

The Chevalley Group $F_4(\mathbb{Z})$ Is Generated by Three Involutions, Two of Which Commute

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Abstract

In Problem 15.67 of the Kourovka Notebook (2002), Ya. N. Nuzhin asked which adjoint Chevalley groups of normal type over the integers are generated by three involutions, two of which commute. By work of Tamburini and Zucca, Nuzhin, Timofeenko, and Kazakova and Nuzhin, the answer was known for the types A_l , $B_2 = C_2$, G_2 , E_6 , E_7 and E_8 ; the types B_l , C_l ($l \geq 3$), D_l ($l \geq 4$) and F_4 were open. Our result is a partial answer: the adjoint Chevalley group $F_4(\mathbb{Z})$ is generated by three involutions two of which commute (computer-assisted proof with an exact certificate); the types B_l , C_l ($l \geq 3$) and D_l ($l \geq 4$) stay open. The involutions are explicit and follow the pattern of Timofeenko's triples for E_l : b and c lift the two halves of the bipartite Coxeter element, and a is a product of commuting root elements and a torus element (four root elements of both lengths here, two in Timofeenko's triples). A direct argument shows that they are involutions and reduces $ab = ba$ to four signs of structure constants. Generation is proved by a straight-line program with 1064 nodes which produces the 48 root elements $x_r(1)$; a checker that uses only the Python standard library verifies it exactly in a few seconds, and three independent programs confirm it. The formulas refer to a Chevalley basis that we describe by its extraspecial signs; the existence statement does not depend on this choice. As consequences, the reductions modulo every prime p generate $F_4(p)$ in the same way, and the least number of generating involutions of $F_4(\mathbb{Z})$ with product 1 is 5. No printed answer for F_4 was found in the sources searched (zbMATH, arXiv, OpenAlex, Crossref, Semantic Scholar, web searches); a search that finds nothing is not a proof of novelty. This is an unrefereed note.

1 Introduction

1.1 The problem

Let Φ be a reduced irreducible root system and $\Phi(\mathbb{Z})$ the adjoint Chevalley group of type Φ and of normal (that is, untwisted) type over the ring of integers; the definition is recalled in Section 2.3. In Problem 15.67 of the Kourovka Notebook [KM26], Ya. N. Nuzhin asked which of these groups are “generated by three involutions two of which commute”, that is, for which Φ there are elements $a, b, c \in \Phi(\mathbb{Z})$ of order 2 with $ab = ba$ and $\langle a, b, c \rangle = \Phi(\mathbb{Z})$. For a non-soluble group such as $F_4(\mathbb{Z})$ (Section 6) the conditions $a^2 = b^2 = c^2 = 1$, $ab = ba$ and $\langle a, b, c \rangle = \Phi(\mathbb{Z})$ already force $a, b, c \neq 1$: if one of them were 1, the group would be generated by two elements of order at most 2, hence would be a quotient of the infinite dihedral group and soluble. The problem appeared in the 15th issue (2002). Its text and its comment are the same in versions 48 (6 October 2026) and 49 (20 October 2026) of the Notebook, where it is not marked as solved and the comment does not mention F_4 , and it is record KOU-15.67 of the UnsolvedMath collection [ula26], where it is listed as open.

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The comment in the Notebook records the following results. Tamburini and Zucca [TZ97] proved that $SL_n(\mathbb{Z})$ is generated by three involutions two of which commute for $n > 13$, hence so is $PSL_n(\mathbb{Z})$. Nuzhin [Nuz08] proved that $PSL_n(\mathbb{Z})$ has this property if and only if $n > 4$; so for the type A_l the answer is positive exactly when $l \geq 4$. The group $PSp_4(\mathbb{Z})$, which is the adjoint group of type $B_2 = C_2$ [Tim17, p. 808], does not have it, because its quotient $PSp_4(3)$ does not [KM26, Archive, 7.30]. Timofeenko proved the property for the type G_2 [Tim15] and for the types E_6 , E_7 and E_8 [Tim17]; for E_7 his involutions lie in the adjoint group, and he notes that the analogous question for the universal group of type E_7 stays open [Tim17, p. 808]. In the introduction of [Tim17] he observes that this answers the question for all exceptional types except F_4 . In 2025 Kazakova and Nuzhin, the latter being the proposer, state that the problem has been solved only for the types A_n (the case PSL_{n+1}), E_n and G_2 [KN25, p. 83]; they prove the analogous statement for $G_2(\mathbb{Z} + i\mathbb{Z})$ and remark that their triple with i replaced by 1 works for $G_2(\mathbb{Z})$ [KN25, p. 87]. So, before this note, the open types were B_l , C_l ($l \geq 3$), D_l ($l \geq 4$) and F_4 .

1.2 The result

Theorem 1.1. *The adjoint Chevalley group $F_4(\mathbb{Z})$ is generated by three involutions, two of which commute.*

Our result is a partial answer: the adjoint Chevalley group $F_4(\mathbb{Z})$ is generated by three involutions two of which commute (computer-assisted proof with an exact certificate); the types B_l , C_l ($l \geq 3$) and D_l ($l \geq 4$) stay open.

The involutions are given in Theorem 3.1, relative to a Chevalley basis B_T that is described in Section 2.5 and Appendix A. In Section 4 we prove without computation that they are involutions of $F_4(\mathbb{Z})$, reduce $ab = ba$ to four signs of structure constants, and show that $\langle a, b, c \rangle = F_4(\mathbb{Z})$ as soon as $\langle a, b, c \rangle$ contains the 48 root elements $x_r(1)$. In Section 5 we show that it does: a straight-line program, that is, a compressed list of words in a, b, c , with 1064 nodes produces every $x_r(1)$. Its correctness is a finite set of identities between integral 52×52 matrices, which a checker of about 350 lines, written with the Python standard library, verifies exactly in a few seconds; three independent programs confirm it (Section 8). As consequences (Section 6), the reductions of a, b, c modulo every prime p generate $F_4(p)$ in the same way, and the least number of generating involutions of $F_4(\mathbb{Z})$ whose product is 1 is 5.

1.3 What is new, what is not, and what stays open

The note settles one more type, F_4 , with a positive answer, by a computer-assisted proof whose finite part is an exact certificate; the classical types B_l , C_l ($l \geq 3$) and D_l ($l \geq 4$) are not touched by it (Section 7). The shape of the triple follows Timofeenko’s construction for E_l [Tim17, Table 2], which uses Nuzhin’s method of choosing generating triples of involutions in groups of Lie type over finite fields [Tim17, p. 807]; the change for F_4 is that a contains four root elements of both lengths, where Timofeenko’s first involution contains two. To our knowledge the triple for F_4 and the proof that it generates are new. No printed answer to the question for F_4 was found in the sources searched (zbMATH, arXiv, OpenAlex, Crossref, Semantic Scholar and web searches; see “Scope and priority” in Section 8); a search that finds nothing is not a proof of novelty, and we claim no priority.

2 Chevalley groups over the integers

2.1 The root system F_4

We use Bourbaki's numbering $b_1 - b_2 \Rightarrow b_3 - b_4$ of the simple roots, realized in $\mathbb{R}^4$ as $b_1 = e_2 - e_3$, $b_2 = e_3 - e_4$, $b_3 = e_4$, $b_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$; the roots b_1, b_2 are long, $(b_i, b_i) = 2$, and b_3, b_4 are short, $(b_i, b_i) = 1$. A root $r = \sum r_i b_i$ is written as its coefficient vector $r_1 r_2 r_3 r_4$; for example $1110 = b_1 + b_2 + b_3 = e_2$ and $-0121 = -(b_2 + 2b_3 + b_4)$. There are 48 roots, 24 long and 24 short, and the highest root is 2342; so $|r_1| \leq 2$ for every root, and 2342 is the only positive root with $r_1 = 2$. With $\langle s, r^\vee \rangle = 2(s, r)/(r, r)$ the Cartan matrix gives, for a root $s = s_1 s_2 s_3 s_4$,

$$\langle s, b_1^\vee \rangle = 2s_1 - s_2, \quad \langle s, b_2^\vee \rangle = -s_1 + 2s_2 - s_3, \quad \langle s, b_3^\vee \rangle = -2s_2 + 2s_3 - s_4, \quad \langle s, b_4^\vee \rangle = -s_3 + 2s_4, \quad (1)$$

so that modulo 2 these numbers are $s_2, s_1 + s_3, s_4$ and s_3 . The Cartan matrix has determinant 1; hence the root lattice $\mathbb{Z}\Phi$ is equal to the weight lattice. We write w_r for the reflection in the root r .

2.2 Chevalley bases and root elements

Let L be the simple complex Lie algebra of type F_4 . A Chevalley basis [Che55, Ste16, Car72] consists of elements $h_1, \dots, h_4$ of a Cartan subalgebra H and root vectors e_r ($r \in \Phi$) with

$$[h_i, h_j] = 0, \quad [h_i, e_r] = \langle r, b_i^\vee \rangle e_r, \quad [e_r, e_{-r}] = h_r, \quad [e_r, e_s] = N_{rs} e_{r+s} \quad (r + s \in \Phi), \quad N_{rs} = \pm(p+1),$$

and $[e_r, e_s] = 0$ if $r + s \notin \Phi \cup \{0\}$. Here $h_r = \sum_i r_i \frac{(b_i, b_i)}{(r, r)} h_i$ is the coroot of r (so $h_{b_i} = h_i$), and $p \geq 0$ is the largest integer such that $s - pr$ is a root. The $\mathbb{Z}$ -span $L_{\mathbb{Z}}$ of a Chevalley basis is a Lie ring. For $r \in \Phi$ and $t \in \mathbb{Z}$ the map

$$x_r(t) = \exp(t \operatorname{ad} e_r) = 1 + t \operatorname{ad} e_r + \frac{t^2}{2} (\operatorname{ad} e_r)^2$$

preserves $L_{\mathbb{Z}}$ [Che55], and we regard it as an integral 52×52 matrix. For $\epsilon = \pm 1$ put $n_r(\epsilon) = x_r(\epsilon) x_{-r}(-\epsilon) x_r(\epsilon)$, $n_r = n_r(1)$ and $h_r(-1) = n_r^2$. For a homomorphism $\chi: \mathbb{Z}\Phi \rightarrow \{\pm 1\}$ let $h(\chi)$ be the linear map with $h(\chi)e_s = \chi(s)e_s$ and $h(\chi)h_i = h_i$. We use the following standard facts. (R1) $x_r(t)x_r(u) = x_r(t+u)$, since $\operatorname{ad} e_r$ is nilpotent; hence $x_r(t) = x_r(1)^t$ for $t \in \mathbb{Z}$. (R2) $h(\chi)$ is an automorphism of $L_{\mathbb{Z}}$, and $h(\chi)x_r(t)h(\chi)^{-1} = x_r(\chi(r)t)$, since $h(\chi)\operatorname{ad}(e_r)h(\chi)^{-1} = \operatorname{ad}(h(\chi)e_r) = \chi(r)\operatorname{ad} e_r$. (R3) $n_r(\epsilon)$ maps H onto H , acting there as w_r , and maps e_s to $\pm e_{w_r(s)}$; hence $n_r(\epsilon)h(\chi)n_r(\epsilon)^{-1} = h(\chi \circ w_r)$ and $n_r(\epsilon)x_s(t)n_r(\epsilon)^{-1} = x_{w_r(s)}(\pm t)$; moreover $n_r(\epsilon)^2 = h_r(-1) = h(\chi_r)$ with $\chi_r(s) = (-1)^{\langle s, r^\vee \rangle}$ [Car72, §6.4]. (R4) If $r \neq -s$ and no $ir + js$ with $i, j \geq 1$ is a root, then $x_r(t)$ and $x_s(u)$ commute (Chevalley's commutator formula, [Car72, Ch. 5]).

2.3 The group of the question

Over a field K the adjoint Chevalley group $\Phi(K)$ is the group generated by the $x_r(t)$, $t \in K$, acting on $L_{\mathbb{Z}} \otimes K$ [Che55]. Over $\mathbb{Z}$ we use the definition of the proposer's school: Timofeenko [Tim17, p. 810] defines $\Phi(\mathbb{Z})$ as the set of elements of $\Phi(\mathbb{Q})$ whose coefficients with respect to the lattice $\mathbb{Z}(\Phi)$, which he calls the lattice of roots, are integers. As $\Phi(\mathbb{Q})$ acts on $L_{\mathbb{Z}} \otimes \mathbb{Q}$, we read this lattice as $L_{\mathbb{Z}}$; so $\Phi(\mathbb{Z})$ is the stabilizer of $L_{\mathbb{Z}}$ in $\Phi(\mathbb{Q})$ (the elements of $\Phi(\mathbb{Q})$ have determinant 1, and an integral matrix of determinant 1 has an integral inverse). His Lemma 3 [Tim17, p. 810] states that $\Phi(\mathbb{Z})$ is generated by the elements $x_r(1)$, $r \in \Phi$; he cites p. 107 of the Russian edition of Steinberg's lectures [Ste16], and Kazakova and Nuzhin [KN25, pp. 83–84] cite the same page for the fact that Chevalley groups over a Euclidean ring are generated by their root subgroups. This is Steinberg's theorem that a Chevalley

group of simply connected type over a Euclidean ring is generated by its root elements. It applies to F_4 : the root lattice equals the weight lattice, so the simply connected and the adjoint group coincide (Timofeenko [Tim17, p. 808] notes that over $\mathbb{Z}$ every adjoint Chevalley group of exceptional type other than E_7 coincides with the universal one), and $\mathbb{Z}$ is Euclidean. We did not read Steinberg's text; we use the theorem as it is quoted in [Tim17, KN25]. So

$$F_4(\mathbb{Z}) = \{g \in F_4(\mathbb{Q}) : gL_{\mathbb{Z}} = L_{\mathbb{Z}}\} = E(F_4, \mathbb{Z}) := \langle x_r(1) : r \in \Phi \rangle = \langle x_r(t) : r \in \Phi, t \in \mathbb{Z} \rangle, \quad (2)$$

the last equality by (R1). The second equality, Steinberg's theorem, is the step that connects our theorem, which is a statement about $\langle x_r(1) \rangle$, with the group of the question.

2.4 Tables of structure constants and changes of basis

Lemma 2.1. *Let $N = (N_{rs})$ be a table of integers indexed by the pairs of roots r, s with $r+s \in \Phi$, and let L_N be the $\mathbb{Q}$ -vector space with basis $h_1, \dots, h_4, e_r$ ($r \in \Phi$) and the bracket given by the formulas of Section 2.2 with these N_{rs} . If the bracket is antisymmetric and satisfies the Jacobi identity, and $|N_{rs}| = p+1$ for all r, s , then $L_N \otimes \mathbb{C}$ is the simple Lie algebra of type F_4 , $H = \text{span}(h_i)$ is a Cartan subalgebra with root system Φ , and $\{h_i, e_r\}$ is a Chevalley basis.*

Proof. In $L = L_N \otimes \mathbb{C}$ every $\text{ad } h, h \in H$, is diagonal, with eigenvalue $r(h)$ on e_r , where $r(h_i) = \langle r, b_i^\vee \rangle$; the 48 functionals r are distinct and nonzero because the Cartan matrix is invertible. Let $I \neq 0$ be an ideal. Being H -stable, I is the sum of $I \cap H$ and of the lines $\mathbb{C}e_r$ that it contains. If $0 \neq h \in I \cap H$, choose r with $r(h) \neq 0$; then $e_r = r(h)^{-1}[h, e_r] \in I$. If $e_r \in I$, then $h_r = [e_r, e_{-r}] \in I$ and $e_s = \langle s, r^\vee \rangle^{-1}[h_r, e_s] \in I$ whenever $\langle s, r^\vee \rangle \neq 0$. Since F_4 is irreducible, its roots are not the union of two nonempty orthogonal sets, so repeating this step gives all e_s , hence all h_s , and $I = L$. So L is simple. The subalgebra H is abelian, consists of semisimple elements and is its own centralizer, so it is a Cartan subalgebra, and its roots are the images of the 48 roots of Φ under a linear isomorphism; by the isomorphism theorem for simple Lie algebras, L is the simple Lie algebra of type F_4 . The basis has the defining properties of a Chevalley basis. $\square$

For a Chevalley basis also $N_{-r, -s} = -N_{rs}$ [Car72, Thm. 4.1.2]; for the tables used below this holds by construction (Section 2.5).

Lemma 2.2. *Let $\{h_i, e_r\}$ and $\{h'_i, e'_r\}$ be Chevalley bases of simple Lie algebras L and L' of type F_4 , with the roots labelled by the same Dynkin diagram. Then there are an isomorphism $\tau: L \rightarrow L'$ and signs $\epsilon_r = \epsilon_{-r} = \pm 1$ with $\tau(h_i) = h'_i$ and $\tau(e_r) = \epsilon_r e'_r$. Consequently $\tau x_r(t) \tau^{-1} = x'_r(\epsilon_r t)$, and τ maps the group $\langle x_r(1) \rangle$ onto the group $\langle x'_r(1) \rangle$.*

Proof. By the isomorphism theorem there is an isomorphism $\sigma: L \rightarrow L'$ mapping H onto H' and each root space onto the root space with the same label; then $\sigma(e_r) = c_r e'_r$ and $\sigma(h_r) = h'_r$, both being the coroot of r . The relations give $c_r c_{-r} = 1$ and $c_r c_s / c_{r+s} = N_{rs} / N'_{rs} = \pm 1$. Let $\chi: \mathbb{Z}\Phi \rightarrow \mathbb{C}^\times$ be the homomorphism with $\chi(b_i) = c_{b_i}$, and $d_r = c_r / \chi(r)$. Then $d_{b_i} = 1$, $d_r d_{-r} = 1$ and $d_r d_s / d_{r+s} = \pm 1$; as every positive root of height at least 2 is a positive root of smaller height plus a simple root, induction gives $d_r = \pm 1$, and $d_{-r} = d_r$. With the automorphism t_χ of L' given by $t_\chi(e'_r) = \chi(r) e'_r$, $t_\chi(h'_i) = h'_i$, the map $\tau = t_\chi^{-1} \circ \sigma$ has $\tau(e_r) = d_r e'_r$; put $\epsilon_r = d_r$. Finally $\tau \exp(t \text{ad } e_r) \tau^{-1} = \exp(t \epsilon_r \text{ad } e'_r)$. $\square$

2.5 The bases B_T and B_0

Order the positive roots by height and, within a height, lexicographically by the coefficient vector, ascending: 0001, 0010, 0100, 1000; 0011, 0110, 1100; 0111, 0120, 1110; 0121, 1111, 1120; 0122, 1121, 1220; 1122, 1221; 1222, 1231; 1232; 1242; 1342; 2342 (semicolons separate the heights). This is the

order in which $r \prec s$ if $s - r$ is positive in the lexicographic order of $(\text{ht}, r_1, r_2, r_3, r_4)$, so it is compatible with addition, as required in [Car72, §4.2]. For a positive root γ of height at least 2, the pairs (ξ, ζ) of positive roots with $\xi + \zeta = \gamma$ and $\xi \prec \zeta$ are the special pairs of γ ; the one with the smallest ξ is the extraspecial pair $(\xi_\gamma, \zeta_\gamma)$. There are 20 extraspecial pairs (Table 1). By [Car72, §4.2] the signs of the 20 constants $N_{\xi_\gamma \zeta_\gamma}$ can be chosen freely, and they determine all N_{rs} by a recursion.

Definition 2.3. B_T is the Chevalley basis with $N_{\xi_\gamma \zeta_\gamma} = +(p+1)$ for $\gamma \in \{1100, 1342, 2342\}$ and $N_{\xi_\gamma \zeta_\gamma} = -(p+1)$ for the other 17 extraspecial pairs. B_0 is the Chevalley basis with all 20 signs $+1$.

All statements below about B_T depend only on its table of structure constants, which is the file `f4_structure_constants.json` (816 entries) of the certificate. The program `check_note.py` computes the tables of B_T and B_0 from the signs of Definition 2.3 by the recursion of [Car72, §4.2], in a version adapted from a program of the second independent verification run, and finds the table of B_T equal to the file, entry by entry; the recursion is not trusted, since the hypotheses of Lemma 2.1 are checked for both tables (Section 5.2). The two bases are related by a sign map: let $\eta_r = \eta_{-r} = -1$ for the eleven roots 0011, 0110, 1110, 0121, 1121, 1220, 1222, 1231, 1242, 1342 and 2342, and $\eta_r = 1$ for the other positive roots. Then $N_{rs}^T = \eta_r \eta_s \eta_{r+s} N_{rs}^0$ for all r, s , so the map $e_r^T \mapsto \eta_r e_r^0$, $h_i \mapsto h_i$ is an isomorphism of Lie rings. The function η is unique up to multiplication by a homomorphism $\mathbb{Z}\Phi \rightarrow \{\pm 1\}$ (the argument of Lemma 2.2); it is normalized by $\eta(b_i) = 1$.

3 The theorem

Theorem 3.1. *In the Chevalley basis B_T let*

$$\begin{aligned} a &= x_{1000}(1) x_{1100}(1) x_{1110}(1) x_{1111}(1) \cdot h_{b_2}(-1) h_{b_4}(-1), \\ b &= n_{b_2} n_{b_4} \cdot h_{b_1}(-1) h_{b_3}(-1), \\ c &= n_{b_1} n_{b_3} \cdot h_{b_2}(-1). \end{aligned}$$

Then $a^2 = b^2 = c^2 = 1$, $ab = ba$, $a, b, c \neq 1$, and $\langle a, b, c \rangle = F_4(\mathbb{Z})$.

In the coordinates of $\mathbb{R}^4$ the four roots in a are $e_2 - e_3$, $e_2 - e_4$ (long) and e_2 , $\frac{1}{2}(e_1 + e_2 - e_3 - e_4)$ (short). By (R3) and (1) the three torus factors are $h(\chi_a)$, $h(\chi_b)$ and $h(\chi_c)$, where

$$\chi_a(s) = (-1)^{s_1}, \quad \chi_b(s) = (-1)^{s_2+s_4}, \quad \chi_c(s) = (-1)^{s_1+s_3}. \quad (3)$$

Theorem 3.1 contains Theorem 1.1; its proof is completed in Section 5.3.

Remark 3.2 (The basis B_0). In B_0 the statement of Theorem 3.1 holds for

$$a_0 = x_{1000}(1) x_{1100}(1) x_{1110}(-1) x_{1111}(1) h_{b_2}(-1) h_{b_4}(-1)$$

and for b and c given by the same formulas as in Theorem 3.1: the isomorphism $e_r^T \mapsto \eta_r e_r^0$ of Section 2.5 maps $x_r^T(t)$ to $x_r^0(\eta_r t)$ and $n_{b_i}^T$ to $n_{b_i}^0$ (as $\eta(b_i) = 1$), fixes the torus elements, and among the roots in a only 1110 has $\eta = -1$. In the transported certificate the target nodes of the 22 roots with $\eta_r = -1$ take the values $x_r^0(-1)$; they are replaced by their inverses, built from the nodes ($\text{inv}(\text{mul } i j) = \text{mul}(\text{inv } j, \text{inv } i)$, $\text{inv}(\text{pow } i k) = \text{pow}(\text{inv } i, k)$, and a generator is its own inverse). This certificate (2079 nodes) and the table of B_0 are accepted by the checker (`check_note.py`, Part C; the files are in `b0_basis/`).

The formulas depend on the basis; the theorem does not. The proofs of $a^2 = b^2 = c^2 = 1$ in Section 4 use no structure constants, but $ab = ba$ depends on four signs (Proposition 4.1(e)). Read in another Chevalley basis $\{h_i, \epsilon_r e_r\}$ of the same lattice ($\epsilon_r = \epsilon_{-r} = \pm 1$), the formulas of Theorem 3.1 become those of B_T with $x_r(1)$ replaced by $x_r(\epsilon_r)$ for $r \in \{1000, 1100, 1110, 1111\}$ and n_{b_i} by $n_{b_i}(\epsilon_{b_i})$; so they depend on the signs ϵ_r , $r \in R_7 = \{b_1, b_2, b_3, b_4, 1100, 1110, 1111\}$. Patterns that differ by a homomorphism $\chi: \mathbb{Z}\Phi \rightarrow \{\pm 1\}$ give triples conjugate under $h(\chi) \in F_4(\mathbb{Z})$, by (R2); all 16 elements $h(\chi)$ are products of the $h_{b_i}(-1)$, as the Cartan matrix is invertible modulo 2. So the 128 patterns form 8 classes of 16, distinguished by $(\epsilon_{1100}\epsilon_{b_1}\epsilon_{b_2}, \epsilon_{1110}\epsilon_{b_1}\epsilon_{b_2}\epsilon_{b_3}, \epsilon_{1111}\epsilon_{b_1}\epsilon_{b_2}\epsilon_{b_3}\epsilon_{b_4})$; the class of B_T is $(1, 1, 1)$. The involution identities hold in all patterns, but $ab = ba$ holds only in the classes $(1, 1, 1)$ and $(1, -1, -1)$, and the certificate works only in the class of B_T : in each of the other seven classes, its target node with the shortest word is not a root element $x_r(\pm 1)$ (Part E of `check_note.py`), and the second independent verification run found that none of its 48 target nodes is one. So the explicit formulas depend on the basis, and the table of B_T , or Definition 2.3, is part of their statement. The existence statement does not: for any Chevalley basis B' of the simple Lie algebra of type F_4 , the isomorphism τ of Lemma 2.2 maps a, b, c to involutions that generate $\langle x'_r(1) \rangle$, the first two commuting; in B' they are given by the formulas of Theorem 3.1 with the parameters multiplied by the signs ϵ_r .

4 The involutions

Let $u = x_{1000}(1)x_{1100}(1)x_{1110}(1)x_{1111}(1)$, $m = n_{b_2}n_{b_4}$ and $m' = n_{b_1}n_{b_3}$, so that $a = u h(\chi_a)$, $b = m h(\chi_b)$ and $c = m' h(\chi_c)$, and let $w = w_{b_2}w_{b_4}$ and $w' = w_{b_1}w_{b_3}$. By (1), w and w' act on coefficient vectors by

$$w(s) = (s_1, s_1 - s_2 + s_3, s_3, s_3 - s_4), \quad w'(s) = (s_2 - s_1, s_2, 2s_2 - s_3 + s_4, s_4). \quad (4)$$

Proposition 4.1. (a) $a, b, c \in E(F_4, \mathbb{Z})$. (b) $a^2 = 1$. (c) $b^2 = 1$ and $c^2 = 1$. (d) $a, b, c \neq 1$. (e) w permutes the four roots in a as $1000 \leftrightarrow 1100$, $1110 \leftrightarrow 1111$; $b h(\chi_a) b^{-1} = h(\chi_a)$; and $ab = ba$ holds if and only if

$$b x_r(1) b^{-1} = x_{w(r)}(1) \quad \text{for } r = 1000, 1100, 1110, 1111. \quad (5)$$

Parts (a)–(d) and the first two statements of (e) hold, with the same proofs, for the formulas read in any Chevalley basis. The four identities (5) hold in B_T .

Proof. (a) The elements n_r and $h_r(-1) = n_r^2$ are products of the $x_{\pm r}(\pm 1)$, and $x_r(-1) = x_r(1)^{-1}$ by (R1).

(b) The four roots in u have $r_1 = 1$, and $|s_1| \leq 2$ for every root s ; so a root of the form $ir + js$ with $i, j \geq 1$, $r \neq s$ among them, would have $i = j = 1$. No sum of two of the four roots is a root: the six sums have first coefficient 2 and differ from 2342. By (R4) the four factors of u commute. Since $\chi_a(r) = (-1)^{r_1} = -1$ for each of them, (R2) gives $h(\chi_a) u h(\chi_a)^{-1} = u^{-1}$, and as $h(\chi_a)^2 = 1$,

$$a^2 = u h(\chi_a) u h(\chi_a) = u (h(\chi_a) u h(\chi_a)^{-1}) = u u^{-1} = 1.$$

(c) The roots b_2 and b_4 are orthogonal, of squared lengths 2 and 1, so $\pm i b_2 \pm j b_4$ ($i, j \geq 1$) has squared length $2i^2 + j^2 \geq 3$ and is not a root. By (R4) every $x_{\pm b_2}(t)$ commutes with every $x_{\pm b_4}(t')$; hence $n_{b_2}n_{b_4} = n_{b_4}n_{b_2}$ and $m^2 = h_{b_2}(-1)h_{b_4}(-1) = h(\chi_a)$. By (R3), $m^{-1}h(\chi)m = h(\chi \circ w)$ for every χ . Therefore

$$b^2 = m h(\chi_b) m h(\chi_b) = m^2 \cdot (m^{-1}h(\chi_b)m) \cdot h(\chi_b) = h(\chi_a \cdot (\chi_b \circ w) \cdot \chi_b),$$

and by (3) and (4) the exponent of -1 in this character is $s_1 + ((s_1 - s_2 + s_3) + (s_3 - s_4)) + (s_2 + s_4) = 2s_1 + 2s_3$, which is even; so $b^2 = 1$. In the same way $b_1 \perp b_3$, the squared lengths are 2 and 1,

$m'^2 = h_{b_1}(-1)h_{b_3}(-1) = h(\chi_b)$, and $c^2 = h(\chi_b \cdot (\chi_c \circ w') \cdot \chi_c)$ with exponent $(s_2 + s_4) + ((s_2 - s_1) + (2s_2 - s_3 + s_4)) + (s_1 + s_3) = 4s_2 + 2s_4$; so $c^2 = 1$.

(d) The element u is unipotent and $h(\chi_a) \neq 1$ is diagonal, so $a = u h(\chi_a) \neq 1$ (otherwise $u = h(\chi_a)$ would have the eigenvalue -1). By (R3), b maps e_{b_2} to $\pm e_{w(b_2)} = \pm e_{-b_2}$ and c maps e_{b_1} to $\pm e_{-b_1}$, so $b, c \neq 1$.

(e) The permutation follows from (4), and $\chi_a \circ w = \chi_a$ because $w(s)_1 = s_1$; by (R3), $b h(\chi_a) b^{-1} = m h(\chi_a) m^{-1} = h(\chi_a \circ w) = h(\chi_a)$. Hence $bab^{-1} = (bub^{-1}) h(\chi_a)$, and $ab = ba$ if and only if $bub^{-1} = u$. By (R2) and (R3), $b x_r(1) b^{-1} = x_{w(r)}(\sigma_r)$ with $\sigma_r = \pm 1$ for each of the four roots, so bub^{-1} is the product of the four commuting elements $x_{w(r)}(\sigma_r)$. Such a product $\prod_s x_s(t_s)$ over the four roots determines the t_s : since no sum of two of the roots is a root, it maps $h \in H$ to $h - \sum_s t_s s(h) e_s$. So $bub^{-1} = u$ if and only if all $\sigma_r = 1$, which is (5). The signs σ_r depend on the structure constants and on the torus factor of b (conjugation by $h(\chi_b)$ alone changes the parameters of x_{1100} and x_{1110} by the factor -1), and they cannot be determined without the table; that they are all $+1$ in B_T is a computation, done by the checker of Section 5 in the form $ab = ba$ and by `check_note.py` in the form (5). $\square$

Lemma 4.2. *If $\langle a, b, c \rangle$ contains $x_r(1)$ for every root r , then $\langle a, b, c \rangle = F_4(\mathbb{Z})$.*

Proof. By (2), $\langle a, b, c \rangle$ then contains $E(F_4, \mathbb{Z}) = F_4(\mathbb{Z})$, and it is contained in it by Proposition 4.1(a). $\square$

5 The certificate

5.1 The straight-line program

A straight-line program over (a, b, c) is a list of nodes; node k is `gen  $x$` with $x \in \{a, b, c\}$, or `inv  $j$` with j a gen node, or `mul  $i j$` with $i, j < k$, or `pow  $i e$` with $i < k$ and $e \geq 1$. Its value is the generator x ; the inverse of the generator of node j ; the product of the values of nodes i and j ; or the e -th power of the value of node i . Every value is a word in a, b, c . The certificate, the file `certificate_F4.json`, contains the generator data (the formulas of Theorem 3.1), 1064 nodes (3 gen, 3 inv, 832 mul and 226 pow, with exponents from 2 to 327) and, for each of the 48 roots r , a target node. Written out without cancellation, the 48 target words have lengths between 3144 and about $5 \cdot 10^{14}$, and for the roots $\pm b_i$ between about $8 \cdot 10^8$ and $7 \cdot 10^{13}$; this is why they are not printed.

5.2 The checker

The program `verify_certificate.py` (about 350 lines, Python standard library only, exact integer arithmetic with sparse matrices) (1) generates the 48 roots of F_4 from the description in $\mathbb{R}^4$, converts them to coefficient vectors and checks that $b_1, \dots, b_4$ is a base; (2) builds the 52-dimensional algebra from the table (816 entries N_{rs}) and verifies antisymmetry, the Jacobi identity on all $52^3 = 140608$ triples of basis vectors, and $|N_{rs}| = p + 1$, so that by Lemma 2.1 the table is that of a Chevalley basis of the simple Lie algebra of type F_4 , whatever its origin; (3) computes $x_r(t) = 1 + t \operatorname{ad} e_r + \frac{t^2}{2} (\operatorname{ad} e_r)^2$ after checking that $(\operatorname{ad} e_r)^3 = 0$ and that $(\operatorname{ad} e_r)^2$ has even entries; (4) builds a, b, c from the generator data and checks that each torus factor is the stated product of elements $n_{b_i}^2 = h_{b_i}(-1)$; (5) checks $a^2 = b^2 = c^2 = 1$, $a, b, c \neq 1$ and $ab = ba$; (6) evaluates the program exactly, the inv nodes as the generators themselves, which is correct because a, b, c were verified to be involutions in step (5), and checks that the 48 target nodes are the 48 elements $x_r(1)$. Every matrix entry is compared with the bound 10^{12} (the valid certificate stays far below it: its largest entry is 408096); rejecting an input with larger entries is sound. On a laptop (Apple arm64, Python 3.9.6) the check takes about 3.5 seconds.

5.3 Proof of Theorem 3.1

By Proposition 4.1, a, b, c are involutions of $F_4(\mathbb{Z})$ different from 1, and $ab = ba$ once the four signs (5) are known. The checker establishes by exact computation that the table of B_T satisfies the hypotheses of Lemma 2.1, that $ab = ba$ (and again that $a^2 = b^2 = c^2 = 1$), and that each $x_r(1)$ is the value of a node of the program, hence a word in a, b, c . By Lemma 4.2, $\langle a, b, c \rangle = F_4(\mathbb{Z})$. $\square$

The only parts of the proof that rest on computation are the four signs (5) and the 48 identities of the program; both are finite identities between integral matrices.

5.4 Independent checks and rejection tests

Four implementations agree on all identities: the checker; the exact program of the first independent verification run (root system generated by reflections, the Jacobi identity in the form $\text{ad}[x, y] = [\text{ad}x, \text{ad}y]$, checks of $N_{sr} = -N_{rs}$, $N_{-r, -s} = -N_{rs}$ and of the Killing form, and inverses of the generators computed as genuine inverses); the program of the second independent verification run, which builds its own model of F_4 over $\mathbb{Z}$ from the Cartan matrix by the extraspecial-pair recursion (the basis B_0), connects the table of B_T to it by the sign map η and evaluates the program in this model; and the program of the third independent verification run, which realizes B_T and B_0 from their extraspecial signs inside the 27-dimensional minuscule module of the Lie algebra of type E_6 , folded to type F_4 , finds both tables equal to the files, and evaluates the program exactly in the adjoint representation and again in the 27-dimensional module. The search program checks the 48 identities a fifth time, in 64-bit arithmetic. The checker rejects each of 11 deliberate corruptions made by `rejection_tests.py` (among them changed torus factors, a changed parameter in a , exchanged arguments of a product node, an exponent changed by 1, a deleted target, a sign change of one pair N_{rs}, N_{sr} and a doubled N_{rs}) and accepts the original; the three verification runs made 5, 13 and 8 negative tests of their own, all rejected.

5.5 How the triple and the program were found

Nothing in this subsection is needed for the proof. The triple has the shape of Timofeenko's triples for E_l [Tim17, Table 2]: b and c are involutive lifts, with torus factors, of the two halves w and w' of the bipartite Coxeter element ww' , and a is a product of commuting root elements (two in Timofeenko's triples) and a torus element that commutes with b . In F_4 the root elements in a were taken of both lengths: commutators of short root elements give long root elements only with even parameters (the structure constant of two short roots with a long sum is ± 2), and the following computation shows how a produces an odd one.

Remark 5.1 (The key step). For the short root $s = 0110$, exactly in B_T ,

$$a x_s(1) a x_s(-1) = x_{1110}(-1) x_{1220}(-3) x_{1221}(-1)$$

(`check_note.py`, Part D; `hand_step_check.py`). As $\chi_a(s) = 1$, the left side is the commutator $u x_s(1) u^{-1} x_s(-1)$; the factor $x_{1000}(1)$ of u contributes through the roots $1000 + s = 1110$ (short) and $1000 + 2s = 1220$ (long), and $x_{1110}(1)$ contributes $x_{1220}(\pm 2)$, which together give the odd parameter -3 . So once $x_{0110}(\pm 1)$, $x_{1110}(\pm 1)$, $x_{1221}(\pm 1)$ and $x_{1220}(2)$ lie in $\langle a, b, c \rangle$, so does $x_{1220}(1)$; the program does all such bookkeeping.

The search program `search_certificate.py` forms $\zeta = bc$, a lift of the Coxeter element, of order 12; the conjugates $a_k = \zeta^k a \zeta^{-k}$ ($0 \leq k \leq 11$); and the commutators $(a_i a_j)^2$, of which 36 of the 66 with $i < j$ are unipotent. For eight positive systems Ψ it computes in Mal'cev (echelon) coordinates the subgroup of $U_\Psi(\mathbb{Z})$ generated by those of these elements that lie in U_Ψ , by the root elements

already found and by their conjugates under the a_k that normalize U_Ψ , and reads off root elements $x_r(\pm 1)$ and $x_r(\pm 2)$; conjugation by ζ , by b and by the n_r for which $x_{\pm r}(\pm 1)$ are known completes the list. One round of 21 seconds gives all 48 elements $x_r(\pm 1)$; the pruned program keeps 1064 of 32210 nodes, and a second run reproduces the certificate byte for byte. The algebra of type F_4 of the search program is obtained by folding the algebra of type E_6 , constructed with the sign function of Frenkel and Kac [FK80].

6 Consequences

Corollary 6.1. *For every prime p the reductions $\bar{a}, \bar{b}, \bar{c}$ of a, b, c modulo p are involutions of the Chevalley group $F_4(p)$, $\bar{a}\bar{b} = \bar{b}\bar{a}$, and $\langle \bar{a}, \bar{b}, \bar{c} \rangle = F_4(p)$.*

Proof. Reduction modulo p is a homomorphism from $F_4(\mathbb{Z}) \subset GL_{52}(\mathbb{Z})$ to $GL_{52}(\mathbb{F}_p)$, and it maps $x_r(t)$ to the root element $\bar{x}_r(\bar{t})$ of Chevalley's group $F_4(p)$ acting on $L_{\mathbb{Z}} \otimes \mathbb{F}_p$ [Che55]. Since $\bar{x}_r(\bar{t}) = \bar{x}_r(\bar{1})^t$ for $t \in \{0, 1, \dots, p-1\}$, the images of the $x_r(1)$ generate $F_4(p)$; so the image of $F_4(\mathbb{Z})$, which by Theorem 3.1 is $\langle \bar{a}, \bar{b}, \bar{c} \rangle$, is $F_4(p)$. The relations $a^2 = b^2 = c^2 = 1$ and $ab = ba$ reduce, and $\bar{a}, \bar{b}, \bar{c} \neq 1$ because the entries of each of $a-1$, $b-1$, $c-1$ have greatest common divisor 1 (`check_note.py`, Part D). $\square$

For $p = 2$ this agrees with Nuzhin's theorem [Nuz90] as it is stated in its zbMATH review (Zbl 0725.20015): a simple Chevalley group of normal or twisted type over a finite field of characteristic 2 is generated by three involutions two of which commute if and only if its type is not A_2 , ${}^2A_2$, A_3 or ${}^2A_3$. For odd p we have not read the classification in odd characteristic [Nuz97a, Nuz97b] and claim no comparison.

Let $n(G)$ be the least number of involutions that generate a group G and whose product is 1; Nuzhin asked for this number for the finite simple groups [KM26, Problem 14.69 a)], a part of that problem which is marked as solved there.

Proposition 6.2. $n(F_4(\mathbb{Z})) = 5$.

Proof. The five involutions a, b, ab, c, c generate $F_4(\mathbb{Z})$, and $a \cdot b \cdot ab \cdot c \cdot c = (ab)^2 c^2 = 1$ because $ab = ba$; here ab is an involution since $a \neq b$ (`check_note.py`, Part D). So $n(F_4(\mathbb{Z})) \leq 5$. Conversely, let $x_1, \dots, x_k$ be involutions with product 1 which generate a group G . If $k \leq 3$, then $|G| \leq 4$: for $k = 2$ we get $x_2 = x_1$, and for $k = 3$ the element $x_3 = x_2 x_1$ is an involution only if x_1 and x_2 commute. If $k = 4$, then G is a quotient of $\Gamma = \langle x, y, z \mid x^2, y^2, z^2, (xyz)^2 \rangle$. In Γ the elements $u = xy$ and $v = yz$ commute, since $xyzxyz = 1$ gives $vu = yzxy = x^{-1}z^{-1} = xz = uv$. The words of even length form a subgroup of index at most 2 (all relators have even length), and it is generated by the products of two generators, which are 1, $u^{\pm 1}$, $v^{\pm 1}$ and $(uv)^{\pm 1}$; so it is abelian, and Γ and all its quotients are soluble. (Γ , the orbifold group of the sphere with four cone points of order 2, is virtually $\mathbb{Z}^2$.) But $F_4(\mathbb{Z})$ is not soluble: by Corollary 6.1 it maps onto $F_4(2)$, a simple non-abelian group [Che55]. Hence $k \geq 5$. $\square$

Kazakova and Nuzhin prove the analogous equality $n(G_2(\mathbb{Z} + i\mathbb{Z})) = 5$ [KN25, Corollary] in the same way: the upper bound comes from the five involutions $\alpha, \beta, \beta\alpha, \gamma, \gamma$ of their triple, and the lower bound from a simple finite quotient and a lemma of Nuzhin, quoted there as [KN25, Lemma 5], by which a group G with $n(G) = 4$ has a nontrivial cyclic normal subgroup. Our argument for the lower bound, through the soluble group Γ , is self-contained.

Remark 6.3 (Scott's inequality). If a group G generated by $g_1, \dots, g_k$ with $g_1 \cdots g_k = 1$ acts on a finite-dimensional vector space V , then $\sum_i \dim[V, g_i] \geq \dim[V, G] + \dim[V^*, G]$ with $[V, g] = (g-1)V$ [Sco77]. For $V = L_{\mathbb{Q}}$ and the five involutions above the right side is at most 104, and the left side is

$5 \cdot 28 = 140$, since $\dim[V, g] = 28$ for $g = a, b, ab, c$ and c occurs twice (`check_note.py`, Part F). This necessary condition for generation is a consistency check only.

7 The open types, and conclusion

The status of Problem 15.67 after this note is a partial answer: the adjoint Chevalley group $F_4(\mathbb{Z})$ is generated by three involutions two of which commute (computer-assisted proof with an exact certificate); the types B_l, C_l ($l \geq 3$) and D_l ($l \geq 4$) stay open.

A non-result in small rank. We tried the same design for B_3, C_3 and D_4 over $\mathbb{Z}$ (adjoint representations of dimensions 21, 21 and 28, built by folding simply-laced algebras and checked like the one of type F_4): b and c lifts of the two halves of the bipartite Coxeter element, and $a = ut$ with u a product of commuting root elements $x_r(\pm 1)$ and t a torus element of the elementary group; then $a = um$ with m a monomial involution and u with Mal'cev coordinates in $\{-1, 0, 1\}$; then seeds from all words of length at most 12. In the first design there is no admissible a for B_3 and C_3 (only 4 of the 2^l torus characters occur in the elementary group, the Cartan matrix being singular modulo 2); for D_4 (30 of 128 triples in the first design, 6 in the last) and B_3 (32 triples in the second design, 8 in the last) no root element was found. The words of length at most 12 in a, b, c, ab give 3074 distinct elements for B_3 and 1760 or 1943 for D_4 , against 3641 in the free product $(\mathbb{Z}/2 \times \mathbb{Z}/2) * \mathbb{Z}/2$, so these triples satisfy further relations; we did not determine the subgroups they generate. This is a failure of these designs, not evidence against generation (programs and outputs in `exploratory/`).

No obstruction is known either. For every open type, reduction modulo 2 maps the group generated by the $x_r(1)$ onto the simple Chevalley group of the same type over the field with two elements, and by Nuzhin's theorem [Nuz90] (as stated in its zbMATH review) such a group is generated by three involutions two of which commute; so the prime 2 gives no obstruction of the kind that settles B_2 . We have not examined the reductions modulo odd primes, for which the classification [Nuz97a, Nuz97b] was not read.

Conclusion. The answer is now known for the types $A_l, B_2 = C_2, G_2, F_4, E_6, E_7$ and E_8 , for F_4 by the computer-assisted proof of this note; the types B_l, C_l ($l \geq 3$) and D_l ($l \geq 4$) stay open. For F_4 a human-readable proof of generation would be desirable, for instance in the style of Timofeenko's proof for E_l [Tim17], which obtains a few root elements by explicit commutators and then conjugates by a lift of the Coxeter element and by monomial elements.

8 Computations and reproducibility

The proof of Theorem 3.1 consists of Proposition 4.1, Lemmas 2.1 and 4.2 and finite identities between integral matrices, checked by the programs of the reproducibility package (folder `reproducibility/` of the source archive), whose `README.md` gives command lines, running times and checksums. It contains the data (the certificate and the table of B_T , with SHA-256 beginning `5a1d1ff2` and `5bab4e71`, and their versions for B_0 in `b0_basis/`), the checker with `rejection_tests.py`, and the program `check_note.py` (standard library, about 8 seconds), which checks (A) the certificate, with the checker; (B) the table of B_T from the signs of Definition 2.3; (C) the table of B_0 , the sign map η and Remark 3.2, with the checker; (D) the facts used in Proposition 4.1, Corollary 6.1, Proposition 6.2 and Remark 5.1; (E) the 128 sign patterns of Section 3; (F) the ranks of Remark 6.3; and (G) the data of Section 5.1. The search program (with `numpy`, about 21 seconds) and `export_structure_constants.py` regenerate the certificate and the table byte for byte, without overwriting them. The folders `exploratory/`,

`independent_runs/` and `independent_run_2/` contain the attempts on B_3 , C_3 and D_4 (non-results), the programs and outputs of the first two independent verification runs, and those of the third. All recorded outputs were reproduced from the package folder, up to running times, and again from the source archive by the third verification run.

Verification. The result was first obtained with the search program and checked with the checker of Section 5.2. Two independent verification runs, both AI-assisted, followed; both examined the first write-up of the result, not this text, and each wrote its own programs. The first run re-derived the parts of the proof that do not depend on signs (Proposition 4.1 up to the four signs, Lemmas 2.1 and 4.2, Corollary 6.1, Proposition 6.2), read [Tim17], [KN25], the zbMATH reviews of [Nuz90, Nuz97b] and version 49 of [KM26], and confirmed all identities of the certificate with its exact program. The second run built its own model of F_4 over $\mathbb{Z}$ by the extraspecial-pair recursion, connected the table of the certificate to it by the sign map η , evaluated the program there, examined the 128 sign patterns, made 13 negative tests, checked the reductions modulo the primes up to 31 and modulo 101, re-ran the programs of the first write-up (identical outputs; the certificate and the table were regenerated byte for byte), checked the bibliography with Crossref, and found the description of B_T by extraspecial signs and the form of the theorem in B_0 . Neither run found a mathematical or computational error. The changes they required, all of wording and presentation, were made; among them are Timofeenko's definition of $\Phi(\mathbb{Z})$ with the role of Steinberg's theorem (Section 2.3), the description of B_T and Remark 3.2, and the dependence on the basis in Section 3. When this note was written, `check_note.py` was added; its Parts B, C and E are new.

A third independent verification run, also AI-assisted, examined this text line by line before its final revision and wrote its own programs (folder `independent_run_2/`). It read the sources again (the Notebook in versions 48 and 49, [Tim17] and [KN25] in the journal files, the zbMATH review of [Nuz90]) and re-derived every proof. Its model of F_4 over $\mathbb{Z}$ is built in a way different from those of the search program and of the first two runs: from the 27-dimensional minuscule module of the Lie algebra of type E_6 , on which the Chevalley generators act by 0/1 matrices, by folding, with the root vectors normalized by the 20 extraspecial signs. In this model it found the tables of B_T and B_0 equal to the files, entry by entry, and it checked exactly, in the adjoint representation and again in the 27-dimensional module, the identities of Theorem 3.1 and all 48 targets of the program; in the adjoint representation also the four signs (5), the form of the theorem in B_0 with its certificate of 2079 nodes, Remark 5.1, and the facts used in Corollary 6.1, Proposition 6.2 and Remark 6.3. It confirmed the statements on the 128 sign patterns in Section 3, evaluating the program modulo a prime in the seven other classes. It rejected eight corrupted inputs, re-ran the whole package from the source archive (identical outputs up to running times), checked the identifiers of the bibliography with Crossref and repeated the literature search. It found no error in a theorem or a proof. It required eight corrections, which were made: an incorrect side remark on $a, b, c \neq 1$ in Section 1; Timofeenko's own wording in his definition of $\Phi(\mathbb{Z})$ (Section 2.3); the bracket $[h_i, h_j] = 0$ in Section 2.2; the description of the commutators $(a_i a_j)^2$ in Section 5.5, which are not all unipotent; the credit to [KN25] for the pattern of Proposition 6.2; the comparison with Timofeenko's triples; the description of the second run; and the record of this run.

Scope and priority. The note settles the type F_4 of Problem 15.67 with a positive answer; the types B_l , C_l ($l \geq 3$) and D_l ($l \geq 4$) stay open, and with them the question as a whole. The problem is due to Ya. N. Nuzhin [KM26]; the known cases are due to Tamburini and Zucca [TZ97], Nuzhin [Nuz08], the result for $PSp_4(\mathbb{Z})$ recorded in [KM26, Archive, 7.30], Timofeenko [Tim15, Tim17] and Kazakova and Nuzhin [KN25]; the shape of our triple follows Timofeenko [Tim17]. To our knowledge the triple of Theorem 3.1 and the proof that it generates $F_4(\mathbb{Z})$ are new, with the consequences in

Section 6: the proposer’s own paper of 2025 [KN25] names A_n , E_n and G_2 as the only solved types, and [Tim17] names F_4 as the remaining exceptional type. Read: [Tim17] and [KN25] completely; Problems 15.67 and 14.69 and Archive 7.30 in versions 48 and 49 of [KM26]; the zbMATH entries of [Nuz08, Nuz90, Sco77]; the Crossref records of [TZ97, Tim15, KN25, Nuz90, Nuz97a, Nuz97b, Che55, Ste16, Sco77, FK80] and the registration of the identifier of [Tim17]. Not read: [TZ97, Nuz08, Tim15], of which only the results quoted in [KM26, Tim17, KN25] are used; [Nuz97a, Nuz97b, Sco77, FK80]; the books [Ste16, Car72] and [Che55]. So Steinberg’s theorem is used as it is quoted in [Tim17, KN25], and the section numbers given for [Car72] were not checked against the book. The searches (22 October 2026, repeated in part by the second and the third verification runs) covered zbMATH Open, arXiv, OpenAlex, Crossref, Semantic Scholar (partly refused) and web searches in English and Russian, with author queries (Nuzhin, Timofeenko and coauthors), phrase and title queries (“three involutions”, “two of which commute”, “generating triples”, involutions with Chevalley or integers, F_4 with involutions) and the works citing [TZ97]. The later work on generation by involutions found there concerns $GL_n(\mathbb{Z})$, $PGL_n(\mathbb{Z})$, groups over $\mathbb{Z} + i\mathbb{Z}$ and finite groups, not $F_4(\mathbb{Z})$ or the types B , C , D over $\mathbb{Z}$. No printed answer to the question for F_4 was found in the sources searched (zbMATH, arXiv, OpenAlex, Crossref, Semantic Scholar and the web searches), and nothing on $B_l(\mathbb{Z})$, $C_l(\mathbb{Z})$ or $D_l(\mathbb{Z})$; a search that finds nothing is not a proof of novelty, conference abstracts and theses of the Krasnoyarsk school that are not indexed cannot be excluded, and we claim no priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

A The bases B_T and B_0

Table 1 lists the 20 extraspecial pairs (for the order of Section 2.5) with the signs that define B_T ; in B_0 all signs are +1. `check_note.py` prints this table (Part B).

$\gamma = \xi + \zeta$	$p + 1$	sign in B_T	$\gamma = \xi + \zeta$	$p + 1$	sign in B_T
0011 = 0001 + 0010	1	−	1121 = 0001 + 1120	1	−
0110 = 0010 + 0100	1	−	1220 = 0100 + 1120	1	−
1100 = 0100 + 1000	1	+	1122 = 0001 + 1121	2	−
0111 = 0001 + 0110	1	−	1221 = 0001 + 1220	1	−
0120 = 0010 + 0110	2	−	1222 = 0001 + 1221	2	−
1110 = 0010 + 1100	1	−	1231 = 0010 + 1221	1	−
0121 = 0001 + 0120	1	−	1232 = 0001 + 1231	1	−
1111 = 0001 + 1110	1	−	1242 = 0010 + 1232	2	−
1120 = 0010 + 1110	2	−	1342 = 0100 + 1242	1	+
0122 = 0001 + 0121	2	−	2342 = 1000 + 1342	1	+

Table 1: The extraspecial pairs $(\xi_\gamma, \zeta_\gamma)$ and the signs of $N_{\xi_\gamma \zeta_\gamma} = \pm(p + 1)$ in the basis B_T .

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