

Non-Word-Representable Single-Triangle Graphs with Word-Representable Line Graphs

Alper Ferudun*

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Abstract

For every odd integer $m \geq 5$, let G_m be the Mycielski graph of the cycle C_m with an additional edge joining the shadow copies of two adjacent cycle vertices. We prove that G_m is connected, has exactly one triangle, and is a minimal non-word-representable graph under induced-subgraph containment, while its line graph is word-representable. Indeed, it is 4-vertex-critical. This gives connected examples of clique number three for a follow-up question of Mozhui, Dwary and Krishna. The smallest member of the family has eleven vertices and twenty-one edges. The proof combines a restriction lemma for edge additions inside an independent set of a triangle-free graph with a one-vertex extension of the explicit line-graph orientation of Mozhui, Dwary and Krishna.

1 The question and the result

A finite simple graph is *word-representable* if a word over its vertex set makes two distinct letters alternate exactly when their vertices are adjacent. An orientation is *semi-transitive* if it is acyclic and every directed path whose endpoints are adjacent induces a transitive clique. A directed path with adjacent endpoints that does not induce such a clique is a *shortcut*. We use the theorem of Halldórsson, Kitaev and Pyatkin that word-representability is equivalent to semi-transitive orientability [1, Theorem 3].

The line graph $L(G)$ has the edges of G as its vertices; two are adjacent when their root edges share an endpoint. Mozhui, Dwary and Krishna proved that the line graphs of the non-word-representable graphs $\mu(C_m)$ are word-representable for odd $m \geq 5$ [3, Section 3]. Their result answers negatively the older question of whether the line-graph operation preserves non-word-representability. That older question is also recorded as GRAPH-049 in UnsolvedMath [4]; it is not a new claim of this note.

Question 2 in the concluding section of [3] asks for a non-word-representable graph of clique number three whose line graph is word-representable. Without a connectedness requirement, the disjoint union $\mu(C_5) \sqcup K_3$ already gives an immediate example. Indeed, word-representability is hereditary and is closed under disjoint union: semi-transitive orientations may be taken componentwise. The issue addressed here is the stronger connected construction, with only one triangle.

Label the vertices of C_m cyclically by $1, \dots, m$. Its Mycielski graph $H_m = \mu(C_m)$ has these original vertices, shadow vertices $1', \dots, m'$, and an apex 0. Its edges consist of the original cycle edges, the edges ij' whenever i and j are consecutive on the cycle, and all edges $0j'$. Define

$$G_m = H_m + \{1'm'\}.$$

*Mercury Software GmbH. Email: alper@mercurycodelab.com.

The choice of two consecutive shadow labels is immaterial up to a dihedral relabelling of the cycle.

Theorem 1. *For every odd $m \geq 5$, the graph G_m is connected, has $2m + 1$ vertices, $4m + 1$ edges, and exactly one triangle, namely $\{0, 1', m'\}$. It is not word-representable, but $L(G_m)$ is word-representable. Thus these graphs give infinitely many pairwise nonisomorphic connected examples of clique number three.*

The old line-graph orientation in the proof is precisely the one from [3]. The contribution is its extension across the new shadow edge, together with a proof that this edge does not repair word-representability of the root graph. We also prove induced minimality by a general vertex-criticality lemma for a one-edge Mycielski extension. No minimum-order claim is made for the eleven-vertex example.

2 A restriction lemma

The following elementary observation prevents a misleading appeal to edge-monotonicity: word-representability is not generally preserved by arbitrary edge deletions.

Lemma 2. *Let H be triangle-free and let S be an independent set of H . Obtain G from H by adding any set of edges with both endpoints in S . Every semi-transitive orientation of G , restricted to $E(H)$, is a semi-transitive orientation of H . In particular, if H is not word-representable, then neither is G .*

Proof. The restricted orientation is acyclic. Suppose it had a shortcut $v_0 \rightarrow v_1 \rightarrow \dots \rightarrow v_k$, where $k \geq 3$ and $v_0 v_k \in E(H)$. Semi-transitivity in G makes all these vertices a clique in G . Triangle-freeness of H implies that $v_0 v_2$ and $v_1 v_3$ are not edges of H . Both pairs must therefore be added edges, so $v_0, v_1, v_2, v_3 \in S$. This contradicts $v_0 v_1 \in E(H)$ and the independence of S . Hence no shortcut exists. The last assertion follows from the orientation characterization [1]. $\square$

Corollary 3. *For odd $m \geq 5$, the graph G_m is not word-representable.*

Proof. The Mycielski graph of a triangle-free graph is triangle-free, and the shadow set of H_m is independent. Kitaev and Pyatkin proved that $\mu(C_m)$ is not semi-transitively orientable for every odd $m \geq 3$ [2, Theorem 5]. Apply Lemma 2 to the single added edge $1'm'$. $\square$

3 Induced minimality

A graph is *k-vertex-critical* if it has chromatic number k and deleting any vertex leaves a $(k - 1)$ -colorable graph. This refers to vertex deletion, not arbitrary edge deletion.

Lemma 4. *Let H be a k -vertex-critical graph, where $k \geq 3$, and let $xy \in E(H)$. Then $F = \mu(H) + \{x'y'\}$ is $(k + 1)$ -vertex-critical.*

Proof. Recall that $\chi(\mu(H)) = k + 1$. For completeness, copying a proper k -coloring of H to its shadows and giving the apex a new color proves the upper bound. If $\mu(H)$ had a k -coloring with apex color c , no shadow could have color c . Recolor each original vertex of color c with its own shadow's color. Adjacent originals cannot both have had color c , and the cross edges from shadows to original neighbors guarantee that the resulting coloring of H is proper. It avoids c , contradicting $\chi(H) = k$. The same upper-bound coloring works for F , since $x'y'$ copies the edge xy . Thus $\chi(F) = k + 1$.

Use the color labels $0, \dots, k-1$ for the deletion colorings, and write a for the apex. If a is deleted, copy a proper k -coloring of H onto its shadows. This colors every edge, including $x'y'$, properly.

If a shadow r' is deleted, take a $(k-1)$ -coloring of $H-r$ with colors $0, \dots, k-2$. Use it on those originals and their shadows, and give the original r and a color $k-1$. All remaining shadows avoid $k-1$; cross edges are proper by the coloring of $H-r$ or by their endpoint r having the new color. If $x'y'$ survives, its endpoint colors differ because xy is an edge of $H-r$.

Finally, delete an original vertex r . Choose $s \in \{x, y\} \setminus \{r\}$, and take a $(k-1)$ -coloring of $H-r$ with s colored 0. Give every shadow color $k-1$, except s' , which receives 0, and give a color 1. Ordinary shadows have a color absent from the surviving originals. The neighbors of s' among those originals are neighbors of s and therefore avoid 0. Exactly one endpoint of $x'y'$ is the special shadow. Since $k \geq 3$, the apex color 1 is distinct from both shadow colors 0 and $k-1$. All edges are proper.

These cases provide a k -coloring after every vertex deletion. $\square$

The condition $k \geq 3$ cannot be dropped. For $H = K_2$, the extension contains the triangle on the apex and the two shadows; this triangle survives deletion of either original vertex.

Corollary 5. *For every odd $m \geq 5$, the graph G_m is 4-vertex-critical and is a minimal non-word-representable graph under induced-subgraph containment.*

Proof. An odd cycle is 3-vertex-critical, so Lemma 4 gives the first assertion. Every three-colorable graph is word-representable: orient its edges from smaller to larger color. The orientation is acyclic and has no directed path with three edges, hence it is semi-transitive; use [1]. By Corollary 3, G_m itself is not word-representable. Every proper induced subgraph lies in some $G_m - v$ and is therefore three-colorable and word-representable. $\square$

This is not edge-deletion minimality: removing the new shadow edge recovers the non-word-representable graph H_m . For comparison, Kenkireth, Sajith and Sasidharan classify minimal non-word-representable graphs that are also minimal non-comparability graphs, and those with a universal vertex [5]. The graphs here belong to neither subclass. Their maximum degree is $m < 2m$, so they have no universal vertex. They contain a proper induced odd cycle C_m , which is not a comparability graph: in a triangle-free comparability graph every vertex must be a source or a sink, giving a bipartition. Thus G_m is not a minimal non-comparability graph.

4 The old orientation and its boundary

Write a_{ij} for the line vertex corresponding to the root edge ij' , retaining 0 for the apex. The possible pairs are

$$(0, j) \ (1 \leq j \leq m), \quad (i, j) \ (1 \leq i \leq m, \ j = i-1 \text{ or } i+1 \text{ mod } m).$$

Write c_{ij} , with $i < j$, for the line vertex corresponding to an original cycle edge. Let B_m be the spanning bipartite subgraph of H_m consisting of all edges represented by the a_{ij} . Line vertices of type a are adjacent when either their first indices or their second indices agree.

Use the following orientation D of $L(H_m)$, established to be semi-transitive in [3, Section 3, Steps I–III]:

$$\begin{aligned}
a_{ij} &\longrightarrow a_{ik} & (j < k), \\
a_{ij} &\longrightarrow a_{kj} & (i > k), \\
c_{i+1,i+2} &\longrightarrow c_{i,i+1} & (1 \leq i \leq m-2), \\
c_{1m} &\longrightarrow c_{12}, & c_{m-1,m} \longrightarrow c_{1m}.
\end{aligned}$$

Every edge joining an a -vertex and a c -vertex is directed from the former to the latter. These rules specify every edge of $L(H_m)$. We need only a small boundary of this known orientation:

$$A = a_{01}, \quad B = a_{21}, \quad C = a_{m1}, \quad D_* = a_{1m}, \quad E = a_{m-1,m}, \quad F = a_{0m}.$$

The notation D_* denotes a vertex, not the orientation D . Set $I = \{A, B, C, D_*, E\}$.

Lemma 6. *The orientation D has the following properties.*

1. *The vertex F is a sink.*
2. *Every directed path with both endpoints in I is a subsequence of the transitive clique $C \rightarrow B \rightarrow A$, a subsequence of the transitive clique $E \rightarrow D_*$, or a single vertex.*
3. *Among old neighbors of F , only A, D_*, E can reach I , allowing a path of length zero. The possible paths from such a neighbor to a vertex of I are single vertices and $E \rightarrow D_*$.*

Proof. There is no directed edge from a c -vertex to an a -vertex. A path with both endpoints of type a therefore stays in $L(B_m)$. Along such a path the first index is nonincreasing and the second index is nondecreasing. Each step keeps one index fixed.

The possible second indices in the nonzero rows are

row	second indices
1	2, m
$i, 2 \leq i \leq m-1$	$i-1, i+1$
m	1, $m-1$.

Row zero contains all second indices $1, \dots, m$.

Starting at A , a path stays in row zero, so it cannot meet another vertex of I . From B , the column-one route goes to A ; the other route enters a_{23} and then row zero at a second index at least three. It cannot return to I . From C , the column-one routes use only C, B, A . The other route enters $a_{m,m-1}$ and then can only move to $a_{m-2,m-1}$ or $a_{0,m-1}$ before remaining in row zero. The second index $m-1$ is already maximal in row $m-2$. Thus this branch also misses I . Finally, D_* can only move to F , and E can only move to D_* or F . This proves the second assertion, including the claim about all intermediate vertices. The indicated chains are cliques because their second index is fixed.

The neighbors of F are

$$\{a_{0j} : 1 \leq j < m\} \cup \{D_*, E\}.$$

All their incident edges to F point toward F ; there are no c -neighbors of F . Thus F is a sink. A path from a_{0j} stays in row zero and never decreases its second index. It can reach I only when $j = 1$, and then only at A itself. The assertions for D_* and E were proved above. $\square$

5 Extending across the new edge

The added root edge $1'm'$ gives one new line vertex z . Its old neighbors are exactly

$$N(z) = I \cup \{F\}.$$

Retain the old orientation D and add the arcs

$$u \longrightarrow z \quad (u \in I), \quad z \longrightarrow F.$$

Call the resulting orientation $\hat{D}$.

Proposition 7. *The orientation $\hat{D}$ is semi-transitive.*

Proof. The old orientation is acyclic. A directed cycle containing z would have to leave it through F , which is a sink. Hence $\hat{D}$ is acyclic.

Any shortcut not containing z would be a shortcut in D . Consider a directed path containing z and having adjacent endpoints. If z is its first vertex, the path has at most one edge, since its only successor is F .

If z is the last vertex, the first and penultimate vertices are in I . Lemma 6(2) puts the old portion of the path in one of the two indicated transitive cliques. All its vertices are adjacent to z , and all those edges point to z . The full path therefore induces a transitive clique.

In the remaining case, z is internal. The path must end at F , and its first vertex is an old neighbor of F that reaches I . By Lemma 6(3), the only possibilities are subsequences of

$$A \rightarrow z \rightarrow F \quad \text{or} \quad E \rightarrow D_* \rightarrow z \rightarrow F.$$

Both displayed vertex sets induce transitive cliques. In the second, the three old vertices share their second index m ; in the first, A and F share their first index zero. Thus these paths are not shortcuts either. $\square$

Proof of Theorem 1. The graph H_m is connected and has $2m + 1$ vertices and $4m$ edges. Adding $1'm'$ preserves connectedness and increases the edge count by one. Since H_m is triangle-free, every triangle of G_m must use this new edge. The old common neighbors of $1'$ and m' consist only of 0: their original-vertex neighborhoods are $\{2, m\}$ and $\{1, m - 1\}$, which are disjoint for $m \geq 5$. This proves uniqueness of the triangle and, in particular, clique number three. Corollary 3 proves non-word-representability of G_m . Proposition 7 and [1, Theorem 3] prove word-representability of $L(G_m)$. Corollary 5 proves induced minimality. The vertex counts distinguish the family members. $\square$

6 The smallest example and reproducibility

For $m = 5$, the root graph has eleven vertices and twenty-one edges, and its line graph has twenty-one vertices and sixty-one edges. More generally,

$$|V(L(G_m))| = 4m + 1, \quad |E(L(G_m))| = \frac{m(m-1)}{2} + 9m + 6.$$

The edge formula follows by summing $\binom{\deg(v)}{2}$ over root vertices: adding the shadow edge raises two degrees from three to four, contributing six additional line edges.

The supplementary programs build these graphs from the definitions. They verify the orientation and all boundary-path conditions for sixteen odd cycle lengths between five and fifty-one. For the first three lengths, a separate direct enumeration of directed paths agrees with the reachability-based shortcut checker. The construction at $m = 3$, excluded by the theorem, fails the semi-transitivity check.

A separate C++ program enumerates every orientation of the eleven-vertex graphs. For H_5 it checks all 2^{20} orientations, and for G_5 all 2^{21} orientations; neither has a semi-transitive orientation. Every acyclic orientation is checked both by reachability intervals and by literal path enumeration. Known positive and negative graphs provide controls. These finite checks support the implementation but are not the proof for all m ; that proof is given above using the stated results of [2, 3].

The explicit deletion-coloring construction is separately checked for nineteen odd cycle lengths from five to 201, and for the complete-graph bases $K_3, \dots, K_7$. All 1668 deletion colorings are checked edge by edge. An independent exact coloring search checks the eleven-vertex case; the $k = 2$ case and a corrupted coloring are rejected. Normal and optimized Python runs agree.

The construction does not decide whether the line graph of every triangle-free graph is word-representable, the other question at the end of [3]. It does not classify all word-representable line graphs or assert a smallest possible connected example.

This unrefereed preprint was developed with substantial assistance from an OpenAI model in deriving the construction, writing the proof and implementing the checks. The verification is a self-audit, not independent human peer review or formal certification. The cited authors' theorems and orientation are credited explicitly; no absolute-priority claim is made.

References

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