^b = a \cdot a^4$. $\square$

6.3 Nilpotent and supersoluble groups

Remark 6.4. For $\mathfrak{F} = \mathfrak{N}$ take $f(p) = \{1\}$; by Proposition 5.3(b) every proper subgroup of $\tilde{H}$ is trivial, so $|\tilde{H}|$ is prime, and Proposition 5.3(f), that is, the theorem of [ZS87] and [MS73], gives the answer yes. Nothing here is new.

Theorem 6.5. *Let G be a soluble minimal non-supersoluble group whose supersoluble residual P is a Sylow p -subgroup. Then P is elementary abelian, or $P \cong U(p)$, or $|\tilde{H}|$ is prime (notation of Section 5); in every case P is isomorphic to a factor group of $U(p^n)$ for some n . The last case uses Proposition 5.3(f), hence Theorem 1 of [ZS87] as stated in its review.*

This is not claimed as new; it may follow from the descriptions of minimal non-supersoluble groups in [Doe66, BBER07], which we have not compared.

Proof. Apply Proposition 5.3 with $f = u$, $u(l) = \mathfrak{A}(l-1)$: $\tilde{H} \notin \mathfrak{A}(p-1)$, the proper subgroups of $\tilde{H}$ are abelian of exponent dividing $p-1$, $\tilde{N} \neq 1$, and if P is nonabelian, V carries a nonzero $\tilde{N}$ -invariant alternating form B . We may assume that P is nonabelian and $m \geq 3$ (Proposition 5.3(d),(e)).

Case 1: $\tilde{H}$ abelian. Then $\tilde{H} = \langle \zeta \rangle \cong C_k$ with $\zeta \in \mathbb{F}_{p^m}$ acting by multiplication, $k \nmid p-1$, and every proper divisor of k divides $p-1$. If two primes divided k , the prime-power parts of k would divide $p-1$, and so would k ; so $k = r^{b+1}$ with $r^b \parallel p-1$, and $r \neq p$. If $r = 2$ and $b = 0$, then $p = 2$ and $\tilde{H} \cong C_2$ is not a p' -group: impossible. If r is odd and $b \geq 1$, the subgroup of order r of $\tilde{H}$ lies in $\tilde{N}$ and is generated by an element $w \in \mathbb{F}_p^*$ (as $r \mid p-1$), a scalar; then $B(x, y) = B(wx, wy) = w^2 B(x, y)$ with $w^2 \neq 1$, so $B = 0$, a contradiction. If r is odd and $b = 0$, then $|\tilde{H}| = r$ is prime. If $r = 2$ and $b \geq 1$, then p is odd, $p \equiv 1 + 2^b \pmod{2^{b+1}}$ and $p^2 \equiv 1 \pmod{2^{b+1}}$, so $m = \text{ord}_{2^{b+1}}(p) = 2$: excluded.

Case 2: $\tilde{H}$ nonabelian. Then $\tilde{H}$ is not cyclic, every element lies in a proper subgroup and has order dividing $p-1$, and every abelian subgroup A of $\tilde{H}$ is diagonalizable over $\mathbb{F}_p$. Then $m \leq |\tilde{H} : A|$: if L is an A -invariant line, $\sum_t tL$ over a transversal t of A is a nonzero submodule. Let $M \leq \tilde{N}$ be a minimal normal subgroup of $\tilde{H}$, an elementary abelian l -group, and $V = \bigoplus_\chi V_\chi$ its decomposition into eigenspaces, $\chi: M \rightarrow \mathbb{F}_p^*$; the trivial χ does not occur. B is M -invariant, so $B(V_\chi, V_\psi) = 0$ unless $\psi = \chi^{-1}$. All proper subgroups of $\tilde{H}$ are abelian, so if $\tilde{H}$ is not nilpotent it is a Schmidt group $\tilde{H} = Q \rtimes \langle y \rangle$, with Q a normal Sylow s -subgroup, here abelian, and $\langle y \rangle$ a cyclic Sylow r -subgroup [Hup67, III.5.2]. Then $Q\langle y^r \rangle$ is a proper, hence abelian, subgroup, so $y^r \in Z(\tilde{H})$; and $Q = [Q, y] \times C_Q(y)$ by (F3), where $[Q, y] = [[Q, y], y]$, so y does not centralize $[Q, y]$ and $[Q, y]\langle y \rangle$, being nonabelian, equals $\tilde{H}$; hence $C_Q(y) = 1$ and $Z(\tilde{H}) \cap Q = 1$.

2a: M central. The eigenspaces of M are submodules, so M acts by one faithful character χ , and $B \neq 0$ forces $\chi = \chi^{-1}$: $M = \langle -1 \rangle$ and $l = 2$. If $\tilde{H}$ is nilpotent, it has prime-power order (else it is the direct product of proper, hence abelian, Sylow subgroups), so it is a 2-group whose maximal

subgroups are abelian, and $m \leq 2$. Otherwise the central involution lies outside Q ; so the prime s of Q is odd, the involution lies in a conjugate of $\langle y \rangle$, $r = 2$, and the abelian subgroup $Q\langle y^2 \rangle$ of index 2 gives $m \leq 2$. Both are excluded.

2b: M not central. Then $\tilde{H}$ is not nilpotent (in a nilpotent group minimal normal subgroups are central), and $M \leq Q$: an r -subgroup normal in $\tilde{H}$ is a proper subgroup of $\langle y \rangle$, hence lies in the central subgroup $\langle y^r \rangle$. The abelian subgroup $A = Q\langle y^r \rangle$ has index r , so $m \leq r$. $\tilde{H}$ permutes the V_χ transitively, A fixes each of them, and the orbit has r elements (otherwise M would act by scalars and be central); so $m = r$ and $\dim V_\chi = 1$. If $l = 2$, then $\chi^{-1} = \chi$ and B vanishes on $V_\chi \times V_\chi$ for all χ , so $B = 0$. If l is odd, then $\chi^{-1} = \chi^{y^j}$ with $y^j \notin A$ and $y^{2j} \in A$, so $r = 2 = m$. Both are excluded. $\square$

7 Conclusion

For Problem 15.98 as posed, the answer is no: there is a saturated formation $\mathfrak{F}$ and a finite soluble minimal non- $\mathfrak{F}$ -group G of order 2304 whose $\mathfrak{F}$ -residual is a Sylow 2-subgroup isomorphic to no factor group of a Sylow 2-subgroup of $\text{PSU}(3, 2^{2n})$; 2304 is the least order of such a counterexample (Theorems 4.2 and 5.4). The reason is that the nonabelian factor groups of $U(q)$ are semi-extraspecial, while the special groups that occur as residuals in Lemma 4.1 need not be. The answer stays no under the restrictions of Propositions 6.1 and 6.3, and it is yes for $\mathfrak{F} = \mathfrak{N}$ and $\mathfrak{F} = \mathfrak{U}$; both rest on the theorem of Zhurtov and Syskin (for $p = 2$, of Mazurov and Syskin) as stated in its zbMATH review. Which saturated formations give the answer yes is left open.

8 Computations and reproducibility

No proof in this note depends on a computation. The reproducibility package is the folder `reproducibility/` of the source archive; it has four parts.

The program of this note, `check_note.py` (standard library only; 62 checks, a few seconds). It builds $\mathbb{F}_{64} = \mathbb{F}_2[x]/(x^6 + x + 1)$ with $\zeta = x^7$ and checks: the group axioms of P , with associativity on all 256^3 triples; $Z(P) = P' = \Phi(P)$ of order 4; $[(1, 0), P]$ of order 2, and that 36 non-central elements have 2 commutators and 216 have 4; the 111 involutions of P ; h of order 9 with h^3 trivial on $\Phi(P)$, and the irreducibility of $P/\Phi(P)$; all 105 h^3 -invariant and the 3 h -invariant subgroups of P . Every subgroup of G of order divisible by 3 is conjugate to $K \rtimes \langle h^3 \rangle$ or $K \rtimes \langle h \rangle$ with K an invariant subgroup of P (Sylow and Schur–Zassenhaus); for these 108 groups the program decides membership in $\mathfrak{F}$ from the definition, on a computed chief series, and finds that only G is outside $\mathfrak{F}$. It also finds the normal subgroups of G (orders 1, 4, 2^8 , $2^8 \cdot 3$, $2^8 \cdot 3^2$) and $G^\mathfrak{F} = P$. Further: $U(8)$ is the set of all unitriangular matrices preserving the form (all 64^3 candidates are tested), Lemma 3.1 holds for it, and each of its seven factor groups of order 2^8 has $[[y, Q]] = 4$ for all non-central y and 39 involutions; the invariants of witness B used in Proposition 4.3; and the statements of Lemma 5.1 for C_7 on $\mathbb{F}_8$, C_5 on $\mathbb{F}_{16}$ and C_9 on $\mathbb{F}_{64}$. Deliberate errors (a changed product, a changed γ , the exponent 9 or 1 instead of 3 in the test of $f(2)$) make it fail; these tests were made on modified copies, which are not part of the package.

The programs with which the results were first obtained, with their recorded outputs, in the subfolder `certificates/`: the program `cert_lemma1.py` checks Lemma 3.1 for $q = 2, 3, 4, 5, 7, 8, 9$; the program `cert_2304.py` checks witness A (it needs NumPy); the program `cert_104976.py` checks witness B; the program `cert_small_cases.py` computes invariant forms; and the program `cert_restricted.py` checks the data of Proposition 6.3. In the package their comments refer to this note, and they print to standard output instead of overwriting the recorded outputs.

The programs of the third verification run (see below), with their recorded outputs and a file `README.md`, in the subfolder `independent_run_2/`.

A file `README.md` with the command lines and running times.

Verification. The results were first obtained with proofs and with the five programs above. Two independent verification runs, both AI-assisted, followed; both examined the first write-up of the result, not this text, and each re-derived the proofs line by line and wrote its own programs (both with models of $\mathbb{F}_{64}$ different from ours). The first run re-derived Lemmas 3.1 and 4.1, both witnesses, the structure proposition, the least-order argument, the case $\mathfrak{F} = \mathfrak{U}$ and the witnesses of Proposition 6.3; it recomputed witness A (associativity on all triples, the commutator sets, the invariant subgroups, the involution counts 111 and 39) and $U(q)$ for $q \leq 9$ with all factor groups by central subgroups. Both runs found the argument that makes the case $p = 2$, $m = 4$ of Theorem 5.4 independent of [MS73, ZS87], the first through coinvariants and the norm map, the second through invariant quadratic forms. The second run built the multiplication table of G and tested membership in $\mathfrak{F}$ from the definition for all subgroups up to conjugacy, with wrong local definitions as a test of its test; it checked that $U(8)$ consists of all unitriangular unitary matrices, enumerated the odd-order irreducible subgroups of $\mathrm{GL}_3(2)$ and $\mathrm{GL}_4(2)$, re-ran the five programs (identical outputs), and checked the bibliography with Crossref. Neither run found a mathematical error. The changes they required were made: the direct argument in the case $p = 2$, $m = 4$; the sub-case $r = 2$, $b = 0$ and the words “the prime s of Q is odd” in Theorem 6.5; the convention that $[x, X]$ is a set; the remark on the universe of soluble groups; the involution count; the remark that $\mathfrak{X}$ need not be subgroup-closed in Lemma 4.1; the references; and a test of minimality in the package (`check_note.py`). When this note was written, the proof of Theorem 5.4 was shortened: the cases $p \geq 5$, $(p, m) = (3, 3)$ and $(2, 5)$ now use the parity statement of Proposition 5.3(c), the case $(3, 4)$ uses Lemma 5.2, and the cases $(2, 4)$ and $(2, 6)$ use Lemma 5.1(b) and Lemma 5.2. The two runs did not examine these rearrangements.

A third run, also AI-assisted, examined the final text of this note, and the shortened proof of Theorem 5.4 first. It re-derived every proof in the wording of the note, among them each case of Theorem 5.4 (in the case $p = 2$, $m = 4$, $\tilde{H} \cong C_5$ it confirmed that Proposition 5.3(f) is not used), Lemmas 5.1 and 5.2, Proposition 5.3, Theorem 6.5 and Section 6; read the sources again (versions 48 and 49 of the Notebook, the record of the problem, the zbMATH entries of [ZS87] and [MS73], Sections 4 and 5 of [Lew17] and the abstracts of [BBERR05, BBER07, BCL25]); checked the references with Crossref and zbMATH; and repeated the literature search. Its programs, written before those of the package were run and with a fourth model of $\mathbb{F}_{64}$, check Lemma 3.1 for $q \leq 16$ (all unitriangular matrices tested for $q \leq 9$, all factor groups by central subgroups); witness A from the definitions, including the minimal non- $\mathfrak{F}$ property and $G^{\mathfrak{F}} = P$, decided from the definition of $\mathrm{LF}(f)$ for all subgroups up to conjugacy (with $f(2) = \mathfrak{A}(9)$ and $f(2) = \mathfrak{A}(1)$ as tests of the test), the normal subgroups of G and the counts 111 and 39; the invariants of witness B; Lemma 5.1 for the groups used, and Lemma 5.2 in the instances $U(4)/N_0$, $U(8)/N_0$ and $U(9)/N_0$ of Theorem 5.4; the odd-order irreducible subgroups of $\mathrm{GL}_3(2)$ and those of order less than 72 of $\mathrm{GL}_4(2)$, and the facts on $\mathrm{GL}_3(3)$ used in the case $(3, 3)$; and the data of Proposition 6.3. It ran the package again from the source archive (identical outputs) and repeated the four deliberate-error tests. It found no mathematical error and required eleven corrections, which were made: two missing lines (the abelian case in Lemma 3.1(b) and the solubility of the members of $\mathfrak{F}$ in Lemma 6.2); the credit to Beisiegel in Remark 3.3; the remark that the least order refers to the first reading; the dependence on [ZS87] in the conclusion; and six corrections of wording, among them this account of the runs.

Scope and priority. Problem 15.98 was posed by L. A. Shemetkov [KM26]. The case $\mathfrak{F} = \mathfrak{N}$ is due to Mazurov and Syskin [MS73] ($p = 2$) and Zhurtov and Syskin [ZS87]. We have not read these two papers, only their zbMATH entries; Theorem 1 of [ZS87] is used, in the form stated in its review (Zbl 0622.20010), only in Proposition 5.3(f), Remark 6.4 and Theorem 6.5. The Notebook

cites the translation of [ZS87] with volume 26; zbMATH and Crossref give volume 28. The standard tools (local formations and the theorem of Gaschütz, Lubeseder and Schmid, Sylow’s theorems, the Schur–Zassenhaus theorem, Maschke’s theorem, Schur’s lemma, coprime action, the three subgroups lemma, Burnside’s basis theorem, Schmidt’s theorem on minimal non-nilpotent groups, the Hall–Higman lemma) are taken from [DH92, BBE06, Hup67]; these books were not available to us, and the chapter and theorem numbers given for them were not checked. The negative answer uses only the easy direction of the theorem of Gaschütz, Lubeseder and Schmid; Theorem 5.4 uses the converse. To our knowledge, Lemma 4.1, the witnesses, Theorem 5.4 and Propositions 6.1 and 6.3 are not in print; Lemma 4.1 is elementary and may be known to specialists, and we claim no priority.

The searches were made on 21 October 2026, in zbMATH Open (API), Crossref, the arXiv API and OpenAlex (including the full lists of works citing [BBERR05], [BBER07], [MS73], [ZS87] and [Sem79]), with a few web searches; the third run repeated them. The search of Semantic Scholar refused automated access (HTTP 429). No printed answer to the question was found in these sources; a search that finds nothing is not a proof of novelty. The books [DH92, BBE06, She78] and the full text of [Sem79] and of Semenchuk’s later papers were not read; of [BBERR05, BBER07, BCL25] we read the abstracts, of [Lew17] the first five sections, and of [Bei77] only what [Lew17] reports.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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