

Quotients of Subdirect Products and Regular p -Groups (Kourovka Notebook Problem 13.19)

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Abstract

Let Q be a subgroup of a direct product $G_1 \times \cdots \times G_n$ of groups and H a normal subgroup of Q such that Q and H both project onto every factor. Yu. M. Gorchakov asked in the Kourovka Notebook (Problem 13.19) whether Q/H is a regular p -group whenever it is a p -group; in 2009 he stated this as a conjecture for quotients of class exactly $n - 1$. We show that Q/H is necessarily regular if and only if $n \leq p$. The positive half is classical: Q/H has class at most $n - 1$ by Gorchakov's lemma of 1965, and p -groups of class less than p are regular by a theorem of P. Hall. For the negative half we show, with Magnus's theorem on free groups, that every nilpotent group of class at most $n - 1$ is such a quotient with n free factors, and that a finite p -group whose n -th modular dimension subgroup is trivial is such a quotient with n isomorphic finite p -group factors. With Jennings's theorem this gives irregular quotients of class exactly $n - 1$ for every $n \geq p + 1$, for instance $C_p \wr C_p$ for $n = p + 1$; so the conjecture fails as well. Our smallest example has $p = 2$, $n = 3$, $|Q| = 512$, $|G_i| = 32$ and $Q/H \cong D_8$. For $p = 2$ an irregular quotient with four perfect factors can be extracted from a 2018 theorem of Kearnes, Mayr and Ruškuc, who do not discuss regularity; whether examples of D. M. Evans (1993) give one was not determined. What this note adds is the threshold $n \leq p$ for every prime, in particular for the odd primes, the two realization theorems and the explicit examples. This is an unrefereed note.

1 Introduction

1.1 The problem

Let $G_1, \dots, G_n$ be groups and let $\pi_i: G_1 \times \cdots \times G_n \rightarrow G_i$ be the projections. A subgroup S of the direct product is *subdirect* if $\pi_i(S) = G_i$ for $i = 1, \dots, n$. We call $(G_1, \dots, G_n; Q, H)$ a *configuration of length n* if Q is a subdirect subgroup of $G_1 \times \cdots \times G_n$ and H is a normal subgroup of Q which is also subdirect; the group Q/H is its *quotient*. No finiteness is assumed. Problem 13.19 of the Kourovka Notebook, posed by Yu. M. Gorchakov in the 13th issue (1995), considers exactly this situation and asks the following.

Problem 1.1 ([KM26, Problem 13.19]). “If Q/H is a p -group, is Q/H a regular p -group?”

Regularity is P. Hall's notion [Hal34, §4]. For a group X let X' be its derived subgroup and $\mathcal{U}_1(X)$ the subgroup generated by the p -th powers of its elements.

Definition 1.2. A p -group P is *regular* if for all $a, b \in P$ there is $c \in \mathcal{U}_1(\langle a, b \rangle')$ with $(ab)^p = a^p b^p c$.

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Hall defines regularity for finite p -groups; Problem 1.1 does not require finiteness, and we use the same definition for all p -groups (our negative answers use finite groups only). The condition for a, b involves only $\langle a, b \rangle$, so subgroups of regular p -groups are regular.

The current version of the Notebook (No. 21, arXiv version 48 of 6 October 2026) prints Problem 1.1 without comment [KM26]. An earlier question of Gorchakov in the same setting, Problem 7.8, now in the Archive of the Notebook, asked whether the nilpotency class of the quotient increases with n ; Khukhro answered it positively [Khu82]. That the quotient is nilpotent of class at most $n - 1$ is Gorchakov's lemma [Gor65, Lemma 1] (Theorem 3.1 below). In 2009 Gorchakov proved that this bound is attained for every n [Gor09], and right after his theorem he conjectured that such a quotient of class exactly $n - 1$ is regular whenever it is a p -group (p. 801 of the Russian original).

1.2 Results

Theorem 1.3. *Let p be a prime and $n \geq 1$.*

- (a) *If $n \leq p$, then the quotient of every configuration of length n which is a p -group is regular.*
- (b) *If $n \geq p + 1$, then there is a configuration of length n whose quotient is an irregular finite p -group of nilpotency class exactly $n - 1$. The factors G_i can be chosen to be pairwise isomorphic finite p -groups, or free groups of finite rank. For $n = p + 1$ the quotient can be chosen isomorphic to the wreath product $C_p \wr C_p$.*

So the quotient is necessarily regular if and only if $n \leq p$, and the answer to Problem 1.1 is negative; the first case is $p = 2$, $n = 3$. Part (b) also shows that the conjecture of [Gor09] holds exactly for $n \leq p$. The smallest example in this note (Proposition 6.3) has $p = 2$, $n = 3$, $|Q| = 512$ and $Q/H \cong D_8$, and its factors G_i are extraspecial groups of order 32.

Part (a) is Corollary 3.2. Part (b) rests on two realization theorems. By Theorem 4.2, every nilpotent group P of class at most $n - 1$, generated by a set Y , is the quotient of a configuration of length n whose factors are free groups of rank $(n - 1)|Y|$; with Gorchakov's lemma, the quotients of configurations of length n are thus, up to isomorphism, exactly the nilpotent groups of class at most $n - 1$. By Theorem 5.1, a group P with $D_n(P; \mathbb{Z}) = 1$, for instance a finite p -group with $D_n(P; \mathbb{F}_p) = 1$ (Section 2.3), is such a quotient with pairwise isomorphic factors, each embedded in $P^{2^{n-1}-1}$. Both proofs rest on one inclusion–exclusion identity in the integral group ring of a free group (Lemma 4.1). They are applied to $C_p \wr C_p$ and to its direct products with unitriangular groups in Section 6.

1.3 What is known and what is new

Part (a) of Theorem 1.3 is classical. The class bound is Gorchakov's [Gor65, Lemma 1], as cited in [Gor09] and [KMR18]; Khukhro [Khu82] gives the same argument, citing Gorchakov's book [Gor78, Proposition 4.7], and Kearnes, Mayr and Ruškuc [KMR18, Theorem 2.1] found it again (the addendum to the arXiv version of their paper credits Gorchakov). That a p -group of class less than p is regular is P. Hall's theorem [Hal34, §4] (see also [Hup67, Kap. III, §10]).

The examples of high class known to Gorchakov were regular: his own [Gor09] have $n = p$ factors for an odd prime p and class $p - 1$, and Khukhro's [Khu82] have exponent p and class less than p . Kearnes, Mayr and Ruškuc [KMR18] constructed configurations with *perfect* factors whose quotients have arbitrary class; from their Theorem 4.2 and Lemma 6.1 one obtains, with four factors, a quotient which is a non-abelian 2-group of class 2, hence irregular (Remark 7.1). Their paper does not mention p -groups, regularity or the Notebook; still, for $p = 2$ a negative answer to Problem 1.1 has been implicit in print since 2018. A lemma of E. Hrushovski, as quoted in the review of a 1993 paper of D. M. Evans [Eva93], contains Gorchakov's setting, and Evans's examples give finite non-abelian

nilpotent quotients (Remark 7.2). We have read only the review and could not determine whether this yields a 2-group, in which case the case $p = 2$ would be implicit in print since 1993. For odd primes we found no irregular quotient in print.

Magnus's theorem [Mag37] (see also [Wit37]) and Jennings's theorem [Jen41] are classical, and the construction of Theorem 5.1 is a variant of that of [KMR18, Section 3]. What this note adds is the threshold $n \leq p$ for every prime p , in particular for the odd primes, Theorems 4.2 and 5.1, and the explicit examples. Apart from the implicit examples of Remarks 7.1 and 7.2, no printed answer to Problem 1.1 was found in the sources searched (see "Scope and priority" at the end); a search that finds nothing is not a proof of novelty.

2 Preliminaries

2.1 Commutators and regular p -groups

We write $[x, y] = x^{-1}y^{-1}xy$ and $[x_1, \dots, x_k] = [[x_1, \dots, x_{k-1}], x_k]$, and for subgroups A, B we let $[A, B]$ be the subgroup generated by the $[a, b]$ with $a \in A, b \in B$. The lower central series is $\gamma_1(G) = G$, $\gamma_{k+1}(G) = [\gamma_k(G), G]$, and G has class at most c if $\gamma_{c+1}(G) = 1$. We use the following standard facts: if A and B are normal in G , then $[A, B]$ is normal and $[A, B] \leq A \cap B$; a surjective homomorphism φ satisfies $\varphi([A, B]) = [\varphi(A), \varphi(B)]$; and $\gamma_k(A \times B) = \gamma_k(A) \times \gamma_k(B)$.

Lemma 2.1 (Hall–Petrescu formula; see [Hup67, Kap. III, §9]). *For elements x, y of a group and $m \geq 1$ there are elements $c_i \in \gamma_i(\langle x, y \rangle)$, $2 \leq i \leq m$, such that $x^m y^m = (xy)^m c_2^{\binom{m}{2}} c_3^{\binom{m}{3}} \dots c_m^{\binom{m}{m}}$.*

Lemma 2.2 (P. Hall [Hal34, §4]). *A p -group of class less than p is regular.*

Proof. Let a, b be elements of such a group and $K = \langle a, b \rangle$. By Lemma 2.1 with $m = p$ we have $a^p b^p = (ab)^p c$ with $c = c_2^{\binom{p}{2}} \dots c_{p-1}^{\binom{p}{p-1}} c_p$ and $c_i \in \gamma_i(K)$. Here $c_p \in \gamma_p(K) = 1$, and for $2 \leq i \leq p-1$ the binomial coefficient $\binom{p}{i}$ is divisible by p , so $c_i^{\binom{p}{i}}$ is a p -th power of an element of $\gamma_i(K) \leq K'$. Hence c and c^{-1} lie in $\mathcal{U}_1(K')$, and $(ab)^p = a^p b^p c^{-1}$. Finiteness is not used. $\square$

Lemma 2.3. (a) *A 2-group of class at most 2 is regular if and only if it is abelian.* (b) *If P is a finite non-abelian 2-group, then $P/\gamma_3(P)$ is a non-abelian group of class 2, hence not regular.*

Proof. (a) Abelian groups are regular. Let P have class at most 2 and $a, b \in P$. Then $[b, a]$ is central, $ba = ab[b, a]$, and $(ab)^2 = a(ba)b = a^2 b^2 [b, a]$. Moreover $\langle a, b \rangle'$ is the normal closure of $[a, b]$ in $\langle a, b \rangle$, which is $\langle [a, b] \rangle$; so $\mathcal{U}_1(\langle a, b \rangle') = \langle [a, b]^2 \rangle$. If P is regular, then $[b, a] = [a, b]^{-1}$ lies in $\langle [a, b]^2 \rangle$, so $[a, b]^{2k+1} = 1$ for some integer k , and $[a, b] = 1$ because its order is a power of 2. (b) P is nilpotent. If $P/\gamma_3(P)$ were abelian, then $\gamma_2(P) = \gamma_3(P) = [\gamma_2(P), P]$, and nilpotency would give $\gamma_2(P) = 1$. Now apply (a). $\square$

2.2 Configurations

Lemma 2.4. *Let $(G_1, \dots, G_n; Q, H)$ be a configuration and $M_i = Q \cap \ker \pi_i$.*

(a) *$Q = M_i H$ for $i = 1, \dots, n$.*

(b) *If $H \leq H_1$ and H_1 is normal in Q , then $(G_1, \dots, G_n; Q, H_1)$ is a configuration. In particular every quotient of the group Q/H is the quotient of a configuration of length n .*

Proof. (a) Given $q \in Q$, choose $h \in H$ with $\pi_i(h) = \pi_i(q)$; then $qh^{-1} \in M_i$. (b) $\pi_i(H_1) \supseteq \pi_i(H) = G_i$. $\square$

2.3 Dimension subgroups

Let P be a group and R a commutative ring ($R = \mathbb{Z}$ or $R = \mathbb{F}_p$). The augmentation ideal $\Delta_R(P)$ of the group ring RP is the kernel of the ring homomorphism $RP \rightarrow R$ that maps every $g \in P$ to 1; it is spanned by the elements $g - 1$, $g \in P$. The n -th *dimension subgroup* is

$$D_n(P; R) = \{ g \in P : g - 1 \in \Delta_R(P)^n \}.$$

We use four elementary facts: (D1) $D_n(P; R)$ is a subgroup containing $\gamma_n(P)$; (D2) $D_n(P; \mathbb{Z}) \leq D_n(P; \mathbb{F}_p)$; (D3) a homomorphism $\varphi: P \rightarrow P_1$ maps $D_n(P; R)$ into $D_n(P_1; R)$; (D4) $D_{n+1}(P; R) \leq D_n(P; R)$. For (D1), use $gh - 1 = (g - 1)(h - 1) + (g - 1) + (h - 1)$, $g^{-1} - 1 = -g^{-1}(g - 1)$, and $[g, h] - 1 = g^{-1}h^{-1}((g - 1)(h - 1) - (h - 1)(g - 1))$, which lies in Δ^{k+1} if $g - 1 \in \Delta^k$; then induct on n . For (D2) and (D3), the ring homomorphisms $\mathbb{Z}P \rightarrow \mathbb{F}_pP$ and $RP \rightarrow RP_1$ map augmentation ideals into augmentation ideals, hence their n -th powers into the n -th powers. (D4) holds because $\Delta^{n+1} \subseteq \Delta^n$.

Theorem 2.5 (Magnus [Mag37]; see also [Wit37]). *Let F be the free group on a set X , let $\mathbb{Z}\langle\langle T \rangle\rangle$ be the ring of formal power series with integer coefficients in non-commuting indeterminates t_x , $x \in X$, and let μ be the homomorphism from F to the units of this ring with $\mu(x) = 1 + t_x$. Then for $n \geq 1$ an element $w \in F$ lies in $\gamma_n(F)$ if and only if $\mu(w) - 1$ has no terms of degree less than n .*

Corollary 2.6. *For a free group F and $n \geq 1$, $D_n(F; \mathbb{Z}) = \gamma_n(F)$.*

Proof. The map μ extends to a ring homomorphism $\mathbb{Z}F \rightarrow \mathbb{Z}\langle\langle T \rangle\rangle$, which maps every $g - 1$ to a series without constant term, hence $\Delta_{\mathbb{Z}}(F)^n$ into the series without terms of degree less than n . So $D_n(F; \mathbb{Z}) \leq \gamma_n(F)$ by Theorem 2.5, and the reverse inclusion is (D1). $\square$

Theorem 2.7 (Jennings [Jen41]). *Let P be a finite p -group and $k \geq 1$. Then $D_k(P; \mathbb{F}_p)$ is the product of the subgroups $\gamma_i(P)^{p^j}$ over all $i \geq 1$, $j \geq 0$ with $ip^j \geq k$, where X^m denotes the subgroup generated by the m -th powers of the elements of X .*

3 The class bound and the positive answer for $n \leq p$

Theorem 3.1 (Gorchakov [Gor65, Lemma 1]). *In every configuration $(G_1, \dots, G_n; Q, H)$ of length $n \geq 1$ we have $\gamma_n(Q) \leq H$. Hence Q/H is nilpotent of class at most $n - 1$.*

Proof. Let $M_i = Q \cap \ker \pi_i$, a normal subgroup of Q , and let bars denote images in Q/H . By Lemma 2.4(a), $\overline{M_i} = Q/H$ for every i , so $\gamma_n(Q/H) = [\overline{M_1}, \dots, \overline{M_n}]$, the image of $[M_1, \dots, M_n]$. Since the M_i are normal, induction on k gives $[M_1, \dots, M_k] \leq M_1 \cap \dots \cap M_k$. Finally $M_1 \cap \dots \cap M_n = Q \cap \ker \pi_1 \cap \dots \cap \ker \pi_n = 1$. So $\gamma_n(Q/H) = 1$. $\square$

This is the argument of Khukhro [Khu82] and of Kearnes, Mayr and Ruškuc [KMR18, proof of Theorem 2.1]; we have not seen [Gor65] itself. No finiteness is used.

Corollary 3.2 (Theorem 1.3(a)). *Let $n \leq p$. If the quotient of a configuration of length n is a p -group, then it is regular.*

Proof. By Theorem 3.1 the quotient has class at most $n - 1 \leq p - 1$; apply Lemma 2.2. $\square$

For $n = 1$ the quotient is trivial and for $n = 2$ it is abelian. Problem 1.1 has content only for $n \geq p + 1$, where the class bound $n - 1$ is at least p .

4 Inclusion–exclusion and free factors

Throughout this section and the next, $n \geq 1$, $[n] = \{1, \dots, n\}$, X is the disjoint union of sets $X_1, \dots, X_n$ (the *blocks*), and F is the free group on X . For $T \subseteq [n]$ let ρ_T be the endomorphism of F with $\rho_T(x) = 1$ for $x \in X_i$, $i \in T$, and $\rho_T(x) = x$ for the other $x \in X$; we use the same letter for its extension to a ring endomorphism of $\mathbb{Z}F$. For $w \in F$ put

$$\omega(w) = \sum_{S \subseteq [n]} (-1)^{n-|S|} \rho_{[n] \setminus S}(w) \in \mathbb{Z}F.$$

Lemma 4.1. *For every $w \in F$, $\omega(w) \in \Delta_{\mathbb{Z}}(F)^n$.*

Proof. Write $w = l_1 l_2 \cdots l_m$ with letters $l_j \in X \cup X^{-1}$, and let $\beta(j) \in [n]$ be the block of l_j . In $\mathbb{Z}F$,

$$w = \prod_{j=1}^m (1 + (l_j - 1)) = \sum_{J \subseteq [m]} \pi_J,$$

where π_J is the product of the factors $l_j - 1$, $j \in J$, in increasing order of j , and $\pi_\emptyset = 1$. For $T \subseteq [n]$ we have $\rho_T(l_j - 1) = 0$ if $\beta(j) \in T$ and $\rho_T(l_j - 1) = l_j - 1$ otherwise; since ρ_T is a ring endomorphism, $\rho_T(w) = \sum \pi_J$ over the J with $\beta(J) \cap T = \emptyset$, where $\beta(J) = \{\beta(j) : j \in J\}$. Hence $\omega(w) = \sum_J c_J \pi_J$ with

$$c_J = \sum_{\beta(J) \subseteq S \subseteq [n]} (-1)^{n-|S|} = \sum_{R \subseteq [n] \setminus \beta(J)} (-1)^{n-|\beta(J)|-|R|} = (1-1)^{n-|\beta(J)|},$$

which is 1 if $\beta(J) = [n]$ and 0 otherwise. If $\beta(J) = [n]$, then $|J| \geq n$, and π_J is a product of at least n elements of $\Delta_{\mathbb{Z}}(F)$. $\square$

Theorem 4.2. *Let P be a nilpotent group of class at most $n-1$, generated by a set Y . Then there is a configuration of length n with quotient isomorphic to P in which every G_i is a free group of rank $(n-1)|Y|$.*

Proof. For $n=1$ the group P is trivial and all groups can be taken trivial; let $n \geq 2$. Let $X_i = \{x_{i,y} : y \in Y\}$ be the i -th copy of Y , let $\psi : F \rightarrow P$ be the homomorphism with $\psi(x_{i,y}) = y$, which is onto, and $K = \ker \psi$. Write $\rho_i = \rho_{\{i\}}$ and $N_i = \ker \rho_i$. Since ρ_i maps F onto the free subgroup generated by $X \setminus X_i$ and is the identity on it, $G_i := F/N_i$ is free of rank $(n-1)|Y|$. Let $\theta : F \rightarrow G_1 \times \cdots \times G_n$, $\theta(w) = (wN_1, \dots, wN_n)$, and put $Q = \theta(F)$ and $H = \theta(K)$.

Clearly Q is subdirect, and H is normal in Q . For $j \neq i$ the element $x_{j,y} x_{i,y}^{-1}$ lies in K and $x_{i,y}$ lies in N_i , so KN_i contains X and equals F ; hence $\pi_i(H) = KN_i/N_i = G_i$, and $(G_1, \dots, G_n; Q, H)$ is a configuration. Its quotient is $Q/H \cong F/(K \cdot \ker \theta)$, and $\ker \theta = N_1 \cap \cdots \cap N_n$.

We claim that $N_1 \cap \cdots \cap N_n \leq \gamma_n(F)$. Let w lie in every N_i . For $\emptyset \neq T \subseteq [n]$ choose $i \in T$; then $\rho_T = \rho_{T \setminus \{i\}} \circ \rho_i$ (both sides agree on X), so $\rho_T(w) = 1$. Hence only the term $S = [n]$ of $\omega(w)$ differs from ± 1 , and $\omega(w) = w + \sum_{S \neq [n]} (-1)^{n-|S|} = w - 1$, because $\sum_{S \subseteq [n]} (-1)^{n-|S|} = (1-1)^n = 0$. By Lemma 4.1, $w - 1 \in \Delta_{\mathbb{Z}}(F)^n$, and $w \in \gamma_n(F)$ by Corollary 2.6.

Since P has class at most $n-1$, $\psi(\gamma_n(F)) = \gamma_n(P) = 1$, so $\ker \theta \leq \gamma_n(F) \leq K$ and $Q/H \cong F/K \cong P$. $\square$

With Theorem 3.1 this shows that the quotients of configurations of length n are, up to isomorphism, exactly the nilpotent groups of class at most $n-1$.

5 Finite factors

Let P be a group generated by a set Y , and let $\mathcal{P}$ be the set of all subsets of $[n]$. We work in the direct power $P^{\mathcal{P}}$, whose elements are the functions $f: \mathcal{P} \rightarrow P$, and write $\text{ev}_C(f) = f(C)$. For $i \in [n]$ and $y \in Y$ let $\delta_i(y) \in P^{\mathcal{P}}$ be given by $\delta_i(y)(C) = y$ if $i \in C$ and $\delta_i(y)(C) = 1$ if $i \notin C$. Put

$$L = \langle \delta_i(y) : i \in [n], y \in Y \rangle, \quad H = \{f \in L : f([n]) = 1\},$$

$$K_i = \{f \in L : f(C) = 1 \text{ for all } C \in \mathcal{P} \text{ with } i \notin C\}, \quad G_i = L/K_i \quad (i \in [n]).$$

Every $f \in L$ has $f(\emptyset) = 1$.

Theorem 5.1. *Suppose that $D_n(P; \mathbb{Z}) = 1$; by (D2) this holds if P is a finite p -group with $D_n(P; \mathbb{F}_p) = 1$. Then $K_1 \cap \dots \cap K_n = 1$, so $f \mapsto (fK_1, \dots, fK_n)$ embeds L into $G_1 \times \dots \times G_n$. Identifying L with its image Q , the tuple $(G_1, \dots, G_n; Q, H)$ is a configuration of length n with quotient isomorphic to P . Moreover L embeds in P^{2^n-1} , every G_i embeds in $P^{2^{n-1}-1}$, the groups $G_1, \dots, G_n$ are pairwise isomorphic, and if P is a finite p -group, then so are L , H and the G_i .*

Proof. Let $X_i = \{x_{i,y} : y \in Y\}$ and F be as in Section 4, let $\psi: F \rightarrow P$ be the homomorphism with $\psi(x_{i,y}) = y$, and $\theta: F \rightarrow P^{\mathcal{P}}$ the homomorphism with $\theta(x_{i,y}) = \delta_i(y)$, so that $L = \theta(F)$. For $C \in \mathcal{P}$,

$$\text{ev}_C \circ \theta = \psi \circ \rho_{[n] \setminus C}, \quad (1)$$

because both sides are homomorphisms that agree on the generators: both map $x_{i,y}$ to y if $i \in C$ and to 1 otherwise.

Step 1: $K_1 \cap \dots \cap K_n = Z$, where $Z = \{f \in L : f(C) = 1 \text{ for all } C \neq [n]\}$. If $C \neq [n]$, choose $i \notin C$; an element of K_i is trivial at C . Conversely, every C with $i \notin C$ differs from $[n]$.

Step 2: $Z = 1$. Let $f = \theta(w) \in Z$ and extend ψ to a ring homomorphism $\mathbb{Z}F \rightarrow \mathbb{Z}P$. By (1),

$$\psi(\omega(w)) = \sum_{S \subseteq [n]} (-1)^{n-|S|} f(S) = f([n]) + \sum_{S \neq [n]} (-1)^{n-|S|} f(S) = f([n]) - 1.$$

By Lemma 4.1, $\psi(\omega(w)) \in \psi(\Delta_{\mathbb{Z}}(F)^n) \subseteq \Delta_{\mathbb{Z}}(P)^n$. So $f([n]) \in D_n(P; \mathbb{Z}) = 1$, and $f = 1$.

Step 3: the configuration. By Steps 1 and 2 the map $L \rightarrow \prod L/K_i$ is injective, and it is onto each factor; so Q is subdirect. $H = \ker(\text{ev}_{[n]}|_L)$ is normal in L , and $L/H \cong \text{ev}_{[n]}(L) = P$, because $\text{ev}_{[n]}(\delta_i(y)) = y$. For each i , $\delta_i(y) \in K_i$ and $\delta_j(y)\delta_i(y)^{-1} \in H$, so the subgroup HK_i contains all generators of L ; hence $HK_i = L$ and H projects onto $L/K_i = G_i$.

Step 4: the remaining statements. K_i is the kernel of the projection of L to the coordinates $C \not\ni i$, and the coordinate $\emptyset$ is constant; so G_i embeds in $P^{2^{n-1}-1}$, and likewise L in P^{2^n-1} . A transposition τ of i and j in $[n]$ permutes $\mathcal{P}$ and induces an automorphism of $P^{\mathcal{P}}$ which maps $\delta_k(y)$ to $\delta_{\tau(k)}(y)$; it maps L onto L and K_i onto K_j , so $G_i \cong G_j$. If P is a finite p -group, so is $P^{\mathcal{P}}$, and with it L , H and the G_i . $\square$

Remark 5.2. (1) The construction is a variant of the subdirect products $\Gamma_k(G, N) \leq G^{\mathcal{P}}$ of [KMR18, Section 3], with only the generators attached to one-element subsets and with P in place of their normal subgroup N . (2) Since $g^{p^j} - 1 = (g - 1)^{p^j}$ in $\mathbb{F}_p P$, the condition $D_n(P; \mathbb{F}_p) = 1$ implies $P^{p^j} = 1$ for $p^j \geq n$, and $\gamma_n(P) = 1$ by (D1). It is not necessary for finite factors: for $P = C_{p^2}$ and $n = 2$, the group $P \times P$ and its diagonal subgroup form a configuration with quotient P , although $D_2(P; \mathbb{F}_p) \supseteq P^p \neq 1$. This example does not bear on the condition $D_n(P; \mathbb{Z}) = 1$, which holds here: $g - 1 \mapsto gP'$ defines an additive map $\Delta_{\mathbb{Z}}(P) \rightarrow P/P'$ that vanishes on $\Delta_{\mathbb{Z}}(P)^2$, so $D_2(P; \mathbb{Z}) \leq P' = 1$. (3) Without any hypothesis, the proof shows that L/Z , its subgroup HZ/Z and the G_i form a configuration with quotient $P/\text{ev}_{[n]}(Z)$, and $\text{ev}_{[n]}(Z) \leq D_n(P; \mathbb{Z})$.

6 The answer to Problem 13.19

6.1 Irregular groups of small class

Let $W_p = C_p \wr C_p$ be the semidirect product $B \rtimes \langle a \rangle$, where $B = \mathbb{F}_p[x]/(x^p - 1)$, written additively, and a has order p and acts by $aca^{-1} = xc$ for $c \in B$. Put $b_0 = 1 \in B$. Since $x^p - 1 = (x - 1)^p$ in $\mathbb{F}_p[x]$, the ideals $(x - 1)^k B$, $0 \leq k \leq p$, have index p^k in B ; they are normal subgroups of W_p .

Lemma 6.1. (a) $|W_p| = p^{p+1}$, and $\gamma_{k+1}(W_p) = (x - 1)^k B$ for $k \geq 1$. So W_p has class exactly p , and $\gamma_2(W_p) \leq B$ is elementary abelian.

(b) Every p -th power in W_p lies in $\gamma_p(W_p)$; $(b_0 a)^p = (x - 1)^{p-1} \neq 0$; and W_p has exponent p^2 .

(c) $W_p = \langle a, b_0 \rangle$, and W_p is not regular.

(d) $D_{p+1}(W_p; \mathbb{F}_p) = 1$.

(e) Every p -group of order at most p^p is regular, so W_p is an irregular p -group of the smallest possible order p^{p+1} . For $p = 2$, W_2 is the dihedral group D_8 .

Proof. (a) For $c \in B$ and $j \in \mathbb{Z}$ we have $a^{-j} c a^j = x^{-j} c$, so $[c, a^j] = (x^{-j} - 1)c$ in additive notation; note that $x^{-1} - 1$ is $x - 1$ times a unit. So $[B, \langle a \rangle] = (x - 1)B$; this is a normal subgroup, and modulo it the images of B and a commute, so $\gamma_2(W_p) = (x - 1)B$. If $M = (x - 1)^k B$ with $k \geq 1$, then $[M, B] = 1$ and $[M, W_p] = [M, \langle a \rangle] = (x - 1)M$; induction gives $\gamma_{k+1}(W_p) = (x - 1)^k B$, which is non-zero for $k = p - 1$ and zero for $k = p$.

(b) For $c \in B$ and j not divisible by p , induction on m gives $(ca^j)^m = (\sum_{k < m} x^{jk})c \cdot a^{jm}$, so $(ca^j)^p = (1 + x + \cdots + x^{p-1})c = (x - 1)^{p-1}c \in \gamma_p(W_p)$, because $(x^p - 1)/(x - 1) = (x - 1)^{p-1}$ in $\mathbb{F}_p[x]$; and $c^p = 0$. As $\gamma_p(W_p)$ has exponent p and $(x - 1)^{p-1} \neq 0$ in B , the exponent is p^2 .

(c) The conjugates $a^k b_0 a^{-k} = x^k$ span B , so $\langle a, b_0 \rangle = W_p$. If W_p were regular, then, since $a^p = b_0^p = 1$ and $\langle a, b_0 \rangle' = (x - 1)B$ has exponent p , we would get $(ab_0)^p = 1$. But $(ab_0)^p = a(b_0 a)^p a^{-1} \neq 1$ by (b).

(d) In Theorem 2.7 with $k = p + 1$, the factors with $j = 0$ have $i \geq p + 1$ and are trivial by (a); those with $j = 1$ have $i \geq 2$ and lie in $\gamma_2(W_p)^p = 1$; and those with $j \geq 2$ lie in $W_p^{p^2} = 1$ by (b).

(e) Let P be a p -group of class $c \geq 2$ with upper central series $1 = Z_0 < Z_1 < \cdots < Z_c = P$. If P/Z_{c-1} were cyclic, then P/Z_{c-2} , whose centre is Z_{c-1}/Z_{c-2} , would be abelian, and $Z_{c-1} = P$. So $|P/Z_{c-1}| \geq p^2$ and $|P| \geq p^{c+1}$. Hence a p -group of order at most p^p has class less than p , and it is regular by Lemma 2.2. Finally, W_2 has order 8 and is generated by the involutions a and b_0 , whose product has order 4 by (b). $\square$

For $n \geq 2$ let $R_n = \text{UT}_n(\mathbb{F}_p)$ be the group of upper unitriangular $n \times n$ matrices over $\mathbb{F}_p$, and let $t_k = 1 + E_{k,k+1}$, where the E_{ij} are the matrix units.

Lemma 6.2. (a) $D_n(R_n; \mathbb{Z}) = 1$.

(b) $[t_1, \dots, t_{n-1}] = 1 + E_{1n}$. So R_n has class exactly $n - 1$.

(c) Let $n \geq p + 1$. Then $W_p \times R_n$ is an irregular finite p -group of class exactly $n - 1$, generated by the $n + 1$ elements $a, b_0, t_1, \dots, t_{n-1}$, and $D_n(W_p \times R_n; \mathbb{Z}) = 1$. For $n = p + 1$ the same holds for W_p itself.

Proof. (a) The inclusion of R_n in the matrix ring $M_n(\mathbb{F}_p)$ extends to a ring homomorphism $\mathbb{Z}R_n \rightarrow M_n(\mathbb{F}_p)$, which maps every $g - 1$ to a strictly upper triangular matrix, and hence $\Delta_{\mathbb{Z}}(R_n)^n$ to 0. So $g - 1 \in \Delta_{\mathbb{Z}}(R_n)^n$ forces $g = 1$. (b) For $2 \leq k \leq n - 1$ a direct computation with $E_{ij}E_{jl} = E_{il}$

gives $[1 + E_{1k}, 1 + E_{k,k+1}] = 1 + E_{1,k+1}$, and the formula follows by induction. So $\gamma_{n-1}(R_n) \neq 1$, while $\gamma_n(R_n) = 1$ by (D1) and (a). (c) The group contains W_p , so it is irregular (Section 1), and its class is $\max(p, n-1) = n-1$. If (w, r) lies in $D_n(W_p \times R_n; \mathbb{Z})$, then by (D3), applied to the two projections, $w \in D_n(W_p; \mathbb{Z}) \leq D_n(W_p; \mathbb{F}_p) \leq D_{p+1}(W_p; \mathbb{F}_p) = 1$ by (D2), (D4) and Lemma 6.1(d), and $r \in D_n(R_n; \mathbb{Z}) = 1$. The last statement follows from Lemma 6.1(a), (c), (d) in the same way. $\square$

Proof of Theorem 1.3. Part (a) is Corollary 3.2. For (b) let $n \geq p+1$, and let $P = W_p$ if $n = p+1$ and $P = W_p \times R_n$ if $n \geq p+2$. By Lemma 6.2(c), P is an irregular finite p -group of class exactly $n-1$, generated by a finite set Y , with $D_n(P; \mathbb{Z}) = 1$. Theorem 5.1 gives a configuration of length n with quotient P whose factors are pairwise isomorphic finite p -groups, and Theorem 4.2 gives one with free factors of rank $(n-1)|Y|$. $\square$

6.2 Explicit examples

Proposition 6.3. *Let $V = \mathbb{F}_2^6$ with basis $u_1, v_1, u_2, v_2, u_3, v_3$, let $\alpha_i(x)$ and $\beta_i(x)$ denote the coordinates of $x \in V$ at u_i and v_i , and let $C = \mathbb{F}_2^3$ with basis f_1, f_2, f_3 . Let $\varepsilon_{ij} = \varepsilon_{ji} \in C$ be given by*

$$\varepsilon_{11} = \varepsilon_{12} = f_1, \quad \varepsilon_{22} = \varepsilon_{23} = f_2, \quad \varepsilon_{33} = \varepsilon_{13} = f_3,$$

and $\gamma(x, y) = \sum_{i,j} \alpha_i(x)\beta_j(y)\varepsilon_{ij}$. On $Q = V \times C$ put $(x, c)(y, d) = (x+y, c+d+\gamma(x, y))$. Let A_i be the normal closure of $\{U_i, V_i\}$, where $U_i = (u_i, 0)$ and $V_i = (v_i, 0)$, let $G_i = Q/A_i$, and let H be the kernel of the homomorphism $\psi: Q \rightarrow D_8$ defined below. Then Q is a group of order 512, $q \mapsto (qA_1, qA_2, qA_3)$ embeds Q into $G_1 \times G_2 \times G_3$, and $(G_1, G_2, G_3; Q, H)$ is a configuration of length 3 with $|G_i| = 32$, $|H| = 64$ and $Q/H \cong D_8$. In particular Q/H is not regular.

Proof. Since γ is bilinear, both $((x, c)(y, d))(z, e)$ and $(x, c)((y, d)(z, e))$ equal $(x+y+z, c+d+e+\gamma(x, y)+\gamma(x, z)+\gamma(y, z))$; the identity is $(0, 0)$, and $(x, c)^{-1} = (x, c+\gamma(x, x))$. Commutators are $[(x, c), (y, d)] = (0, \gamma(x, y)+\gamma(y, x))$; so $0 \times C$ is central, and $[U_i, V_j] = (0, \varepsilon_{ij})$.

Let C_i be the span of $\varepsilon_{i1}, \varepsilon_{i2}, \varepsilon_{i3}$, that is $C_1 = \langle f_1, f_3 \rangle$, $C_2 = \langle f_1, f_2 \rangle$, $C_3 = \langle f_2, f_3 \rangle$. The set $\langle u_i, v_i \rangle \times C_i$ is a subgroup (for $x, y \in \langle u_i, v_i \rangle$ we have $\gamma(x, y) \in \langle \varepsilon_{ii} \rangle$); it is normal, because by the commutator formula and the symmetry of (ε_{ij}) its commutators with elements of Q lie in $0 \times C_i$; and it is generated by U_i, V_i and the commutators $[U_i, V_j]$. So $A_i = \langle u_i, v_i \rangle \times C_i$, of order 16. The first components of A_1, A_2, A_3 intersect trivially, and $C_1 \cap C_2 \cap C_3 = 0$; hence $A_1 \cap A_2 \cap A_3 = 1$, and Q embeds in $G_1 \times G_2 \times G_3$ as a subdirect subgroup, with $|G_i| = 32$.

Realize D_8 as the set $\mathbb{F}_2^3$ with $(s, t, z)(s', t', z') = (s+s', t+t', z+z'+ts')$: the elements $a = (1, 0, 0)$ and $b = (0, 1, 0)$ are involutions, $(ab)^2 = (0, 0, 1) \neq 1$, and they generate the whole group of order 8. Let $\sigma(x) = \sum_i \alpha_i(x)$, $\tau(x) = \sum_i \beta_i(x)$, let $\lambda(c)$ be the sum of the coordinates of c , and put $\psi(x, c) = (\sigma(x), \tau(x), \lambda(c) + \sigma(x)\tau(x))$. As $\lambda(\varepsilon_{ij}) = 1$ for all i, j , we have $\lambda(\gamma(x, y)) = \sigma(x)\tau(y)$, and expanding $\sigma(x+y)\tau(x+y)$ shows that ψ is a homomorphism. It maps U_i to a and V_i to b , so $\psi(A_i) = D_8$ for every i ; hence $HA_i = Q$, that is, H projects onto every G_i . Finally $|H| = 64$ and $Q/H \cong D_8 = W_2$, which is not regular by Lemma 6.1(c) or Lemma 2.3. $\square$

The cyclic permutation $u_i \mapsto u_{i+1}$, $v_i \mapsto v_{i+1}$, $f_k \mapsto f_{k+1}$ (indices modulo 3) preserves (ε_{ij}) , so it is an automorphism of Q permuting A_1, A_2, A_3 , and the G_i are isomorphic. They are extraspecial: modulo C_1 , the form $\gamma(x, y) + \gamma(y, x)$ induces on $V/\langle u_1, v_1 \rangle \cong \mathbb{F}_2^4$ the alternating form $\alpha_2(x)\beta_2(y) + \alpha_2(x)\beta_3(y) + \alpha_3(x)\beta_2(y) + (x \leftrightarrow y)$ with values in $C/C_1 \cong \mathbb{F}_2$, which is non-degenerate; so $Z(G_1) = G'_1$ is the image of C , of order 2, and $G_1/Z(G_1)$ is elementary abelian.

Example 6.4. The program of this note (Section 8) checks the conclusion of Theorem 5.1 directly, by exact group orders, in two cases with $Y = \{a, b_0\}$. For $P = W_2 = D_8$ and $n = 3$: $|L| = 2^{12}$, $|H| = 2^9$, $|K_i| = 2^5$ and $|G_i| = 2^7$. For $P = W_3$ and $n = 4$: $|L| = 3^{32}$, $|H| = 3^{28}$, $|K_i| = 3^{13}$ and $|G_i| = 3^{19}$; this

is a configuration of length 4 of finite 3-groups whose quotient $C_3 \wr C_3$ is irregular of class $3 = n - 1$. As negative controls, $(P, n) = (D_8, 2)$ and $(W_3, 3)$, where the class of P exceeds $n - 1$, give $|Z| = 2$ and 3.

7 Remarks, earlier examples and open questions

Remark 7.1 (an irregular quotient in [KMR18]). We use the numbering of arXiv version 4 of [KMR18]; the journal version was not compared. For a group G , a normal subgroup N and $d \geq 1$, Kearnes, Mayr and Ruškuc define subgroups $\Gamma_1 \geq \dots \geq \Gamma_{d+1}$ of $G^{\mathcal{P}}$, where now $\mathcal{P}$ is the set of subsets of $[d]$; all contain the constant functions, and $S = \Gamma_1(G, N)$ is subdirect [KMR18, Section 3, Lemma 3.3]. If G is perfect, then $\gamma_k(S) = \Gamma_k$ for $k \leq d+1$ [KMR18, Lemma 4.1], and if moreover $\gamma_d(N)$ properly contains $[G, N, \dots, N]$ (d copies of N), then $S/\gamma_{d+1}(S)$ has class exactly d [KMR18, Theorem 4.2]. Take $d = 2$ and the perfect group $G \leq \mathrm{SL}_6(\mathbb{F}_4)$ with the normal subgroup N of [KMR18, Section 6], for which $[N, N] > [G, N, N]$ [KMR18, Lemma 6.1], and put $H = \gamma_3(S) = \Gamma_3$. As H contains the constant functions, $(G, G, G, G; S, H)$ is a configuration of length 4, and S/H has class exactly 2. Modulo the constant functions, S is generated by functions with values in N , a group of block unitriangular matrices over $\mathbb{F}_4$ whose exponent divides 4; so the finite nilpotent group S/H is a non-abelian 2-group of class 2, and not regular by Lemma 2.3(a). The paper does not mention p -groups, regularity or the Notebook. The family of [KMR18, Section 7] in $\mathrm{SL}_{2m}(\mathbb{F}_4)$, whose proof is given there in outline, yields in the same way 2-group quotients of class d with 2^d factors for $d < m$. We found no analogue for odd primes in print.

Remark 7.2 (Hrushovski's lemma and Evans's examples). The zbMATH review of [Eva93] quotes a lemma of Hrushovski: if a subgroup H of $\prod_{i \in I} G_i$ maps onto $G_i \times G_j$ for all $i \neq j$, and for every i there is a finite set $J(i) \ni i$ such that $f \in H$ and $f(j) = 1$ for all $j \in J(i) \setminus \{i\}$ imply $f(i) = 1$, then G_i has class at most $|J(i)| - 2$. This contains Gorchakov's setting: for fixed i let Q_i be the projection of H to the coordinates in $J' = J(i) \setminus \{i\}$; then $f(i)$ depends only on the projection of f , which defines a homomorphism ψ_i from Q_i onto G_i , and Q_i and $\ker \psi_i$ are subdirect (for $j \in J'$ and $g \in G_j$ there is $f \in H$ with $f(j) = g$ and $f(i) = 1$). So G_i is the quotient of a configuration of length $|J(i)| - 1$, and Theorem 3.1 gives the lemma. According to the review, Evans constructs finite non-abelian groups G_i in this situation (with an additional symmetry condition). By Lemma 2.4(b) their Sylow subgroups are also quotients of configurations, and a non-abelian Sylow 2-subgroup would give an irregular quotient by Lemma 2.3(b), that is, a negative answer for $p = 2$ in print since 1993. We have read only the review, which does not name the primes, and could not decide this.

Remark 7.3 (small factors). In a configuration of length 3 with non-abelian quotient every G_i is non-abelian. Indeed, with M_i as in Lemma 2.4, $(Q/H)'$ is the image of $[M_1, M_2] \leq M_1 \cap M_2$; the projection π_3 is injective on $M_1 \cap M_2$ and maps $[M_1, M_2]$ into G'_3 ; so if G_3 were abelian, Q/H would be abelian, and similarly for G_1 and G_2 . Exhaustive computer searches (Section 8) enumerated all subgroups of $G_1 \times G_2 \times G_3$ for every choice of the G_i among D_8 and Q_8 and found that every configuration with these factors has an abelian quotient. Since D_8 and Q_8 are the non-abelian groups of order 8, no configuration of length 3 with three factors of order 8 has an irregular quotient, whereas the factors in Proposition 6.3 have order 32. We do not know the smallest possible order of the factors for $p = 2$ and $n = 3$; factors of order 16, and factors of different orders, were not examined exhaustively. Two further questions are open: whether every finite p -group of class at most $n - 1$ is the quotient of a configuration of length n with finite factors (by Theorem 5.1, it is if $D_n(P; \mathbb{Z}) = 1$), and whether there are irregular quotients with perfect factors for odd primes p .

8 Computations and reproducibility

No proof in this note depends on a computation. The program `check_note.py` of the reproducibility package (Python standard library only; about ten seconds) checks: (A) Proposition 6.3 exhaustively (the cocycle identity on all 64^3 triples, ψ on all 512^2 pairs), that the G_i are extraspecial, and that Hall's identity fails in D_8 for 24 of the 64 ordered pairs; (B) Lemma 6.1 for $p = 2, 3, 5$, with the values of Jennings's formula and, for $p = 2, 3$, the dimension subgroups computed directly in $\mathbb{F}_p W_p$, independently of Theorem 2.7; (C) Example 6.4, by a deterministic Schreier–Sims algorithm: $Z = 1$ if and only if $|L|$ equals the order of the projection of L to the coordinates $C \neq [n]$, and $HK_i = L$ if and only if the projection to $[n]$ and the $C \not\ni i$ has order $|P| |G_i|$; (D) Lemma 4.1 mapped to $\mathbb{F}_p P$, on random words; (E) Lemma 6.2(b) for small n and p , and $D_n(R_n; \mathbb{F}_p) = 1$, which implies Lemma 6.2(a), for the groups R_n of order at most 125; (F) the searches of Remark 7.3: D_8^3 , $D_8^2 \times Q_8$, $D_8 \times Q_8^2$ and Q_8^3 have 55,338, 22,746, 13,354 and 9,434 subgroups and 16,594, 13,010, 16,402 and 28,306 configurations, all with abelian quotient. All checks pass.

Verification. The results were first obtained with written proofs and with programs for Proposition 6.3, the case $(D_8, 3)$ of Theorem 5.1, dimension subgroups and the search for D_8^3 . Two independent verification runs, both AI-assisted, followed; each re-derived all proofs and wrote its own programs, and neither read the other's report. The first compared the statement with the Notebook and with [Gor09] in Russian, read [KMR18] (arXiv version 4) and [Khu82], confirmed Remark 7.1, and recomputed Proposition 6.3, the configurations of Theorem 5.1 for $(D_8, 3)$, $(Q_8, 3)$, $(D_8, 4)$ and $(C_3 \wr C_3, 4)$ by Schreier–Sims orders, and $C_p \wr C_p$ for $p \leq 5$. The second built an example of order 512 of its own design, recomputed Proposition 6.3 and the same configurations with its own Schreier–Sims implementation, compared dimension subgroups with Jennings's formula, made the searches of Remark 7.3, tested the class bound on about 14,600 configurations, re-ran the original programs (identical outputs), repeated the literature search, and found the relation with Hrushovski's lemma (Remark 7.2). Neither run found a mathematical error. Their required corrections (references to sections instead of textbook theorem numbers quoted from memory, dates and identifiers of sources, the credit to [Gor78], [Khu82] and [Eva93], the description of Problem 7.8, and the wording of Remark 5.2(2), Lemma 6.1(e) and Remark 7.3) were all made. Lemmas 2.3 and 6.2 with the choice of P for $n \geq p + 2$, the derivation of Theorem 4.2 from Lemma 4.1, the argument for the extraspecial factors and the program of this note were added when the note was written, after these two runs.

A further AI-assisted verification run then examined the final text, these additions first, and the package; its programs and outputs are in the folder `independent_run_2/` of the reproducibility package. It re-derived every proof line by line, compared the statements with the Notebook, with [Gor09] in Russian, with the arXiv source of [KMR18], with the scan of [Khu82] and with the review of [Eva93], and wrote its own programs before re-running the program of this note from the archived package (identical output). Its programs checked Proposition 6.3 in a different encoding, with the factors built as quotient groups (isomorphic by the cyclic automorphism, extraspecial, the form of rank 4) and with Q/H built as a quotient and tested against Definition 1.2; Lemma 2.3 on all subgroups of $D_8 \times D_8$ and $D_8 \times Q_8$ and on eight further 2-groups; Lemma 6.1 for $p \leq 5$ in a permutation model; Lemma 6.2 by matrix computations, by the lower central series and by linear algebra in the group algebras, including $D_4(W_2 \times \text{UT}_4(\mathbb{F}_2); \mathbb{F}_2) = 1$; Lemma 4.1 in the free group itself, through the Magnus embedding, and exactly in $\mathbb{Z}D_8$, $\mathbb{Z}Q_8$ and $\mathbb{Z}C_4$; Theorem 5.1 by exact orders with a second Schreier–Sims implementation, for the cases of Example 6.4 and for $P = W_2 \times \text{UT}_4(\mathbb{F}_2)$, $n = 4$ (a configuration of length 4 of finite 2-groups with $|Q| = 2^{64}$, $|G_i| = 2^{40}$ and an irregular quotient of class exactly 3), with negative controls; the searches of Remark 7.3 with its own enumeration (the same numbers); and Theorem 3.1 on 360 random configurations. It repeated the literature search. It found no mathematical error; it required the wording of item (E) above, of this paragraph and of the

search paragraph below to be made precise, which was done.

Scope and priority. The note answers Problem 1.1 completely. The question and the 2009 conjecture are Gorchakov’s [KM26, Gor09]; the class bound is Gorchakov’s [Gor65, Lemma 1] (see also [Gor78, Proposition 4.7], as cited in [Khu82]), with the same argument in [Khu82] and [KMR18, Theorem 2.1]; the regularity of p -groups of class less than p is P. Hall’s theorem [Hal34]; the dimension subgroups of free groups and of finite p -groups are due to Magnus [Mag37] (see also [Wit37]) and to Jennings [Jen41]. For $p = 2$ a negative answer is implicit in [KMR18] (Remark 7.1) and possibly in [Eva93] (Remark 7.2). The contribution of this note is the threshold $n \leq p$ for every prime, in particular for the odd primes, Theorems 4.2 and 5.1, and the explicit examples; no priority is claimed.

What was read: [Gor09] in the Russian original (EMIS archive) and [KMR18] in arXiv version 4, completely; [Khu82] in the scan of Math-Net.Ru, whose text layer is partly garbled, the key passages being legible; of [Eva93] only the zbMATH review. [Gor65] and [Gor78] were not available to us; what we say about them is taken from [Gor09], [KMR18] and [Khu82]. The works [Hal34, Mag37, Wit37, Jen41, Hup67] were not consulted; they are cited for classical statements, at the level of sections.

The searches were made on 17 and 18 October 2026, the last ones by the final verification run: queries to the arXiv, zbMATH Open (also for works citing [Gor09] and [KMR18]), Crossref, Math-Net.Ru, EuDML, EMIS, OpenAlex and Semantic Scholar, and seven web searches. Apart from the implicit examples of Remarks 7.1 and 7.2, no printed answer to Problem 1.1 was found in these sources; a search that finds nothing is not a proof of novelty. zbMATH, Crossref and Semantic Scholar listed no work citing [Gor09] or [KMR18]; OpenAlex counted one work citing [KMR18], which could not be identified because OpenAlex refused our requests (rate limiting). A preprint of July 2026 with solutions to eight problems of the Notebook [vDJMM26] does not treat Problem 1.1; of it only the abstract was read.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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