

Partial Answers to Kourovka Problems 11.18 and 17.80: Infinite Groups $G(a, b)$ and Their Two-Dimensional Images

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Abstract

R. Brandl asked in the Kourovka Notebook (Problem 11.18, 1990) whether the group $G(a, b) = \langle x, y \mid x = [x, {}_a y], y = [y, {}_b x] \rangle$ is finite; he noted that $G(1, b) = 1$ and $G(2, 2) = 1$ and that nothing is known about $G(2, 3)$. O. Macedońska asked (Problem 17.80, 2010) whether $M_n = G(n, n)$ is infinite for every $n > 2$ (UnsolvedMath records KOU-11.18 and KOU-17.80). We give partial answers: $G(a, b)$ is infinite whenever m divides both a and b for some $m \in \{18, 20, 26, 28, 30, 42, 44\}$, so M_n is infinite whenever one of these numbers divides n (for $m = 18$ this is in print); every homomorphism of $G(a, b)$ into $\mathrm{SL}(2, K)$ or $\mathrm{PSL}(2, K)$ factors through $G(d, d)$ with $d = \gcd(a, b)$; the case $G(2, 3)$ and the general questions stay open. That M_{18} is infinite follows from a result of Quek, Wong and Wong (2014), proved there by a computation credited to the referee, as A. Abdollahi's zbMATH review points out; we know that paper only from the review, and we prove the result again. For each m we give an explicit epimorphism of $G(m, m)$ onto $\mathrm{PSL}(2, q)$ with $q \in \{8, 13, 29, 41, 43\}$ and integer vectors that define homomorphisms onto $\mathbb{Z}$ of its kernel or of the preimage of a split torus; these subgroups have abelianizations of free rank exactly 7, 2, 14, 4, 4, 22, 12 for $m = 18, 20, 26, 28, 30, 42, 44$. The certificates are machine-readable and are checked by independent programs; those for $m = 20, 28, 30, 42, 44$ are new in this note. Infiniteness passes to $G(a, b)$ because $G(a, b)$ is a quotient of $G(a', b')$ whenever $a \mid a'$ and $b \mid b'$. The second result holds over every field: if $X, Y \in \mathrm{SL}(2, K)$ satisfy $X = [X, {}_a Y]$ and $Y = [Y, {}_b X]$, then $\mathrm{tr} X = \mathrm{tr} Y$, a conjugation in $\mathrm{GL}(2, K)$ swaps X and Y , and the relations hold with $\gcd(a, b)$ in place of a and b . Hence $G(2, 3)$, and every $G(a, b)$ with coprime a, b , has no nontrivial homomorphism into $\mathrm{GL}(2, K)$ or $\mathrm{PGL}(2, K)$ for any field K . Tested evidence: $G(2, 3)$ has no nontrivial homomorphism into A_n , $n \leq 11$. For the cases other than $m = 18$, and for the second result, no source was found in zbMATH, OpenAlex, arXiv and Crossref; two related papers of Quek, Wong and Wong were not accessible to us; we claim no priority. This is an unrefereed note.

1 Introduction

1.1 The problems

Throughout, $[u, v] = u^{-1}v^{-1}uv$, as in the Kourovka Notebook [KM26, Problem 8.11], and Engel commutators are left-normed: $[u, {}_0 v] = u$ and $[u, {}_k v] = [[u, {}_{k-1} v], v]$ for $k \geq 1$, so that $[u, {}_k v] = [u, v, \dots, v]$ with k copies of v , as in [KM26, Problem 17.80]. For integers $a, b \geq 1$ let

$$G(a, b) = \langle x, y \mid x = [x, {}_a y], y = [y, {}_b x] \rangle, \quad M_n = G(n, n). \quad (1)$$

Problem 11.18 of the Notebook, posed by R. Brandl in 1990, asks: “Is $G(a, b)$ finite?” The proposer adds that $G(1, b) = 1$ is easy to show, that one can show $G(2, 2) = 1$, and that “nothing is known about

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$G(2, 3)$ ". His motivation comes from [BW88]: if every minimal simple group were a quotient of some $G(a, b)$, this would give a sequence of words in two variables characterizing the finite soluble groups. Such sequences, made of other words, were later given by Bray, Wilson and Wilson [BWW05] and by Bandman, Greuel, Grunewald, Kunyavskiĭ, Pfister and Plotkin [BGG⁺06]; Bandman, Grunewald and Kunyavskiĭ [BGK10] treat them through periodic points of word maps, studied with the trace map on $\mathrm{SL}(2, q)$. We know [BWW05, BGG⁺06] only from [BGK10] and from their Crossref metadata. Problem 17.80, posed by O. Macedońska in 2010, asks whether M_n is infinite for every $n > 2$, and refers to 11.18. Both problems are printed without any comment in the current version of the Notebook [KM26] (version 49 of 20 October 2026), as in version 48 of 6 October 2026. They are the records KOU-11.18 and KOU-17.80 of the UnsolvedMath collection [ula26], where both are listed as open.

Since a and b are free, Problem 11.18 contains three questions: whether $G(a, b)$ is finite for all $a, b \geq 1$; for which pairs it is finite; and what $G(2, 3)$ is. One infinite $G(a, b)$ answers the first question in the negative and does not touch the other two. Brandl's motivation concerns the finite quotients of the groups $G(a, b)$, and is not settled by an infinite example either. Problem 17.80 asks for all $n \geq 3$; note that $M_1 = M_2 = 1$ by the remarks in 11.18. All statements of this note refer to the convention above; with $[u, v] = uvu^{-1}v^{-1}$ the presentations (1) may define other groups, and our examples were not checked for them.

1.2 Results

Theorem A. *Let $m \in \{18, 20, 26, 28, 30, 42, 44\}$, and let q, X and Y be as in the row of Table 1 for m . Then $X = [X, {}_m Y]$ and $Y = [Y, {}_m X]$ in $\mathrm{SL}(2, q)$, and $x \mapsto \pm X, y \mapsto \pm Y$ defines an epimorphism $\varphi_m: G(m, m) \rightarrow \mathrm{PSL}(2, q)$. Let $S = 1$ for $m \in \{18, 26\}$, and let $S = T$ be the split torus, the image of the diagonal matrices, for the other m . Then the subgroup $H = \varphi_m^{-1}(S)$, of index $|\mathrm{PSL}(2, q) : S|$, has abelianization $H/[H, H] \cong \mathbb{Z}^r \oplus A$ with A finite and r as in Table 1. In particular, $G(m, m)$ is infinite.*

m	q	X	Y	H	$ G(m, m) : H $	r
18	8	$\begin{pmatrix} 0 & 1 \\ 1 & t+1 \end{pmatrix}$	$\begin{pmatrix} 1 & t^2+t \\ t^2+t & t \end{pmatrix}$	$\ker \varphi_{18}$	504	7
20	41	$\begin{pmatrix} 0 & 1 \\ 40 & 38 \end{pmatrix}$	$\begin{pmatrix} 0 & 5 \\ 8 & 38 \end{pmatrix}$	$\varphi_{20}^{-1}(T)$	1722	2
26	13	$\begin{pmatrix} 0 & 1 \\ 12 & 11 \end{pmatrix}$	$\begin{pmatrix} 0 & 12 \\ 1 & 11 \end{pmatrix}$	$\ker \varphi_{26}$	1092	14
28	29	$\begin{pmatrix} 0 & 1 \\ 28 & 19 \end{pmatrix}$	$\begin{pmatrix} 0 & 8 \\ 18 & 19 \end{pmatrix}$	$\varphi_{28}^{-1}(T)$	870	4
30	29	$\begin{pmatrix} 0 & 1 \\ 28 & 4 \end{pmatrix}$	$\begin{pmatrix} 2 & 4 \\ 8 & 2 \end{pmatrix}$	$\varphi_{30}^{-1}(T)$	870	4
42	43	$\begin{pmatrix} 0 & 1 \\ 42 & 20 \end{pmatrix}$	$\begin{pmatrix} 0 & 30 \\ 10 & 20 \end{pmatrix}$	$\varphi_{42}^{-1}(T)$	1892	22
44	43	$\begin{pmatrix} 0 & 1 \\ 42 & 16 \end{pmatrix}$	$\begin{pmatrix} 1 & 10 \\ 10 & 15 \end{pmatrix}$	$\varphi_{44}^{-1}(T)$	1892	12

Table 1: The certificates of Theorem A. For $q = 8$, $\mathbb{F}_8 = \mathbb{F}_2[t]/(t^3 + t^2 + 1)$; for prime q the entries are residues modulo q . The index is $|\mathrm{PSL}(2, q)|$ for the kernels and $q(q+1)$ for the preimages of the split torus; r is the free rank of $H/[H, H]$. For $m = 30$ a second pair, $X = \begin{pmatrix} 0 & 1 \\ 28 & 8 \end{pmatrix}$, $Y = \begin{pmatrix} 1 & 8 \\ 8 & 7 \end{pmatrix}$, gives $r = 10$.

The certificates are machine-readable files with r integer vectors on the Schreier generators of H (Sections 3 and 7). If $a \mid a'$ and $b \mid b'$, then $G(a, b)$ is a quotient of $G(a', b')$ (Lemma 2.2), so the infiniteness passes to multiples.

Corollary 1.1. *If m divides both a and b for some $m \in \{18, 20, 26, 28, 30, 42, 44\}$, then $G(a, b)$ is infinite. In particular M_n is infinite whenever n is divisible by one of 18, 20, 26, 28, 30, 42, 44.*

Theorem B. *Let K be a field, let $a, b \geq 1$, and let $X, Y \in \mathrm{SL}(2, K)$ satisfy $X = [X, {}_a Y]$ and $Y = [Y, {}_b X]$. Put $s = \mathrm{tr} X$ and $d = \gcd(a, b)$. Then*

- (i) $\mathrm{tr} Y = s$ and $\mathrm{tr}(XY) = s(s - 1)$;
- (ii) there is $g \in \mathrm{GL}(2, K)$ with $gXg^{-1} = Y$ and $gYg^{-1} = X$;
- (iii) $X = [X, {}_d Y]$ and $Y = [Y, {}_d X]$;
- (iv) if $d = 1$, then $X = Y = I$.

Corollary 1.2. *Let K be a field and let L be one of $\mathrm{SL}(2, K)$, $\mathrm{PSL}(2, K)$, $\mathrm{GL}(2, K)$, $\mathrm{PGL}(2, K)$. Every homomorphism $G(a, b) \rightarrow L$ factors through the natural epimorphism $G(a, b) \rightarrow G(d, d)$, $d = \gcd(a, b)$. If a and b are coprime, for instance if $(a, b) = (2, 3)$, every homomorphism $G(a, b) \rightarrow L$ is trivial; in particular $G(a, b)$ has no quotient isomorphic to $\mathrm{PSL}(2, q)$ for any prime power q , and no nontrivial linear or projective representation of degree 2 over any field.*

In short, the note gives partial answers: $G(a, b)$ is infinite whenever m divides both a and b for some $m \in \{18, 20, 26, 28, 30, 42, 44\}$, so M_n is infinite whenever one of these numbers divides n (for $m = 18$ this is in print); every homomorphism of $G(a, b)$ into $\mathrm{SL}(2, K)$ or $\mathrm{PSL}(2, K)$ factors through $G(d, d)$ with $d = \gcd(a, b)$; the case $G(2, 3)$ and the general questions stay open. Section 5 adds tested, not proved, evidence: $G(2, 3)$ has no nontrivial homomorphism into the alternating groups A_n with $n \leq 11$, into M_{11} , M_{12} , M_{22} , $\mathrm{PSL}(3, 3)$, $\mathrm{PSL}(3, 4)$ and $\mathrm{PSL}(3, 5)$, and coset enumeration of $G(2, 3)$ was inconclusive.

1.3 What is known, what is new, and what stays open

Quek, Wong and Wong studied quotients of M_n obtained by adding relations; we know their papers from zbMATH. The quotient by $xyx = yxy$ is trivial unless $5 \mid n$, and isomorphic to $\mathrm{SL}(2, 5)$ if $5 \mid n$ [QWW13], so M_5 maps onto $\mathrm{SL}(2, 5)$; in his review [Bra] Brandl relates this to his conjecture $G(5, 5) \cong \mathrm{SL}(2, 5)$. In [QWW14] they studied the quotient G_n of M_n by the relations $xyx^{-2}yx = yxy$ and $xyx^{-2}xy = yxy$. According to A. Abdollahi's review [Abd], they proved $G_n = 1$ for odd n , and the paper contains a GAP proof, credited to the referee, that G_{18} is infinite; the reviewer notes that the authors were not aware of Problems 11.18 and 17.80, and that this shows that $G(18, 18) = M_{18}$ is infinite. This answers the first question of 11.18 in the negative and, with Lemma 2.2, gives the case $18 \mid n$ of 17.80. We have not seen [QWW14]: the publisher's site refused automated access, and this was not worked around; what we say about it, in particular about the referee's proof, is taken from the review and from the abstract in the Crossref metadata. So Theorem A for $m = 18$ is not new; our proof is independent of [QWW14], as the certificate is checked against the relators of $G(18, 18)$ themselves. According to its zbMATH summary, [QWW16] shows that $\mathrm{SL}(2, F)$ contains a pair with $h = [h, {}_n g]$, $g = [g, {}_n h]$, $h \neq 1$, for some n , after an extension of F of degree at most 6, and such pairs with $n = 5$ in characteristic $p \equiv \pm 1 \pmod{5}$. We could not read it either; it may contain trace computations close to those in the proof of Theorem B. With the trace map, Bandman, Grunewald and Kunyavskii [BGK10, Section 5.4] show that for every q there are $x, y \in \mathrm{SL}(2, q)$ with $x \neq \pm I$ and $x = [x, {}_n y]$ for some $n \geq 1$; this is one of the two relations of M_n only, and [BGK10] does not mention the groups $G(a, b)$.

Apart from [QWW14] and its review, no printed answer to either question, and no printed statement of Theorem A for $m \neq 18$, of Theorem B or of Corollary 1.2, was found in the sources searched (zbMATH Open, OpenAlex, arXiv and Crossref; see "Scope and priority" in Section 7). A search that finds nothing is not a proof of novelty, and we claim no priority. The certificates for $m = 20, 28, 30, 42, 44$ are new in this note. The note answers neither question in full: $G(2, 3)$, the

question for which (a, b) the group $G(a, b)$ is finite, and Problem 17.80 for the n not divisible by one of 18, 20, 26, 28, 30, 42, 44 remain open; for instance M_3 , M_4 , M_6 , M_{10} , M_{12} , M_{14} and M_{24} are not decided (Sections 5 and 6).

2 Preliminaries

Lemma 2.1. *Let u, v be elements of a group and $u_k = [u, {}_k v]$. Then $[u, {}_{j+k} v] = [u_j, {}_k v]$ for all $j, k \geq 0$. If $u_a = u$ for some $a \geq 1$, let $p \geq 1$ be least with $u_p = u$, the least period of u under v . Then $u_{k+p} = u_k$ for all $k \geq 0$, and for $n \geq 1$ we have $u_n = u$ if and only if $p \mid n$.*

Proof. The identity holds for $k = 0$, and $[u, {}_{j+k+1} v] = [[u, {}_{j+k} v], v] = [[u_j, {}_k v], v] = [u_j, {}_{k+1} v]$. With $j = p$ it gives $u_{p+k} = [u_p, {}_k v] = [u, {}_k v] = u_k$. Hence $u_{\ell p+r} = u_r$ for $\ell, r \geq 0$. If $u_n = u$ and $n = \ell p + r$ with $0 \leq r < p$, then $u_r = u$, so $r = 0$ by the minimality of p ; and $u_{\ell p} = u_0 = u$. $\square$

Lemma 2.2 (monotonicity). *If $a \mid a'$ and $b \mid b'$, then $x \mapsto x$, $y \mapsto y$ induces an epimorphism $G(a', b') \rightarrow G(a, b)$. In particular $G(d, d)$ is a quotient of $G(a, b)$ for $d = \gcd(a, b)$, M_n is a quotient of M_{kn} for $k \geq 1$, and if $G(a, b)$ is infinite, so is $G(a', b')$.*

Proof. In $G(a, b)$ we have $x = [x, {}_a y]$, so $x = [x, {}_{a'} y]$ by Lemma 2.1, and likewise $y = [y, {}_b x]$. Thus the defining relations of $G(a', b')$ hold in $G(a, b)$, and the map is a homomorphism by von Dyck's theorem; it is onto since x and y generate $G(a, b)$. $\square$

Lemma 2.3. (a) $G(a, b)$ is perfect, and so is each of its quotients.

(b) If u, v are elements of a group with $u = [u, v]$ and $v = [v, {}_b u]$ for some $b \geq 1$, then $u = v = 1$. In particular $G(1, b) = G(b, 1) = 1$.

(c) If $u = [u, {}_a v]$ and $v = [v, {}_b u]$ with $a, b \geq 1$ and $(u, v) \neq (1, 1)$, then the least periods of u under v and of v under u are at least 2.

Proof. (a) $x = [[x, {}_{a-1} y], y]$ and $y = [[y, {}_{b-1} x], x]$ are commutators. (b) From $u = [u, v] = u^{-1}u^v$ we get $[v, u] = [u, v]^{-1} = u^{-1}$, hence $[v, {}_2 u] = [u^{-1}, u] = 1$ and $[v, {}_k u] = 1$ for $k \geq 2$. If $b \geq 2$, then $v = [v, {}_b u] = 1$ and $u = [u, 1] = 1$; if $b = 1$, then $v = [v, u] = u^{-1}$ and $u = [u, u^{-1}] = 1$. Applied to the generators, this gives $G(1, b) = 1$, and $G(b, 1) \cong G(1, b)$ by exchanging x and y . (c) follows from (b), applied to (u, v) and to (v, u) . $\square$

The certificates rest on the Reidemeister–Schreier method [LS01, Chapter II, §4], which we use in the following form. Let $G = \langle x, y \mid r_1, r_2 \rangle$ act transitively on a finite set Ω from the right, let $\omega_0 \in \Omega$, and let H be the stabilizer of ω_0 . The Schreier graph has vertex set Ω and, for every $\omega \in \Omega$, an edge (ω, x) from ω to ω^x and an edge (ω, y) from ω to ω^y . Fix a spanning tree of the underlying undirected graph and let E' be the set of the $|\Omega| + 1$ edges not in it. For a word w and $\omega \in \Omega$ let $c(w, \omega) \in \mathbb{Z}^{E'}$ count, with signs, how often the path of w from ω runs through each edge of E' (a letter x^{-1} at ω runs backwards through $(\omega^{x^{-1}}, x)$), and let R be the integer matrix whose $2|\Omega|$ rows are the vectors $c(r_j, \omega)$.

Lemma 2.4. (a) $H/[H, H] \cong \mathbb{Z}^{E'}/\mathbb{Z}^{2|\Omega|}R$. Hence the free rank of $H/[H, H]$ is $|E'| - \text{rank}_{\mathbb{Q}} R$, which is at most $|E'| - \text{rank}_{\mathbb{F}_p}(R \bmod p)$ for every prime p .

(b) If $v_1, \dots, v_k \in \mathbb{Z}^{E'}$ are linearly independent and $Rv_i = 0$ for all i , then the free rank of $H/[H, H]$ is at least k . If $v \in \mathbb{Z}^{E'}$, $v \neq 0$ and $Rv = 0$, there is a homomorphism of H onto $\delta\mathbb{Z}$, where δ is the greatest common divisor of the entries of v ; in particular H and G are infinite.

Proof. (a) By the Reidemeister–Schreier theorem [LS01, Chapter II, §4], H is generated by the Schreier generators s_e , $e \in E'$, with defining relators the rewritten words $t_\omega r_j t_\omega^{-1}$, where t_ω is the word along the tree path from ω_0 to ω ; the tree edges contribute trivial generators. The rewritten relator is the product of the $s_e^{\pm 1}$ along the path of r_j from ω , so its image in the free abelian group on the s_e is $c(r_j, \omega)$. The inequality holds because $\text{rank}_{\mathbb{F}_p}(R \bmod p) \leq \text{rank}_{\mathbb{Q}} R$. (b) By (a), each v_i defines a homomorphism $H/[H, H] \rightarrow \mathbb{Z}$, $s_e \mapsto (v_i)_e$; independent vectors give independent homomorphisms, so the free rank is at least k . The image of the homomorphism defined by v is generated by the entries of v ; since $|G : H| = |\Omega|$ is finite and H is infinite, so is G . $\square$

The vectors $c(w, \omega)$ obey $c(w\omega', \omega) = c(w, \omega) + c(\omega', \omega^\omega)$ and $c(\omega^{-1}, \omega) = -c(\omega, \omega^{\omega^{-1}})$. These composition rules give the rows of R for the relators $x^{-1}[x, m y]$ and $y^{-1}[y, m x]$ in $O(m|\Omega|)$ vector operations, although these words have $3 \cdot 2^m - 1$ letters as written: 786431 for $m = 18$ (and 524323 after free reduction, the reduced word being cyclically reduced), about $5 \cdot 10^{13}$ for $m = 44$.

3 Infinite cases: proof of Theorem A

Matrices act on row vectors from the right, so that $\text{PSL}(2, q)$ acts faithfully on the projective line $\mathbb{P}^1(\mathbb{F}_q)$; we write $0 = [1 : 0]$ and $\infty = [0 : 1]$, and $|\text{PSL}(2, q)| = q(q^2 - 1)/\gcd(2, q - 1)$. Fix m and the corresponding row of Table 1. The proof consists of five finite computations, each of which is exact and is carried out by the programs described in Section 7.

Step 1: the Engel relations. Computing $X_k = [X_{k-1}, Y]$ and $Y_k = [Y_{k-1}, X]$ in $\text{SL}(2, q)$, starting from $X_0 = X$ and $Y_0 = Y$, gives $X_m = X$ and $Y_m = Y$; moreover $X_k \neq \pm X$ and $Y_k \neq \pm Y$ for $0 < k < m$, so m is the least period of both sequences, in $\text{SL}(2, q)$ and in $\text{PSL}(2, q)$. Hence $x \mapsto X$, $y \mapsto Y$ defines a homomorphism $G(m, m) \rightarrow \text{SL}(2, q)$, and φ_m is its composite with the projection to $\text{PSL}(2, q)$. In every row $\text{tr } Y = \text{tr } X = s$ and $\text{tr}(XY) = s(s - 1)$, as Theorem B predicts.

Step 2: surjectivity. The subgroup of $\text{PSL}(2, q)$ generated by the images of X and Y , enumerated element by element, has $|\text{PSL}(2, q)|$ elements, namely 504, 1092, 12180, 34440 and 39732 for $q = 8, 13, 29, 41, 43$. So φ_m is onto.

Step 3: the coset space. For $S = 1$ let $\Omega = \text{PSL}(2, q)$, on which $G(m, m)$ acts by right multiplication through φ_m ; the stabilizer of the identity is $\ker \varphi_m$. (The programs realize $\text{PSL}(2, q)$ as the permutation group generated by the actions of X and Y on $\mathbb{P}^1(\mathbb{F}_q)$.) For $S = T$ let Ω be the orbit of the ordered pair $(0, \infty)$. Since $\text{PSL}(2, q)$ is 2-transitive on the $q + 1$ points, Ω consists of all $q(q + 1)$ ordered pairs of distinct points; a matrix fixes $[1 : 0]$ and $[0 : 1]$ if and only if it is diagonal, so the stabilizer of $(0, \infty)$ in $\text{PSL}(2, q)$ is T , and its stabilizer in $G(m, m)$ is $H = \varphi_m^{-1}(T)$. In both cases $|G(m, m) : H| = |\Omega|$.

Step 4: the vectors. The points of Ω are numbered in the order in which a breadth-first search from the base point, applying x and then y , meets them; the spanning tree of the Schreier graph is the breadth-first tree from the base point with the neighbours taken in the order x, y, x^{-1}, y^{-1} ; the edge (ω, x) has index 2ω and (ω, y) index $2\omega + 1$, and E' is ordered by index. The certificate gives r vectors $v_1, \dots, v_r \in \mathbb{Z}^{E'}$ with $Rv_i = 0$, computed with the composition rules of Section 2 for the relators $x^{-1}[x, m y]$ and $y^{-1}[y, m x]$ from every point. The vectors are linearly independent (their rank modulo $2^{31} - 1$ is r), and the entries of each have greatest common divisor 1. By Lemma 2.4(b), H maps onto $\mathbb{Z}$, and the free rank of $H/[H, H]$ is at least r .

Step 5: the upper bound. The rank of R modulo $p = 2^{31} - 1$ is $|E'| - r$, namely 498, 1721, 1079, 867, 867, 1871 and 1881 for $m = 18, 20, 26, 28, 30, 42, 44$, with $|E'| = |\Omega| + 1$. By Lemma 2.4(a), the free rank of $H/[H, H]$ is at most r . This completes the proof of Theorem A; Corollary 1.1 follows by Lemma 2.2. $\square$

The certificates. For $m = 18, 26$ the files `cert_G18_18.json` and `cert_G26_26.json` give the permutations of $\mathbb{P}^1(\mathbb{F}_q)$ induced by X and Y and one vector v each; `own_cert_G18.json` and

own_cert_G26.json give bases of 7 and 14 vectors, whose span contains v . For the other m the files newcert_Gm_qq_ss.json ($s = \text{tr } X$) give q , X , Y , the numbered points of Ω and the r vectors. Each file is checked by two independent programs, which recompute everything from X and Y (Section 7). For $m = 18$ only the relators of $G(18, 18)$ are used, not those of the quotient G_{18} of [QWW14]. The pairs were found among the Y with $\text{tr } Y = s$ and $\text{tr}(XY) = s(s-1)$ (Theorem B(i)); the preimage of T , of index $q(q+1)$ instead of $|\text{PSL}(2, q)|$, made $q = 29, 41, 43$ accessible.

4 Two-dimensional images: proof of Theorem B

Let R be a commutative ring. For $B \in \text{SL}(2, R)$ the Cayley–Hamilton theorem gives $B^2 - \text{tr}(B)B + I = 0$, hence, for all $M \in M_2(R)$,

$$B + B^{-1} = \text{tr}(B)I, \quad \text{tr}(B^{-1}) = \text{tr}(B), \quad \text{tr}(MB) + \text{tr}(MB^{-1}) = \text{tr}(M)\text{tr}(B). \quad (2)$$

Lemma 4.1. *Let (A, B) and (A', B') be pairs in $\text{SL}(2, R)$ with $\text{tr } A = \text{tr } A'$, $\text{tr } B = \text{tr } B'$ and $\text{tr}(AB) = \text{tr}(A'B')$. Then $\text{tr } w(A, B) = \text{tr } w(A', B')$ for every element w of the free group on x and y .*

This is the classical theorem of Fricke and Vogt that $\text{tr } w(A, B)$ is a polynomial with integer coefficients in $\text{tr } A$, $\text{tr } B$ and $\text{tr}(AB)$; see [Hor72, Mag80]. We include the short proof.

Proof. First let w be a positive word, that is, a word in x and y without inverses; we use induction on its length ℓ . For $\ell \leq 1$ the claim is the hypothesis or $\text{tr } I = 2$. Let $\ell \geq 2$. Traces do not change under cyclic permutation of the factors, simultaneously for both pairs. If two cyclically adjacent letters of w are equal, we may assume that $w = uzz$ with $z \in \{x, y\}$; then (2) with $M = u(A, B)z(A, B)$ gives $\text{tr } w(A, B) = \text{tr}(M)\text{tr } z(A, B) - \text{tr } u(A, B)$, the same expression holds for (A', B') , and uz and u are shorter. Otherwise w is a cyclic permutation of $(xy)^k$ with $2k = \ell$; for $k = 1$ the claim is the hypothesis, and for $k \geq 2$, (2) applied to $AB \in \text{SL}(2, R)$ gives $\text{tr}((AB)^k) = \text{tr}(AB)\text{tr}((AB)^{k-1}) - \text{tr}((AB)^{k-2})$, and the same for (A', B') . For an arbitrary w , replace every factor A^{-1} by $\text{tr}(A)I - A$ and every B^{-1} by $\text{tr}(B)I - B$ in the product $w(A, B)$, by (2), and expand: $w(A, B) = \sum_u c_u(\text{tr } A, \text{tr } B)u(A, B)$, a sum over positive words u with integer polynomials c_u that depend on w only. The same expansion holds for (A', B') with the same values of the c_u , and taking traces proves the claim. $\square$

Proof of Theorem B. Put $X_0 = X$, $X_{k+1} = [X_k, Y]$, $Y_0 = Y$, $Y_{k+1} = [Y_k, X]$, so that $X_a = X$ and $Y_b = Y$, and let $t = \text{tr } Y$.

(i) For $k \geq 0$, $X_{k+1}Y^{-1} = X_k^{-1}Y^{-1}X_k$ is conjugate to Y^{-1} , so $\text{tr}(X_{k+1}Y^{-1}) = \text{tr}(Y^{-1}) = t$, and (2) gives $\text{tr}(X_{k+1}Y) = t\text{tr}(X_{k+1}) - t$. For $k = a - 1$ this reads $\text{tr}(XY) = t(s - 1)$. In the same way $\text{tr}(YX) = s(t - 1)$. Since $\text{tr}(XY) = \text{tr}(YX)$, we get $ts - t = st - s$, so $t = s$ and $\text{tr}(XY) = s(s - 1)$. No division was used, so this holds in every characteristic.

(ii) By (i), $(\text{tr } X, \text{tr } Y, \text{tr } XY) = (s, s, s(s - 1)) = (\text{tr } Y, \text{tr } X, \text{tr } YX)$, so Lemma 4.1 gives $\text{tr } w(X, Y) = \text{tr } w(Y, X)$ for every word w . The group $\Gamma = \langle X, Y \rangle$ is perfect, since $X = [X_{a-1}, Y]$ and $Y = [Y_{b-1}, X]$ with $X_{a-1}, Y_{b-1} \in \Gamma$. If X and Y have a common eigenvector in $\overline{K}^2$, where $\overline{K}$ is an algebraic closure of K , then Γ is conjugate in $\text{GL}(2, \overline{K})$ to a group of upper triangular matrices, which is soluble; a perfect soluble group is trivial, so $X = Y = I$ and $g = I$ will do. Otherwise Γ acts irreducibly on $\overline{K}^2$, and by Burnside's theorem [Lan02, Chapter XVII, §3] the $\overline{K}$ -linear span of Γ is $M_2(\overline{K})$. Then the K -linear span of Γ contains four elements that are linearly independent over $\overline{K}$, hence over K , so it is $M_2(K)$, and we may choose words $w_1, \dots, w_4$ such that the matrices $P_i = w_i(X, Y)$ form a basis of $M_2(K)$. Put $P'_i = w_i(Y, X)$. The trace form $(U, V) \mapsto \text{tr}(UV)$ on $M_2(K)$ is nondegenerate in every characteristic (on the matrix units, $\text{tr}(E_{ij}E_{kl}) = \delta_{jk}\delta_{il}$), so the Gram matrix $\Delta = (\text{tr } P_i P_j)_{i,j}$ is invertible; and $\text{tr}(P'_i P'_j) = \text{tr}(w_i w_j)(Y, X) = \text{tr}(w_i w_j)(X, Y) = \text{tr}(P_i P_j)$, so the P'_i have the same Gram matrix and also form a basis. Let ϕ be the K -linear map with $\phi(P_i) = P'_i$. For a word w write

$w(X, Y) = \sum_i c_i P_i$ and $w(Y, X) = \sum_i c'_i P'_i$; then $\Delta c = (\text{tr}(ww_j)(X, Y))_j = (\text{tr}(ww_j)(Y, X))_j = \Delta c'$, so $c = c'$, that is, $\phi(w(X, Y)) = w(Y, X)$ for every word w . The matrices $w(X, Y)$ span $M_2(K)$, and $\phi(w(X, Y)w'(X, Y)) = (ww')(Y, X) = \phi(w(X, Y))\phi(w'(X, Y))$, so ϕ is a bijective K -algebra endomorphism of $M_2(K)$. By the Skolem–Noether theorem it is inner. Directly: the $e_{ij} = \phi(E_{ij})$ satisfy $e_{ij}e_{kl} = \delta_{jk}e_{il}$; choose $u_1 \neq 0$ in the image of e_{11} and put $u_2 = e_{21}u_1$; then $e_{ij}u_k = \delta_{jk}u_i$, so u_1, u_2 are linearly independent and the matrix g with columns u_1, u_2 satisfies $e_{ij}g = gE_{ij}$. Hence $\phi(U) = gUg^{-1}$ for all U , with $g \in \text{GL}(2, K)$, and $gXg^{-1} = Y$, $gYg^{-1} = X$.

(iii) Conjugation by g gives $gX_kg^{-1} = [gXg^{-1}, {}_k gYg^{-1}] = [Y, {}_k X] = Y_k$ for all $k \geq 0$. Hence $X_n = X$ if and only if $Y_n = Y$, and by Lemma 2.1 both hold exactly for the positive multiples of the least period p of X under Y . Since $X_a = X$ and $Y_b = Y$, p divides a and b , hence d ; so $X_d = X$ and $Y_d = Y$.

(iv) If $d = 1$, then $X = [X, Y]$ and $Y = [Y, X]$, and Lemma 2.3(b) gives $X = Y = I$. $\square$

Proof of Corollary 1.2. A homomorphism $\rho: G(a, b) \rightarrow L$ factors through $G(a, b) \rightarrow G(d, d)$ if and only if $\rho(x) = [\rho(x), {}_a \rho(y)]$ and $\rho(y) = [\rho(y), {}_d \rho(x)]$. For $L = \text{SL}(2, K)$ this is Theorem B(iii). For $L = \text{PSL}(2, K)$, let $X', Y' \in \text{SL}(2, K)$ be lifts of $\rho(x), \rho(y)$; then $[X', {}_a Y'] = eX'$ and $[Y', {}_b X'] = fY'$ with $e, f \in \{\pm I\}$. For central e, f we have $[eu, v] = [u, v] = [u, fv]$, hence $[eu, {}_k fv] = [u, {}_k v]$ for $k \geq 1$. So $X = eX'$ and $Y = fY'$ satisfy $[X, {}_a Y] = [X', {}_a Y'] = eX' = X$ and likewise $[Y, {}_b X] = Y$; they lift $\rho(x)$ and $\rho(y)$, and Theorem B(iii) for (X, Y) gives the claim after projection. (In characteristic 2, $\text{SL}(2, K) = \text{PSL}(2, K)$.) For $L = \text{GL}(2, K)$: the image of the perfect group $G(a, b)$ (Lemma 2.3(a)) has trivial image under $\det$, so it lies in $\text{SL}(2, K)$. For $L = \text{PGL}(2, K)$: the determinant induces a homomorphism of $\text{PGL}(2, K)$ to the abelian group $K^\times / (K^\times)^2$ with kernel $\text{PSL}(2, K)$, so the image of $G(a, b)$ lies in $\text{PSL}(2, K)$. If $\gcd(a, b) = 1$, the homomorphism factors through $G(1, 1) = 1$ (Lemma 2.3(b)). A quotient $\text{PSL}(2, q)$ of $G(a, b)$, or a nontrivial linear or projective representation of degree 2 over a field K , would be a nontrivial homomorphism into $\text{PSL}(2, \mathbb{F}_q)$, $\text{GL}(2, K)$ or $\text{PGL}(2, K)$. $\square$

Remark 4.2. Theorem B says nothing about finite simple groups other than the $\text{PSL}(2, q)$; this is why it does not decide $G(2, 3)$. As a consistency check, `check_note.py` lists all Engel pairs in $\text{SL}(2, q)$ for $q \leq 16$ (Section 7); each has the properties that the proof gives.

5 Tested evidence on $G(2, 3)$ and on other cases

The statements of this section are tested, not proved: they rest on exhaustive computer searches within the stated bounds, repeated in part by the independent verification runs (Section 7).

Engel pairs in other finite groups. For each group below, given by permutation generators whose group order was checked, the program `permscan.c` lists all pairs (X, Y) with $X = [X, {}_a Y]$ and $Y = [Y, {}_b X]$ and least periods $a, b \leq K$, with X running over representatives of the conjugacy classes and Y over the whole group. For A_5, A_6 and M_{11} ($K = 60$) only $(a, b) = (5, 5)$ occurs; for $\text{PSL}(3, 3)$ ($K = 60$) only $(16, 16)$ and $(24, 24)$; for A_7, S_7, A_8 and A_9 ($K = 80$) only $(5, 5), (8, 8), (16, 16), (18, 18)$ and $(49, 49)$; for $\text{PSL}(3, 4)$ ($K = 80$) only $(5, 5), (8, 8), (15, 15)$ and $(42, 42)$; for M_{12} ($K = 80$) only $(5, 5)$ and $(37, 37)$; and $M_{22}, \text{PSL}(3, 5), A_{10}$ and A_{11} have no pair with $a, b \leq 3$. So no pair has unequal least periods, and none has a least period 2 or 3. The verification runs obtained the same lists with their own programs for $A_5, A_6, A_7, S_7, A_8, \text{PSL}(3, 3)$ and $\text{PSL}(3, 4)$ (for the last two with $K = 50$). A nontrivial homomorphism of $G(2, 3)$ into a group L maps (x, y) to a pair whose least periods divide 2 and 3 (Lemma 2.1) and are not 1 (Lemma 2.3(c)), so they are 2 and 3. Hence, as tested: $G(2, 3)$ has no nontrivial homomorphism into A_n for $n \leq 11$, into $M_{11}, M_{12}, M_{22}, \text{PSL}(3, 3), \text{PSL}(3, 4)$ or $\text{PSL}(3, 5)$. With Corollary 1.2 and Lemma 2.3(a): if $G(2, 3)$ has a nontrivial finite quotient, it has a nonabelian simple quotient which is none of these groups and not of the form $\text{PSL}(2, q)$.

Coset enumeration. A coset enumerator written for this purpose (`tc.c`, Felsch and Haselgrove–Leech–Trotter strategies; it reproduces the orders of A_5 , $\text{PSL}(2, 7)$, M_{11} and $G(2, 2) = 1$) was run on $G(2, 3)$ over $\langle x \rangle$ and over $\langle y \rangle$ with $6 \cdot 10^7$ cosets, and over $\langle x \rangle$ and over the trivial subgroup with $8 \cdot 10^7$ cosets. All runs overflowed with almost no coincidences, and so did runs for $G(3, 3)$ and $G(4, 4)$ over $\langle x \rangle$. This is inconclusive.

Other cases of Problem 17.80. The scans of $\text{SL}(2, q)$ find Engel pairs of least period 4 in $\text{SL}(2, 13)$ and of least period 6 in $\text{SL}(2, 17)$, which generate $\text{PSL}(2, 13)$ and $\text{PSL}(2, 17)$ modulo $\pm I$; for the pairs $X = \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}$, $Y = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ over $\mathbb{F}_{13}$ and $X = \begin{pmatrix} 0 & 1 \\ -1 & 13 \end{pmatrix}$, $Y = \begin{pmatrix} 2 & 7 \\ 3 & 11 \end{pmatrix}$ over $\mathbb{F}_{17}$ this is checked exactly by `check_note.py`. So M_4 and M_6 map onto $\text{PSL}(2, 13)$ and $\text{PSL}(2, 17)$, and M_5 maps onto $\text{SL}(2, 5)$ by [QWW13]. No pair of least period 3 was found in any group scanned (including $\text{SL}(2, q)$ for $q \leq 32$), so M_3 has no nontrivial homomorphism into any of them. The second verification run searched for further certificates of the kind used in Theorem A: for $n \in \{10, 12, 14, 24\}$ and every Engel pair in $\text{PSL}(2, q)$ with $q \leq 50$ and least period dividing n , the kernel or the preimage of the split torus has finite abelianization (full column rank of R modulo a prime); for larger q only partial checks were made. These computations decide nothing about M_{10} , M_{12} , M_{14} and M_{24} .

6 Conclusion and open questions

The note gives partial answers: $G(a, b)$ is infinite whenever m divides both a and b for some $m \in \{18, 20, 26, 28, 30, 42, 44\}$, so M_n is infinite whenever one of these numbers divides n (for $m = 18$ this is in print); every homomorphism of $G(a, b)$ into $\text{SL}(2, K)$ or $\text{PSL}(2, K)$ factors through $G(d, d)$ with $d = \gcd(a, b)$; the case $G(2, 3)$ and the general questions stay open. The following questions remain.

1. Is $G(2, 3)$ trivial, finite or infinite? By Corollary 1.2 it has no nontrivial image in $\text{GL}(2, K)$ or $\text{PGL}(2, K)$ for any field K , and by the tests of Section 5 none in the alternating groups of degree at most 11 and in the further groups listed there.
2. Do the two least periods of a pair with $u = [u, {}_a v]$ and $v = [v, {}_b u]$ agree in every group? Applied to the pair (x, y) in $G(a, b)$ itself, an affirmative answer would give $G(a, b) = G(d, d)$ for $d = \gcd(a, b)$, and in particular $G(2, 3) = G(1, 1) = 1$. Theorem B proves this for subgroups of $\text{SL}(2, K)$, and the scans of Section 5 found no counterexample.
3. Is M_n infinite for the $n > 2$ not divisible by one of 18, 20, 26, 28, 30, 42, 44? The smallest cases are M_3 , about which we know nothing beyond the tests of Section 5, and M_4 , M_5 , M_6 , which are nontrivial.
4. For which (a, b) is $G(a, b)$ finite? The only finite cases known to us are the trivial groups $G(1, b)$, $G(b, 1)$ and $G(2, 2)$.

7 Computations and reproducibility

Theorem A rests on the exact finite computations of Section 3; Theorem B, the corollaries and the lemmas do not depend on computation. The reproducibility package (its `README.md` gives the command lines and running times) contains the folder `certificates/` with the ten files of Section 3 and two programs written with this note. The program `check_note.py` (Python standard library, a few seconds) repeats Steps 1–4 for every certificate, recomputing everything from X and Y ; it lists all Engel pairs in $\text{SL}(2, q)$ for the ten prime powers $q \leq 16$, with no bound on the periods (X a companion matrix or $\pm I$, Y arbitrary), and checks the conclusions of Theorem B for each; and it verifies the pairs for M_4 and M_6 of Section 5 and the relator lengths of Section 2. The program `check_ranks.py`

(with numpy, about three minutes) repeats Step 5. The folder `first_run/` holds the programs with which the results were first obtained, with their outputs: the checker `verify_cert.py` for $m = 18, 26$, the tests of Theorem B in $\mathrm{SL}(2, p)$ for primes $p \leq 23$ and in $\mathrm{SL}(2, q)$ for $q \leq 49$, the scans of Section 5, and the coset enumerator `tc.c` with the log of its runs. The folders `verification_run_A/` and `verification_run_B/` hold a certificate checker and a test of Theorem B from each of the first two verification runs, and the checker `check_newcert.py` of the five new certificates. The folder `independent_run_2/` holds the programs of the final verification run with their outputs; its `README.md` gives the command lines.

Verification. The certificates for $m = 18$ and $m = 26$, the proof of Theorem B and the tested statements were first obtained with the programs in `first_run/`. Three independent verification runs, all AI-assisted, followed; each wrote its own programs. The first re-derived the proofs of Theorem B, Corollary 1.2 and Lemmas 2.1 to 2.4 line by line in every characteristic; it checked both certificates with a program of its own (for $m = 18$ following the relators, written out with 786431 letters, from all 504 points), tested Theorem B over ten fields with $q \leq 27$, each time finding a swapping g over $\mathbb{F}_q$, reproduced with a coset enumerator of its own $G(2, 2) = 1$ and the order 120 of the quotient G_5 of [QWW13] for our convention (and the order 1 for the other), and compared every statement about the sources with the zbMATH reviews and with Crossref. It found no mathematical error and required five corrections of the text, which were made: the wording on the absence of sources, the general form of Lemma 2.2 with the exact families, the elementary lifting and the extension to $\mathrm{GL}(2, K)$ and $\mathrm{PGL}(2, K)$ in Corollary 1.2, the reference for Lemma 4.1, and the statement that we know [QWW14] only from its review. The second run recomputed both certificates by three methods of its own (composition rules, Fox derivatives, and for $m = 18$ the relators written out and freely reduced to 524323 letters) and determined the free ranks 7 and 14; it scanned the Engel pairs of $\mathrm{SL}(2, q)$ exhaustively for $q \leq 32$ and by their traces for $q \leq 128$, read version 49 of the Notebook, and found the certificates for $m = 20, 28, 30, 42, 44$, using the preimage of the split torus. These certificates are new in this note. It required four corrections, which were made: the relator lengths before and after free reduction, the citation of version 49, the exact free ranks, and the inclusion of the new certificates. The programs `check_note.py` and `check_ranks.py` of this note confirm all certificates and free ranks of Table 1; the programs of the package with a recorded output were run again from a copy of the package (the long scans on part of their range) with the same outputs.

A final verification run then checked the note as written, with programs of its own, written from the definitions before those of the package were read (folder `independent_run_2/`). For every row of Table 1 and for the second pair for $m = 30$ it verified the Engel relations, the least periods and the generation of $\mathrm{PSL}(2, q)$, built the coset space with its own numbering and its own spanning tree, and determined the free rank exactly without the certificate files: the rank of R modulo two other primes, and as many integer vectors of its own, each checked exactly on all rewritten relators. It checked every vector of the ten certificate files exactly in the coordinates of the files, the five new certificates for the first time by a run other than the one that found them, with altered vectors as negative controls. It listed all Engel pairs, with no bound on the periods, in $\mathrm{SL}(2, q)$ for $q \leq 16$, in $\mathrm{GL}(2, q)$ for $q \leq 9$, in $\mathrm{PSL}(2, q)$ for odd $q \leq 13$ and in $\mathrm{PGL}(2, q)$ for odd $q \leq 11$, and confirmed the conclusions of Theorem B and Corollary 1.2 for each; it found no pair of least period 3 in $\mathrm{SL}(2, q)$ for $q \leq 32$, and confirmed the pairs for M_4 and M_6 and the relator lengths of Section 2. It read the sources again, repeated the searches, and ran the package again from its archive, with the recorded outputs. It found no mathematical error and required six corrections, which were made: the mention of [BWW05, BGG⁺06, BGK10], the word “may” in the remark on the other commutator convention, the reference to Lemma 2.2 for the case $18 \mid n$, the heading of the scans of Section 5 (the list contains S_7), the attribution of the Skolem–Noether step, and the description of this run.

Scope and priority. The note answers neither question in full; $G(2, 3)$, the question for which (a, b) the group $G(a, b)$ is finite, and Problem 17.80 for the n not divisible by one of 18, 20, 26, 28, 30, 42, 44 remain open. The problems are due to R. Brandl and O. Macedońska [KM26], the motivation of 11.18 to Brandl and Wilson [BW88]. That M_{18} is infinite is due to Quek, Wong and Wong [QWW14], whose proof is credited there to the referee; the consequence for M_{18} is drawn in Abdollahi’s review [Abd]. Our certificate for $m = 18$ is a second proof. The trace identities are classical [Hor72, Mag80], and the trace map was used for periodic points of word maps, among them the Engel map, in [BGK10]; Burnside’s theorem [Lan02] and the Reidemeister–Schreier method [LS01] are textbook results, and the Skolem–Noether step is proved in the text. What was read: the texts of Problems 11.18 and 17.80 in versions 48 and 49 of the Notebook, the zbMATH reviews of [QWW13, QWW14], the zbMATH summary of [QWW16], and the Crossref metadata of the cited articles, including the abstract of [QWW14]. Of [BGK10] we read, in the arXiv version (v2), whose section numbers we use, Sections 1, 2 and 5.4, and we searched its whole text for the Notebook and for the groups $G(a, b)$. The papers [QWW14, QWW16] were not openly available and were not read: the publisher’s site refused automated access (HTTP 403), and this was not worked around. Neither [QWW13], [BW88], [BWW05] nor [BGG⁺06] was read, nor a thesis of S. G. Quek (2015) whose repository did not answer. The searches were made on 18 October 2026 and repeated on 22 October 2026 by the verification runs: zbMATH Open (the problem numbers with “Kourovka”, “Engel pair”, “Engel pairs”, “verbal dynamical systems”, the authors of the problems and of [QWW14]), OpenAlex (the works citing [BW88] and the papers of Quek, Wong and Wong, and the phrase “Engel pair”), the arXiv API (Engel pairs, the problem numbers with “Kourovka”, recent papers on the Notebook, titles with “Engel” since 2005, the papers of T. Bandman), Crossref, the citation lists of Semantic Scholar, and three web searches; the search of Semantic Scholar refused automated access (HTTP 429). The only work found that cites [QWW14] is [QWW16]. Apart from [QWW14] and [Abd], no printed answer to either question was found in these sources, and no source states Theorem A for $m \neq 18$, Theorem B or Corollary 1.2. A search that finds nothing is not a proof of novelty, and no priority is claimed.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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