

On Plotkin's Question on Two-Variable Identities Characterizing Finite Soluble Groups: Two Infinite Families of $\mathrm{PSL}(2, p)$ and Finite Certificates

Alper Ferudun*

October 22, 2026

Abstract

B. I. Plotkin asked in the Kourovka Notebook (Problem 15.75 b), 2002) whether a finite group is soluble if and only if it satisfies one of the identities $u_n = 1$, where $u_1 = [x, y]$ and $u_{n+1} = [[u_n, x], [u_n, y]]$. We give a *partial answer*: *the question reduces to the minimal simple groups; $\mathrm{PSL}(2, p)$ is settled for all primes $p \equiv \pm 1 \pmod{16}$ and, in the Notebook's commutator convention, for all primes $p \equiv 0, \pm 1 \pmod{7}$, and every minimal simple group of order less than $|\mathrm{PSL}(2, 10007)|$ has a pair whose sequence never reaches 1; the question stays open.* The reduction to Thompson's list of minimal simple groups is standard: it was outlined for this sequence by Nikolova and Plotkin (2001) and carried out by Bandman, Greuel, Grunewald, Kunyavskii, Pfister and Plotkin for another sequence. The families are explicit: if $x, y \in \mathrm{SL}(2, p)$, $\mathrm{tr} x = 0$ and $\mathrm{tr} y = \mathrm{tr} xy$ is a root of $2t^4 - 4t^2 + 1$, then $u_3 = -u_2 \neq \pm 1$ (in the other commutator convention $u_4 = u_3 \neq \pm 1$); if $\mathrm{tr} x = 1$ and $\mathrm{tr} y = \mathrm{tr} xy$ is a root of $t^3 \pm t^2 - 2t \mp 1$, then $u_4 = u_2 \neq \pm 1$ (in the Notebook's convention). By Dirichlet's theorem they contain half of the primes $p > 3$ with $p \equiv \pm 2 \pmod{5}$, for which $\mathrm{PSL}(2, p)$ is minimal simple. Checked finite certificates, pairs with $u_M = u_{M+L} \neq 1$, settle $\mathrm{PSL}(3, 3)$, the 626 groups $\mathrm{PSL}(2, q)$ of Thompson's list with $q \leq 10^4$, and $\mathrm{Sz}(8)$, $\mathrm{Sz}(32)$, $\mathrm{Sz}(128)$; for $\mathrm{PSL}(3, 3)$ and $\mathrm{PSL}(2, p)$, $5 \leq p < 100$, Nikolova and Plotkin had reported such pairs, found by computer. Not decided are the other groups of Thompson's list: $\mathrm{PSL}(2, p)$ for primes $p > 10^4$ outside the families, $\mathrm{PSL}(2, 2^r)$ for primes $r \geq 17$, and $\mathrm{PSL}(2, 3^r)$ and $\mathrm{Sz}(2^r)$ for primes $r \geq 11$. This is computational progress and two infinite families, not a near-solution. No printed answer to the question was found in the sources searched; we claim no priority. This is an unrefereed note.

1 Introduction

1.1 The problem

Throughout, $[x, y] = x^{-1}y^{-1}xy$, the convention of the Kourovka Notebook (written out, for instance, in its Problem 8.11 [KM26]). In the free group $F_2 = \langle x, y \rangle$ put

$$u_1 = [x, y], \quad u_{n+1} = [[u_n, x], [u_n, y]] \quad (n \geq 1). \quad (1)$$

A group G satisfies the identity $u_n = 1$ if $u_n(g, h) = 1$ for all $g, h \in G$. Problem 15.75 of [KM26] (B. I. Plotkin, 15th issue, 2002) has two parts. Part a), on the existence of some sequence of two-variable identities, each implying the next, that characterizes the finite soluble groups, is in the Archive of solved problems, answered by [BGG⁺06, BWW05, Rib09] with other sequences. Part b)

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
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asks whether (1) is such a sequence: is an arbitrary finite group “soluble if and only if it satisfies one of these identities $u_n = 1$ ”? Its text is the same in versions 48 (6 October 2026) and 49 (20 October 2026, the current one) of [KM26], in the main part and without comment. It is record KOU-15.75 of the UnsolvedMath collection [ula26], whose label (partially solved) refers to the archived part a). *This note treats part b) only.*

Soluble groups satisfy some $u_n = 1$ (Lemma 2.2); the content of b) is the converse. The Lie-algebra analogue, with Lie brackets, characterizes the finite-dimensional solvable Lie algebras over infinite fields of characteristic different from 2, 3, 5 [GKNP03] (see [BGG⁺06, Theorem 1.3]). For the sequence (1), Nikolova and Plotkin [NP01] stated the question as Plotkin’s conjecture, in the Notebook’s commutator convention, reduced it to the minimal simple groups through relations $u_k = u_n$, and reported solutions of such relations, found by computer, for $\text{PSL}(3, 3)$ and for $\text{PSL}(2, p)$ with every prime $5 \leq p < 100$. For groups, Bandman, Greuel, Grunewald, Kunyavskii, Pfister and Plotkin proved the characterization for the sequence $w_1 = x^{-2}y^{-1}x$, $w_{n+1} = [xw_nx^{-1}, yw_ny^{-1}]$ [BGG⁺06, Theorem 1.1]. Bandman, Garion and Kunyavskii state that they do not know the answer for Plotkin’s sequence, which resists their algebraic-geometric method for computational reasons [BGK14, Remark 3.5 and Section 7]; and b) is a case of [BGKJ10, Question 6.12], whose conditions hold for (1) by Lemmas 2.1 and 2.2 and because A_5 has pairs that never reach 1.

Two conventions. The papers of Bandman et al. just cited write $[a, b] = aba^{-1}b^{-1}$, whereas [NP01] uses the Notebook’s convention. We call the Notebook’s convention K and the other one O, and write u'_n for the sequence (1) formed with $[a, b]' = aba^{-1}b^{-1}$. The two sequences are genuinely different (Remark 2.6); the question is posed for K, and we treat both.

1.2 Results

We give a *partial answer*: the question reduces to the minimal simple groups; $\text{PSL}(2, p)$ is settled for all primes $p \equiv \pm 1 \pmod{16}$ and, in the Notebook’s commutator convention, for all primes $p \equiv 0, \pm 1 \pmod{7}$, and every minimal simple group of order less than $|\text{PSL}(2, 10007)|$ has a pair whose sequence never reaches 1; the question stays open. Here a group is *settled* if it contains a pair (g, h) with $u_n(g, h) \neq 1$ for all n ; then no finite group with a section isomorphic to it satisfies an identity $u_n = 1$. The four parts are: the reduction to Thompson’s list [Tho68] (Theorem 2.4, the argument of [BGG⁺06, Proposition 2.2]); the families of Theorems 3.4 and 3.5, explicit pairs on two lines of the trace coordinates, which contain half of the primes $p > 3$ with $p \equiv \pm 2 \pmod{5}$, for which $\text{PSL}(2, p)$ is minimal simple (a quarter in convention O, Corollary 3.7); certificates, that is, explicit pairs with a cycle of their sequence avoiding 1, for $\text{PSL}(3, 3)$, the 626 groups $\text{PSL}(2, q)$ of Thompson’s list with $q \leq 10^4$, and $\text{Sz}(8)$, $\text{Sz}(32)$, $\text{Sz}(128)$, in both conventions, each checked by several independent programs (Theorem 4.1); and the consequence that a finite non-soluble group satisfying some $u_n = 1$ has order at least $|\text{PSL}(2, 10007)| = 501\,050\,730\,168$, with all its minimal simple sections in the list (2) (Corollary 4.3). *Not decided*, and open, is whether the following groups are settled:

$$\begin{aligned} & \text{PSL}(2, p) \text{ for primes } p > 10^4, \ p \equiv \pm 2 \pmod{5}, \ p \not\equiv \pm 1 \pmod{16}, \ p \not\equiv \pm 1 \pmod{7}; \\ & \text{PSL}(2, 2^r) \ (r \geq 17), \quad \text{PSL}(2, 3^r) \ (r \geq 11), \quad \text{Sz}(2^r) \ (r \geq 11), \quad r \text{ prime}; \end{aligned} \tag{2}$$

this is the list for convention K, and in convention O the condition $p \not\equiv \pm 1 \pmod{7}$ is to be omitted.

1.3 What is new, what is not, and what stays open

No printed answer to Problem 15.75 b) was found; the searches are described in the paragraph “Scope and priority” at the end. The reduction is the standard argument of [BGG⁺06, Proposition 2.2], with Thompson’s list as quoted there (Theorem 2.1), outlined for Plotkin’s sequence already in [NP01]; the use of equations $u_i = u_j$ with $u_i \neq 1$ comes from [NP01] and [BGG⁺06, (2.1)], the trace coordinates

from [BGG⁺06, BGKJ10], and the Suzuki matrices from [BGG⁺06, Section 4.1]. For $\mathrm{PSL}(3, 3)$ and for $\mathrm{PSL}(2, p)$ with $5 \leq p < 100$, solutions of such equations in convention K were reported in [NP01, Section 5], found by computer and not listed there. To our knowledge, the families of Theorems 3.4 and 3.5, the list (2), Corollary 4.3 and the certificates of Theorem 4.1, except for the groups of [NP01] in convention K, are not in print. Their value is modest: the families come from a short exact computation and the certificates from routine searches. Numerical evidence suggests that fixed relations $u_i = \pm u_j$ have only finitely many solutions in the trace coordinates; if so, the Lang–Weil method of [BGG⁺06] does not apply directly, and the characteristic-2, characteristic-3 and Suzuki families in (2) need a new idea (Section 5).

2 Preliminaries and the reduction

For elements g, h of a group G put $F_{g,h}(u) = [[u, g], [u, h]]$, so that $u_n(g, h) = F_{g,h}^{n-1}(u_1(g, h))$. The pair (g, h) *reaches* 1 if $u_n(g, h) = 1$ for some n , and *never reaches* 1 otherwise. Everything in this section holds verbatim for the sequence u'_n of convention O, with $F'_{g,h}(u) = [[u, g]', [u, h]']'$.

Lemma 2.1 (monotonicity). *$F_{g,h}(1) = 1$. Hence $u_n(g, h) = 1$ implies $u_m(g, h) = 1$ for all $m \geq n$, the identity $u_n = 1$ implies $u_{n+1} = 1$, and a finite group satisfies one of the identities $u_n = 1$ if and only if each of its pairs reaches 1.*

Proof. $F_{g,h}(1) = [[1, g], [1, h]] = [1, 1] = 1$, which gives the first two statements. If every pair of a finite group G reaches 1, choose $n(g, h)$ with $u_{n(g,h)}(g, h) = 1$ and let N be the largest of these numbers; then $u_N(g, h) = 1$ for all g, h . The converse is clear. $\square$

Lemma 2.2. *$u_n(g, h)$ lies in the n -th derived subgroup $G^{(n)}$. Hence a soluble group of derived length d satisfies $u_d = 1$, and in a soluble group every pair reaches 1.*

Proof. $u_1(g, h) \in G'$. If $u_n(g, h) \in G^{(n)}$, then $[u_n, g]$ and $[u_n, h]$ lie in the normal subgroup $G^{(n)}$, and their commutator $u_{n+1}(g, h)$ lies in $G^{(n+1)}$. $\square$

Lemma 2.3 (certificates). (a) *If $u_M(g, h) = u_{M+L}(g, h) \neq 1$ for some integers $M, L \geq 1$, then (g, h) never reaches 1.*

(b) *Let $x, y \in \mathrm{SL}(2, q)$, with images $\bar{x}, \bar{y}$ in $\mathrm{PSL}(2, q) = \mathrm{SL}(2, q)/\{\pm 1\}$. If $u_M(x, y) \neq \pm 1$ and $u_{M+L}(x, y) = \pm u_M(x, y)$, then $(\bar{x}, \bar{y})$ never reaches 1 in $\mathrm{PSL}(2, q)$.*

Proof. (a) If $u_n = 1$ with $n \leq M$, then $u_M = F^{M-n}(1) = 1$ by Lemma 2.1. The sequence $(u_{M+j})_{j \geq 0}$ is periodic with period L ; if $u_n = 1$ with $n > M$, then $u_{M+kL} = 1$ for every k with $M + kL \geq n$, by Lemma 2.1, whereas $u_{M+kL} = u_M \neq 1$. (b) The quotient map is a homomorphism, so $u_n(\bar{x}, \bar{y})$ is the image of $u_n(x, y)$, and an element of $\mathrm{SL}(2, q)$ maps to 1 if and only if it is ± 1 ; apply (a) in $\mathrm{PSL}(2, q)$. $\square$

A pair together with (M, L) as in Lemma 2.3 is a *certificate*: it proves, by a finite computation, that the group is settled. A *minimal simple group* is a non-abelian simple group all of whose proper subgroups are soluble.

Theorem 2.4 (reduction). (a) *Every finite non-soluble group G has a section H/R (R normal in $H \leq G$) that is a minimal simple group.*

(b) *If G satisfies an identity, so does every section of G .*

(c) *The answer to Problem 15.75 b) is yes if and only if every minimal simple group is settled. By Thompson's classification [Tho68, Corollary 1], every minimal simple group is isomorphic to one of*

$$\begin{aligned} &\mathrm{PSL}(2, 2^r) \text{ (} r \text{ prime), } \quad \mathrm{PSL}(2, 3^r) \text{ (} r \text{ an odd prime), } \quad \mathrm{Sz}(2^r) \text{ (} r \text{ an odd prime), } \quad \mathrm{PSL}(3, 3), \\ &\mathrm{PSL}(2, p) \text{ (} p > 3 \text{ a prime with } p \equiv \pm 2 \pmod{5}), \end{aligned} \tag{3}$$

so the answer is yes if and only if every group in (3) is settled. The same holds for convention O.

The list (3) is quoted as in [BGG⁺06, Theorem 2.1], where $\text{PSL}(2, 5) \cong \text{PSL}(2, 4)$ is listed among the $\text{PSL}(2, p)$; the condition on p is often written $p^2 + 1 \equiv 0 \pmod{5}$, which is the same.

Proof. (b) An identity holds for all elements of a subgroup, and $u_n(gR, hR) = u_n(g, h)R$ in a quotient. (a) Let $H \leq G$ be a non-soluble subgroup of least order, so that all proper subgroups of H are soluble, and let R be the soluble radical of H , its largest soluble normal subgroup. Then H/R is not soluble, its proper subgroups K/R ($K < H$) are soluble, and its soluble radical is trivial. Let M/R be a minimal normal subgroup of H/R ; it is a direct power T^k of a simple group T , non-abelian because H/R has no nontrivial soluble normal subgroup. If $M < H$ then M , hence M/R , would be soluble; so $M = H$, and H/R , being a minimal normal subgroup of itself, is simple. (c) By Lemmas 2.1 and 2.2 the answer is yes if and only if every finite non-soluble group is settled. If a finite non-soluble group G is not settled, it satisfies some $u_n = 1$, and so does a minimal simple section of G , by (a) and (b), which is then not settled. Every minimal simple group occurs in (3), and every group in (3) is non-soluble. $\square$

Remark 2.5. In a minimal simple group S , a pair that never reaches 1 generates S : otherwise it lies in a proper, hence soluble, subgroup and reaches 1 by Lemma 2.2. In particular the pairs of Section 3 generate $\text{PSL}(2, p)$ whenever $p \equiv \pm 2 \pmod{5}$, $p > 3$. No argument through proper sections can settle a minimal simple group.

Remark 2.6 (the two conventions). In $A_5 = \text{PSL}(2, 5)$ exactly 480 of the 3600 pairs never reach 1 in convention K, and exactly 720 in convention O (Table 1). If there were an automorphism α of F_2 , elements $w_n \in F_2$ and signs ε_n with $u'_n = w_n \alpha(u_n)^{\varepsilon_n} w_n^{-1}$ for all n , the bijection $\beta(g, h) = (\alpha(x)(g, h), \alpha(y)(g, h))$ of $A_5 \times A_5$ would satisfy $u'_n(g, h) = 1 \iff u_n(\beta(g, h)) = 1$ and map 720 pairs onto 480. So the two questions must be treated separately (the identity $[a, b]' = [a^{-1}, b^{-1}]$ relates the commutators, not the recursions).

3 Two infinite families of $\text{PSL}(2, p)$

3.1 The trace algebra

Lemma 3.1. *Let R be a commutative ring, $x, y \in \text{SL}(2, R)$, and $a = \text{tr } x$, $b = \text{tr } y$, $c = \text{tr } xy$. Let A be the free R -module with basis e_0, e_1, e_2, e_3 and the bilinear multiplication in which e_0 is the identity and*

$$\begin{aligned} e_1 e_1 &= a e_1 - e_0, & e_1 e_2 &= e_3, & e_1 e_3 &= a e_3 - e_2, \\ e_2 e_1 &= -e_3 + b e_1 + a e_2 + (c - ab) e_0, & e_2 e_2 &= b e_2 - e_0, & e_2 e_3 &= e_1 + c e_2 - a e_0, \\ e_3 e_1 &= e_2 + c e_1 - b e_0, & e_3 e_2 &= b e_3 - e_1, & e_3 e_3 &= c e_3 - e_0. \end{aligned}$$

Put $\varphi(v) = v_0 + v_1 x + v_2 y + v_3 xy$, $\tau(v) = 2v_0 + av_1 + bv_2 + cv_3$ and $\iota(v) = \tau(v)e_0 - v$. Then $\varphi(vw) = \varphi(v)\varphi(w)$, $\text{tr } \varphi(v) = \tau(v)$, and $\varphi(\iota(v)) = \varphi(v)^{-1}$ whenever $\det \varphi(v) = 1$.

Proof. By Cayley–Hamilton, $X^2 = \text{tr}(X)X - \det(X)$ for 2×2 matrices over R , and applying this to $X+Y$ gives $XY+YX = \text{tr}(X)Y + \text{tr}(Y)X + (\text{tr } XY - \text{tr } X \text{tr } Y)$. Since x, y and xy have determinant 1, this gives $x^2 = ax - 1$, $y^2 = by - 1$, $(xy)^2 = cxy - 1$ and $yx = -xy + bx + ay + (c - ab)$, and then $x \cdot xy = axy - y$, $xy \cdot y = bxy - x$, $y \cdot xy = (yx)y = x + cy - a$ and $xy \cdot x = x(yx) = y + cx - b$. So φ is multiplicative on basis elements, hence everywhere by bilinearity. The trace is linear with $\text{tr } 1 = 2$, and $X^{-1} = \text{tr}(X) - X$ when $\det X = 1$. $\square$

Computing (1) in A over the polynomial ring $\mathbb{Z}[a, b, c]$, with ι as the inverse, gives vectors $U_n \in \mathbb{Z}[a, b, c]^4$, and U'_n for convention O. By Lemma 3.1, applied at each step to matrices of determinant 1, and since specialization $\mathbb{Z}[a, b, c] \rightarrow R$ is a ring homomorphism,

$$u_n(x, y) = \varphi(U_n(\text{tr } x, \text{tr } y, \text{tr } xy)), \quad u'_n(x, y) = \varphi(U'_n(\text{tr } x, \text{tr } y, \text{tr } xy)) \quad (4)$$

for every commutative ring R and all $x, y \in \text{SL}(2, R)$; in particular $\text{tr } u_1 = \text{tr } u'_1 = \kappa := a^2 + b^2 + c^2 - abc - 2$. On a line $(a, b, c) = (a_0 + a_1 t, b_0 + b_1 t, c_0 + c_1 t)$ the coordinates of U_n lie in $\mathbb{Z}[t]$. If $R = K$ is a field and $\kappa \neq 2$, then $1, x, y, xy$ are linearly independent (the *basis property*): otherwise their span, closed under multiplication, would be a proper subalgebra of $M_2(\bar{K})$, reducible by Burnside's theorem, and $[x, y]$ would be unipotent. We use it only in Remark 3.6.

3.2 Root criteria and pairs with given traces

Let $g(t) = 2t^4 - 4t^2 + 1$, $h(t) = t^3 + t^2 - 2t - 1$ and $h^-(t) = -h(-t) = t^3 - t^2 - 2t + 1$.

Lemma 3.2. *Let p be a prime. (a) For p odd, g has a root in $\mathbb{F}_p$ if and only if $p \equiv \pm 1 \pmod{16}$. (b) h has a root in $\mathbb{F}_p$ if and only if $p = 7$ or $p \equiv \pm 1 \pmod{7}$; the same holds for h^- .*

Proof. (a) Let $p \equiv \pm 1 \pmod{16}$. Then $16 \mid p^2 - 1$, so $\mathbb{F}_{p^2}^\times$ contains an element z of order 16, and $z^p = z^{\pm 1}$. Hence $\theta = z + z^{-1}$ and $s = z^2 + z^{-2}$ lie in $\mathbb{F}_p$. From $z^8 = -1$ we get $z^4 + z^{-4} = 0$, so $s^2 = 2$ (and $s \neq 0$), and $\theta^2 = 2 + s$. For $t = \theta/s$ we have $t^2 = (2 + s)/2$ and $g(t) = (2 + s)^2/2 - 2(2 + s) + 1 = (6 + 4s)/2 - 3 - 2s = 0$. Conversely, let $g(t) = 0$ with $t \in \mathbb{F}_p$. Put $s = 2t^2 - 2$ and $\theta = st$; then $s^2 = 2g(t) + 2 = 2$ and $\theta^2 = 2t^2 = s + 2$. Let $z \in \mathbb{F}_{p^2}$ be a root of $Z^2 - \theta Z + 1$. Then $z^2 + z^{-2} = \theta^2 - 2 = s$ and $z^4 + z^{-4} = s^2 - 2 = 0$, so $z^8 = -1$ and z has order 16. The Frobenius map permutes the roots z, z^{-1} of this polynomial over $\mathbb{F}_p$, so $z^p = z^{\pm 1}$ and $16 \mid p \mp 1$.

(b) Expanding gives $z^3 h(z + z^{-1}) = z^6 + z^5 + \dots + z + 1$. If $p \equiv \pm 1 \pmod{7}$, an element $z \in \mathbb{F}_{p^2}$ of order 7 gives the root $z + z^{-1} \in \mathbb{F}_p$ of h ; if $p = 7$, then $h(2) = 7 = 0$. Conversely, if $p \neq 7$ and $h(t) = 0$ with $t \in \mathbb{F}_p$, a root z of $Z^2 - tZ + 1$ satisfies $z^6 + \dots + 1 = 0$ and $z \neq 1$ (as $7 \neq 0$ in $\mathbb{F}_p$), so z has order 7, and $z^p = z^{\pm 1}$ gives $p \equiv \pm 1 \pmod{7}$. The roots of h^- are the negatives of those of h . $\square$

Lemma 3.3. *Let $p \geq 5$ be a prime and $t \in \mathbb{F}_p$. For each of $x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $x = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ there is $y \in \text{SL}(2, p)$ with $\text{tr } y = \text{tr } xy = t$.*

Proof. Write $y = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. For the first x , take $\gamma = s$, $\beta = t + s$, $\delta = t - \alpha$; then $\text{tr } y = \text{tr } xy = t$, and $\det y = 1$ becomes $\alpha^2 - t\alpha + s^2 + ts + 1 = 0$, which has a solution α if $D(s) = -4s^2 - 4ts + t^2 - 4$ is a square in $\mathbb{F}_p$. For the second x , take $\gamma = s$, $\beta = s + \alpha$, $\delta = t - \alpha$; then $\text{tr } y = \text{tr } xy = t$ and $\det y = 1$ becomes $\alpha^2 + (s - t)\alpha + s^2 + 1 = 0$, solvable if $D(s) = -3s^2 - 2ts + t^2 - 4$ is a square. In both cases $D(s) = As^2 + Bs + C$ with $A \in \{-4, -3\}$ nonzero in $\mathbb{F}_p$. Completing the square shows that D takes exactly $(p+1)/2$ values on $\mathbb{F}_p$, and the squares (including 0) are $(p+1)/2$ elements of $\mathbb{F}_p$; two subsets of $\mathbb{F}_p$ of this size meet. $\square$

This is a special case of Macbeath's theorem [Mac69] that every trace triple over $\mathbb{F}_q$ comes from a pair in $\text{SL}(2, q)$; see [BGKJ10, Theorem 3.4] for a proof in this form.

3.3 The families

Theorem 3.4 (family L1). *Let $p \equiv \pm 1 \pmod{16}$ be a prime, $t \in \mathbb{F}_p$ a root of $g(t) = 2t^4 - 4t^2 + 1$, and $x, y \in \text{SL}(2, p)$ with $\text{tr } x = 0$ and $\text{tr } y = \text{tr } xy = t$ (they exist by Lemma 3.3). Then*

(K) $u_3(x, y) = -u_2(x, y)$ and $\text{tr } u_2(x, y) = 0$, so $u_2(x, y) \neq \pm 1$;

(O) $u'_4(x, y) = u'_3(x, y)$ and $\text{tr } u'_3(x, y) = 0$, so $u'_3(x, y) \neq \pm 1$.

Hence the images of x, y in $\text{PSL}(2, p)$ never reach 1, in both conventions, and no finite group with a section isomorphic to $\text{PSL}(2, p)$ satisfies an identity $u_n = 1$ or $u'_n = 1$.

Theorem 3.5 (family L2). *Let p be a prime with $p = 7$ or $p \equiv \pm 1 \pmod{7}$, $t \in \mathbb{F}_p$ a root of $h(t) = t^3 + t^2 - 2t - 1$ or of $h^-(t) = t^3 - t^2 - 2t + 1$, and $x, y \in \text{SL}(2, p)$ with $\text{tr } x = 1$ and $\text{tr } y = \text{tr } xy = t$ (they exist by Lemma 3.3). Then $u_4(x, y) = u_2(x, y)$ and $\text{tr } u_2(x, y) = 1$, so $u_2(x, y) \neq \pm 1$. Hence the images of x, y in $\text{PSL}(2, p)$ never reach 1 in convention K, and no finite group with a section isomorphic to $\text{PSL}(2, p)$ satisfies an identity $u_n = 1$.*

Proof of Theorems 3.4 and 3.5. On the line $(a, b, c) = (0, t, t)$ the coordinates of $U_3 + U_2$ and of $U'_4 - U'_3$ and the traces $\tau(U_2)$, $\tau(U'_3)$ are polynomials in $\mathbb{Z}[t]$, of degree up to 69 for U_3 and 914 for U'_4 . On the line $(1, t, t)$ the same holds for $U_4 - U_2$ and $\tau(U_2) - 1$, of degree up to 315. We claim:

(F1) g divides the four coordinates of $U_3 + U_2$ and of $U'_4 - U'_3$, and $\tau(U_2)$ and $\tau(U'_3)$, on the line $(0, t, t)$;

(F2) h and h^- divide the four coordinates of $U_4 - U_2$, and $\tau(U_2) - 1$, on the line $(1, t, t)$.

These are identities in $\mathbb{Z}[t]$, verified by an exact computation: since reduction modulo g is a ring homomorphism from $\mathbb{Z}[t]$ to $\mathbb{Q}[t]/(g)$, the recursion can be computed in A over $\mathbb{Q}[t]/(g)$, where all polynomials have degree below 4, and the claimed elements are zero there; likewise for h and h^- . So g (resp. h , h^-) divides them in $\mathbb{Q}[t]$, hence in $\mathbb{Z}[t]$ by Gauss's lemma, since g , h , h^- are primitive. (The computation was done several times with independent programs, and once by a different method, Section 6.)

Now let $t_0 \in \mathbb{F}_p$ be a root of g and x, y as in Theorem 3.4. The homomorphism $\mathbb{Z}[t] \rightarrow \mathbb{F}_p$, $t \mapsto t_0$, kills g , so by (F1) and (4), $u_3(x, y) + u_2(x, y) = 0$, $u'_4(x, y) = u'_3(x, y)$, and $\text{tr } u_2(x, y) = \text{tr } u'_3(x, y) = 0$. As p is odd, $\text{tr}(\pm 1) = \pm 2 \neq 0$, so $u_2, u'_3 \neq \pm 1$. By Lemma 2.3(b) with $(M, L) = (2, 1)$ in convention K and $(3, 1)$ in convention O, the images never reach 1. In the setting of Theorem 3.5, (F2) gives $u_4(x, y) = u_2(x, y)$ and $\text{tr } u_2(x, y) = 1$; here $p \geq 7$, so $1 \neq \pm 2$ in $\mathbb{F}_p$, and Lemma 2.3(b) applies with $(M, L) = (2, 2)$. The statement on sections follows from Theorem 2.4(b) and Lemma 2.1. $\square$

In $\text{PSL}(2, p)$ the image of u_2 (of u'_3 in convention O) is a fixed point of F and an involution in Theorem 3.4, and the image of u_2 lies on a cycle of length dividing 2 and has order 3 in Theorem 3.5. For example, for $p = 17$, $t = 2$, $x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ one gets $u_2 = \begin{pmatrix} 15 & 1 \\ 12 & 2 \end{pmatrix} = -u_3$, and for $p = 7$, $t = 2$, $x = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ one gets $u_2 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = u_4$. On the line $(1, t, t)$, modulo h and modulo h^- , the sequence u'_n satisfies no relation $u'_i = \pm u'_j$ with $1 \leq i < j \leq 4$, so L2 is a family for convention K only.

Remark 3.6 (non-degeneracy by resultants). The non-degeneracy was first proved differently. On the line $(0, t, t)$, $\kappa - 2 = 2(t^2 - 2)$ and $g(t) = 1$ if $t^2 = 2$, so $\kappa \neq 2$ at every root of g in odd characteristic, and the basis property holds. If $u_2 = \pm 1$, all coordinates w_k of $U_2 \mp e_0$ vanish at t_0 ; as the leading coefficient of g is a unit modulo p , p divides every $\text{Res}(g, w_k)$, whose greatest common divisor is 2^{10} for both signs (2^{112} for $U'_3 \mp e_0$), so $p = 2$. On the line $(1, t, t)$, $\kappa = t^2 - 1$ and the corresponding greatest common divisors are 1. The basis property is used only here, never for the identities themselves.

Corollary 3.7. *Let P be the set of primes $p > 3$ with $p \equiv \pm 2 \pmod{5}$; for $p \in P$ the group $\text{PSL}(2, p)$ is minimal simple. In convention K, Theorems 3.4 and 3.5 settle $\text{PSL}(2, p)$ for $p = 7$ and all $p \in P$ with $p \equiv \pm 1 \pmod{16}$ or $p \equiv \pm 1 \pmod{7}$; by Dirichlet's theorem on primes in arithmetic progressions modulo $560 = 5 \cdot 16 \cdot 7$, these primes have relative density $1 - \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$ in P . In convention O, Theorem 3.4 settles the $p \in P$ with $p \equiv \pm 1 \pmod{16}$, of relative density $\frac{1}{4}$. Below 130, the elements of P covered by Theorems 3.4 and 3.5 in convention K are 7, 13, 17, 43, 47, 83, 97, 113, 127, and the others are 23, 37, 53, 67, 73, 103, 107 (settled by Theorem 4.1).*

Proof. The conditions modulo 5, 16 and 7 are independent by the Chinese remainder theorem, and among the units modulo 16 (resp. 7) exactly two of eight (resp. two of six) are ± 1 . The lists follow by inspection. $\square$

The two lines were found by an exploratory numerical search over about 1500 lines with small integer coefficients; it is not part of the reproducibility package and is not used in any proof. For convention K it found no other family, up to obvious symmetries.

4 Finite certificates

Theorem 4.1. *In both conventions, each of the following groups is settled: (a) $\text{PSL}(3, 3)$; (b) $\text{PSL}(2, q)$ for each of the 626 values $q \leq 10^4$ in (3), namely $q = 4, 8, 32, 128, 2048, 8192$, $q = 27, 243, 2187$, and the 617 primes $7 \leq q \leq 9973$ with $q \equiv \pm 2 \pmod{5}$; (c) $\text{Sz}(8)$, $\text{Sz}(32)$ and $\text{Sz}(128)$.*

Proof. For each group and each convention, the files listed in Section 6 contain certificates in the sense of Lemma 2.3: matrices x, y and integers $M, L \geq 1$ with $u_M \neq 1$ and $u_{M+L} = u_M$ (up to sign in $\text{SL}(2, q)$). Each certificate is a finite computation, and each was recomputed by programs independent of the search, Section 6. For $\text{Sz}(q)$ the matrices lie in the group of Lemma 4.2 by construction. $\square$

4.1 The groups $\text{PSL}(2, q)$ and $\text{PSL}(3, 3)$

The certificates were found by drawing random pairs with fixed seeds and following their sequences in $\text{PSL}(2, q)$ with Brent's cycle-finding method [Bre80] until they reached 1 or closed a cycle; for $q = p^k$, $k > 1$, the field is given by an irreducible polynomial printed with the certificate. The first random pair gave a certificate for 580 of the 626 groups in convention K (574 in O), at most 8 pairs were needed, and the largest $M + L$ is 3 765 841 ($q = 9677$, convention K). (For $\text{PSL}(3, 3)$ and $\text{PSL}(2, p)$, $5 \leq p < 100$, in convention K, compare [NP01, Section 5].)

For $\text{PSL}(3, 3) = \text{SL}(3, 3)$ an exhaustive computation over all pairs gives the numbers in Table 1, and three explicit certificates for each convention are recorded; for instance, in convention K, the matrices x and y over $\mathbb{F}_3$ with rows $(1, 2, 1)$, $(2, 0, 1)$, $(0, 2, 1)$, respectively $(1, 2, 0)$, $(1, 2, 2)$, $(2, 2, 1)$, satisfy $u_{64}(x, y) = u_{97}(x, y) \neq 1$.

Table 1: Pairs $(g, h) \in G \times G$ whose sequence never reaches 1 (exhaustive count, both conventions). In $\text{PSL}(2, 2)$ and $\text{PSL}(2, 3)$ there are none, as Lemma 2.2 requires.

G	$ G $	convention K	fraction	convention O	fraction
$A_5 = \text{PSL}(2, 4) = \text{PSL}(2, 5)$	60	480	0.133	720	0.200
$\text{PSL}(2, 7)$	168	6 720	0.238	6 384	0.226
$\text{PSL}(2, 8)$	504	80 136	0.315	84 672	0.333
$\text{PSL}(2, 9) = A_6$	360	30 240	0.233	34 560	0.267
$\text{PSL}(2, 11)$	660	153 120	0.352	137 280	0.315
$\text{PSL}(2, 13)$	1 092	480 480	0.403	452 088	0.379
$\text{PSL}(3, 3)$	5 616	14 657 760	0.465	14 062 464	0.446

4.2 The Suzuki groups

Let $q = 2^n$ with $n = 2m + 1 \geq 3$, and $\theta(a) = a^{2^{m+1}}$, a field automorphism of $\mathbb{F}_q$ with $\theta(\theta(a)) = a^2$. Bandman et al. define $\text{Sz}(q)$ as the subgroup of $\text{SL}(4, q)$ generated by all the matrices $U(a, b)$, $M(c)$

and T below, for $a, b \in \mathbb{F}_q$ and $c \in \mathbb{F}_q^\times$ [BGG⁺06, Section 4.1], where θ is called π ; we use their generators verbatim (for the Suzuki groups see [Suz62]):

$$U(a, b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a\theta(a) + b & \theta(a) & 1 & 0 \\ a^2\theta(a) + ab + \theta(b) & b & a & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$M(c) = \text{diag}(c^{1+2^m}, c^{2^m}, c^{-2^m}, c^{-1-2^m}).$$

Lemma 4.2. *For $q = 8, 32, 128$ the group $G_q = \langle U(a, b), M(c), T \rangle$ has order $q^2(q^2 + 1)(q - 1)$, that is 29 120, 32 537 600 and 34 093 383 680.*

Proof. Using that θ is a field automorphism with $\theta(\theta(a)) = a^2$, a direct computation gives $U(a, b)U(c, d) = U(a + c, b + d + a\theta(c))$ and $M(c)U(a, b)M(c)^{-1} = U(ac^{-1}, bc^{-1}\theta(c)^{-1})$. So the $U(a, b)$ form a group $\mathcal{U}$ of order q^2 , normalized by the group $\mathcal{M} = \{M(c)\}$ of order $q - 1$ ($c \mapsto c^{2^m}$ is injective), and $B = \mathcal{U}\mathcal{M}$ is a group of order $q^2(q - 1)$ of lower triangular matrices. Moreover $T^2 = 1$ and $TM(c)T = M(c^{-1})$. Finally, a computation over all $q^2 - 1$ pairs $(a, b) \neq (0, 0)$ shows

$$TU(a, b)T \in BT\mathcal{U}, \quad (5)$$

with explicit factors (Section 6). Let $S = B \cup BT\mathcal{U}$. Then $S\mathcal{U} = S$; $S\mathcal{M} = S$ because $TM(c) = M(c^{-1})T$ and $\mathcal{M}$ normalizes $\mathcal{U}$; and $ST \subseteq S$ because $BT = BT\mathcal{U}(0, 0)$, $BT\mathcal{U}(0, 0)T = B$, and $BT\mathcal{U}(a, b)T \subseteq BBT\mathcal{U} = BT\mathcal{U}$ for $(a, b) \neq (0, 0)$ by (5). So S contains every product of generators, and $S = G_q$. The union is disjoint, because the $(1, 4)$ entry of bTu is the $(1, 1)$ entry of b , which is nonzero, whereas elements of B are lower triangular. And $bTu = b'Tu'$ implies $T(b'^{-1}b)T = u'u^{-1}$, an upper triangular matrix in $\mathcal{U}$, hence $u = u'$ and $b = b'$. Therefore $|G_q| = |B|(1 + q^2)$. $\square$

For $q = 8$ the closure of the identity under right multiplication by five generators has 29 120 elements and contains all $U(a, b)$ and $M(c)$ (the verification runs certified the three orders independently, Section 6).

The pairs (x, y) for $\text{Sz}(q)$ are random words of length 12 in the generators, printed with each certificate, so they lie in G_q by construction. Of 300, 100 and 8 random pairs for $q = 8, 32, 128$, respectively 218, 97, 7 were certified never to reach 1 in convention K and 207, 91, 8 in convention O; the others reached 1.

Corollary 4.3. *Let G be a finite non-soluble group that satisfies an identity $u_n = 1$ (convention K). Then every minimal simple section of G is isomorphic to a group in (2), and $|G| \geq |\text{PSL}(2, 10007)| = 501\,050\,730\,168$. In particular every minimal simple group of order less than $|\text{PSL}(2, 10007)|$ is settled. The same holds in convention O, with the condition $p \not\equiv \pm 1 \pmod{7}$ omitted from (2).*

Proof. A minimal simple section S of G exists by Theorem 2.4(a), satisfies the same identity by Theorem 2.4(b), and so is not settled; S is in (3). The groups of (3) outside (2) are settled: those with $q \leq 10^4$, $\text{PSL}(3, 3)$ and the three Suzuki groups by Theorem 4.1, and the remaining $\text{PSL}(2, p)$ by Theorems 3.4 and 3.5 (which also contain $p = 7$). The least prime above 10^4 with $p \equiv \pm 2 \pmod{5}$ is 10007 ($10001 = 73 \cdot 137$ and $10003 = 7 \cdot 1429$), and $10007 \equiv 7 \pmod{16}$, $10007 \equiv 4 \pmod{7}$; the order $p(p^2 - 1)/2$ increases with p , and $|\text{PSL}(2, 2^{17})| > 2.2 \cdot 10^{15}$, $|\text{PSL}(2, 3^{11})| > 2.7 \cdot 10^{15}$, $|\text{Sz}(2^{11})| > 3.6 \cdot 10^{16}$. Hence $|G| \geq |S| \geq |\text{PSL}(2, 10007)|$. An unsettled minimal simple group satisfies some $u_n = 1$ (Lemma 2.1), so the first part applies. $\square$

5 What remains open

By Theorems 2.4, 3.4, 3.5 and 4.1, Problem 15.75 b) in convention K has the answer yes if and only if every group in (2) is settled, and the same holds in convention O for the larger list. Nothing in this note decides any of these groups, and *the question stays open*.

Fixed relations (numerical evidence only). For each relation $u_i = \pm u_j$, $1 \leq i < j \leq 6$, we counted the trace triples $(a, b, c) \in \mathbb{F}_q^3$ with $\kappa \neq 2$ at which it holds with $u_i \neq \pm 1$, for all primes $17 \leq q \leq 83$ and for $q = 4, 8, 16, 32, 64, 128, 9, 27, 81, 25, 49$ (convention K). The counts are at most 112 and vanish for some q (for instance, among the primes $17 \leq q \leq 83$, the relation $u_1 = \pm u_2$ has solutions only for $q = 37, 43, 67, 71, 79$, and among the primes $5 \leq q \leq 13$, counted for this relation only, for $q = 7$), whereas a curve of solutions defined over $\mathbb{F}_q$ would give about q points. So these relations *appear to have only finitely many solutions* in the trace coordinates (numerical evidence, not a proof): no “dimension jump” in the sense of [BGG⁺06, Section 2.3], in agreement with [BGG⁺06, Section 2.1] and [BGK14, Section 7]. If so, the method of [BGG⁺06], which needs a positive-dimensional family of solutions of one relation and the Hasse–Weil or Lang–Weil estimates, does not apply directly, and the solutions of a fixed relation in characteristic 2 (or 3) lie in fields of bounded degree, so it cannot settle $\mathrm{PSL}(2, 2^r)$ (or $\mathrm{PSL}(2, 3^r)$) for all large primes r . For the Suzuki groups we made no such count; the same would follow if the corresponding solution sets were finite. As no argument through sections works either (Remark 2.5), these families would then need cycles of $F_{x,y}$ whose length grows with r , or another idea; compare the questions on periodic points of verbal dynamical systems in [BGKJ10, Section 6].

The primes. Finitely many families, each covering the primes $p \equiv \pm 1 \pmod{m_i}$ with $m_i > 6$, cannot cover P : the primes that are 2 modulo the power of 5, 5 modulo the power of 2 and primitive roots modulo the other odd prime powers dividing $5\mathrm{lcm}(m_i)$ lie in P and escape all of them, and there are infinitely many by Dirichlet’s theorem.

Heuristics (not a proof). If $F_{x,y}$ behaved like a random map on the $\approx q^3/2$ points of $\mathrm{PSL}(2, q)$, a sequence would close a cycle after about $q^{3/2}$ steps and meet the about q preimages of 1 with probability of order $q^{-1/2}$. The fractions in Table 1 grow accordingly, and are 0.73 and 0.97 in the samples for $\mathrm{Sz}(8)$ and $\mathrm{Sz}(32)$. This supports the expectation that the answer is yes, and proves nothing.

Conclusion. The note gives a *partial answer*: *the question reduces to the minimal simple groups; $\mathrm{PSL}(2, p)$ is settled for all primes $p \equiv \pm 1 \pmod{16}$ and, in the Notebook’s commutator convention, for all primes $p \equiv 0, \pm 1 \pmod{7}$, and every minimal simple group of order less than $|\mathrm{PSL}(2, 10007)|$ has a pair whose sequence never reaches 1; the question stays open.* Its content is computational progress and two infinite families; the characteristic-2, characteristic-3 and Suzuki families in (2) are untouched.

6 Computations and reproducibility

The reduction, Lemmas 3.1 to 3.3 and the logical part of all other proofs need no computation. Computation enters in the identities (F1), (F2), in condition (5) and in the certificates. The reproducibility package has three parts. *The program of this note, `check_note.py`* (Python 3, standard library only; about 100 seconds), recomputes (F1) and (F2) exactly, Lemma 3.2 and the families with matrices for all odd primes $p \leq 2000$, the constants of Corollaries 3.7 and 4.3 (all 629 minimal simple groups

of order less than $|\text{PSL}(2, 10007)|$ are covered), the counts for A_5 , the certificates for $\text{PSL}(2, q)$ with $q \leq 49$ and 40 other q , for $\text{PSL}(3, 3)$ and for $\text{Sz}(8)$, and Lemma 4.2 with (5) for $q = 8, 32, 128$; all checks pass. *The programs with which the results were first obtained* (`original/`; Python 3 with numpy and sympy) contain the computation over $\mathbb{Z}[t]$ with sympy (with the resultants of Remark 3.6), the certificate search with two independent checking programs, the counts, and the certificate files. *The programs of the independent verification runs* are in `independent_runs/` (the first two runs) and in `independent_run_2/` (the third run, which examined this text).

Verification. The results were first obtained, with proofs and with the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each re-derived the proofs and wrote its own programs. The first run compared the statement and the cited passages with [KM26] (versions 48 and 49), [BGK14], [BGG⁺06] and [BGKJ10]; its programs re-checked all 1252 certificates for $\text{PSL}(2, q)$ (a C program with its own field arithmetic) and all 628 Suzuki certificates, with generators written from the Suzuki–Tits ovoid and found identical with those of [BGG⁺06], checked (5) for $q = 8, 32, 128$, recomputed (F1), (F2) and the resultants with its own trace algebra, built the families for all odd $p \leq 1500$, and recounted Table 1 except two rows. The second run wrote its programs before reading the original ones. It proved (F1) and (F2) in another way, with the general pair with the given traces over $\mathbb{Q}[s][t, \alpha]$ and a bound on the degree in the free parameter s ; found the trace argument for non-degeneracy used here; recomputed Table 1 with independent models; found its own certificates for 57 groups $\text{PSL}(2, q)$, for $\text{PSL}(3, 3)$ and for the three Suzuki groups; certified the Suzuki orders by Schreier generators on the Tits ovoid, for a group that the generators of [BGG⁺06] generate; and re-checked all original certificates, rejecting corrupted inputs. All values agree, and neither run found an error in a proof. Their required corrections were made: the identification of the Suzuki generators with [BGG⁺06, Section 4.1] and the order argument for $q = 32, 128$; the remark that the basis property serves only the non-degeneracy; the qualification as numerical evidence of every statement that fixed relations cannot treat the characteristic-2, characteristic-3 and Suzuki families; Remark 2.5; the prime 7 in Corollary 4.3; version 49 of [KM26]; the relation $u'_4 = u'_3$ (not $u'_3 = -u'_2$) in convention O; the data of [GKNP03]; and the reference [PW22]. This text was written after the two runs, with three changes: the non-degeneracy rests on traces, with the resultant argument kept as Remark 3.6; Lemma 3.3 has an elementary counting proof (the runs checked a proof by a character sum); and the two formulas in the proof of Lemma 4.2 are proved, not only tested. These changes are checked by `check_note.py`, the exact program of this note.

A third verification run, also AI-assisted, examined this text and the package. It re-derived every proof line by line, among them the new proof of Lemma 3.3 and the two formulas in the proof of Lemma 4.2; compared the statement and every cited passage again with [KM26] (versions 48 and 49), [BGG⁺06], [BGK14] and [BGKJ10]; and repeated the literature search, which found [NP01, KPS03] and the author’s version of [GKP13]. Its own programs, written before those of the package were read, proved (F1) and (F2) once more by computing the full polynomials in $\mathbb{Z}[t]$ and dividing them by g, h and h^- with integral quotients; recomputed the resultants of Remark 3.6; checked Lemma 3.2 for all odd primes below $2 \cdot 10^5$, Lemma 3.3 for all t and all primes $5 \leq p \leq 700$, and the families with explicit matrices, also randomly conjugated, for all primes $p \leq 3000$ and 48 primes up to about 10^{18} ; re-checked all 1252 certificates for $\text{PSL}(2, q)$, the six for $\text{PSL}(3, 3)$ and all 628 Suzuki certificates with its own field arithmetic, and found its own certificates for 30 groups $\text{PSL}(2, q)$, for $\text{PSL}(3, 3)$ and for the three Suzuki groups, in both conventions; checked the formulas of Lemma 4.2 and condition (5) for $q = 8, 32, 128$, and for $q = 8$ the order and the element orders of the group; and recounted Table 1 with its own models. It re-ran `check_note.py` (with Python 3.9) and the quick programs of `original/` from the archive, with outputs identical to the recorded ones. It found no error in a proof. Its required corrections were made: the credit to [NP01] and the mention of [KPS03]; the undecided

groups in the abstract restricted to Thompson’s list; the fraction for $\text{PSL}(2, 8)$ in Table 1; the primes $5 \leq q \leq 13$ in the count of Section 5; and five changes of wording.

Scope and priority. The note gives a partial answer to Problem 15.75 b) and leaves it open: the groups in (2) are not decided. The problem is due to B. I. Plotkin [KM26]; Nikolova and Plotkin [NP01] had stated it as Plotkin’s conjecture, outlined the reduction for this sequence and reported solutions of relations $u_k = u_n$, found by computer, for $\text{PSL}(3, 3)$ and $\text{PSL}(2, p)$, $5 \leq p < 100$. The reduction and the use of equations $u_i = u_j$ (see also [NP01]), the trace coordinates (see also [BGKJ10]) and the Suzuki matrices come from [BGG⁺06], Thompson’s list [Tho68] is quoted from [BGG⁺06, Theorem 2.1], the realization of trace triples is Macbeath’s theorem [Mac69] (see [BGKJ10, Theorem 3.4]), and the densities rest on Dirichlet’s theorem. That the question is open is stated in [BGK14, Remark 3.5 and Section 7]. Kunyavskii, Plotkin and Shklyar [KPS03] propose a strategy for such sequences (its zbMATH review does not show which sequence). Popović and Wilson [PW22] study orbits of a different map in connection with the soluble radical, and Erschler and Belov [Ers15, BE19] iterated identities in another sense; neither treats Plotkin’s sequence. We read [KM26] (versions 48, 49), [NP01] (in full), the arXiv versions of [BGG⁺06] (v1: Sections 1, 2 and the beginning of 4.1, to which our section numbers refer), [BGK14] (v1: Section 3 to Remark 3.5, Section 7) and [BGKJ10] (v2: Theorem 3.4, Section 6.2), and the author’s version of [GKP13] (Sections 1–3, 6, 7, with no answer to the question; not compared with the journal version); of [GKNP03] only the bibliographic data, of [KPS03] the review, of [PW22, Ers15, BE19] the abstracts (screened at the first derivation and by the verification runs); [Tho68, Mac69, Suz62] were not opened. The searches (21 and 22 October 2026) used the arXiv API, zbMATH Open, OpenAlex (the 68 distinct works citing [BGG⁺06], [BGK14], [BGKJ10], [Rib09], [GKP13] or [GKNP03]), Semantic Scholar, Crossref and five web searches; the last of these, by the third verification run, found [NP01] and the author’s version of [GKP13]. No printed answer to the question was found in the sources searched; a search that finds nothing is not a proof of novelty, and we claim no priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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