

# Every Seven Positive Reals Contain a Dissociated Four-Element Subset: Small Cases of an Erdős Problem on Dissociated Sets

Alper Ferudun\*

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## Abstract

Let  $f(n)$  be the largest  $k$  such that every set of  $n$  real numbers contains a dissociated subset of size  $k$ , that is, a subset whose  $2^k$  subset sums are pairwise distinct (the empty sum included). Erdős noted that  $f(n) \geq \lfloor \log_3 n \rfloor$  and asked whether  $f(n) \geq \lfloor \log_2 n \rfloor$  for every  $n$  (Erdős Problem #963). The general question remains open. Let  $g^+(K)$  be the largest size of a set of positive reals without a dissociated subset of size  $K$ . We show that the corresponding largest size for sets of reals is  $1 + 2g^+(K)$ , so that the question is equivalent to  $g^+(K) \leq 2^{K-1} - 1$  for all  $K$ , and also to the inequality  $|P| \leq 2^{\dim P} - 1$  for finite sets  $P$  of nonzero integer vectors containing no pair  $x, -x$ , where  $\dim P$  is the largest size of a dissociated subset of  $P$ . Elementary arguments give  $g^+(1), g^+(2), g^+(3) = 0, 1, 3$ . Our main result is  $g^+(4) = 6$ : every seven positive reals contain a dissociated four-element subset, and  $\{1, \dots, 6\}$  contains none. The proof is computer-assisted and is given twice: by a case analysis over sign patterns in which every terminal case is closed by an exact Farkas certificate, checked by independent exact checkers, and by an enumeration of linear forms over a dissociated triple in exact integer arithmetic. Consequently  $f(n)$  is known for all  $n \leq 27$ : it is 0, 1, 1, 2, 2, 2, 2 for  $n \leq 7$ , then 3 for  $8 \leq n \leq 13$  and 4 for  $14 \leq n \leq 27$ ; and  $f(n) \geq \lfloor \log_2 n \rfloor$  holds for all  $n \leq 31$ . The first open case,  $n = 32$ , is equivalent to  $g^+(5) \leq 15$ . The upper bound for  $14 \leq n \leq 27$  uses the set  $\{1, \dots, 10, 12, 13, 15\}$  without dissociated five-element subsets, posted by BAKKAOUI on the problem's forum (2026); we also show  $g^+(6) \geq 24$  by an explicit set. An unrefereed argument for  $f(n) \geq (1 - o(1)) \log_2 n$  was posted on the same forum by KoishiChan (2025). To our knowledge the value  $g^+(4) = 6$  and its consequences for  $n \leq 31$  are new. This is an unrefereed note.

## 1 Introduction

### 1.1 The problem

A finite set  $B$  of real numbers is *dissociated* if the  $2^{|B|}$  sums  $\sum_{b \in S} b$ ,  $S \subseteq B$ , are pairwise distinct; the empty sum 0 is one of them. For a finite set  $A \subset \mathbb{R}$  we write  $\dim A$  for the largest size of a dissociated subset of  $A$  (this is often called the additive dimension of  $A$ ). For  $n \geq 1$  let

$$f(n) = \min\{\dim A : A \subset \mathbb{R}, |A| = n\},$$

that is, the largest  $k$  such that every set of  $n$  real numbers has a dissociated subset of size  $k$ . Problem #963 of the collection of Erdős problems [Blo26] asks to estimate  $f(n)$ , and in particular whether

$$f(n) \geq \lfloor \log_2 n \rfloor \quad \text{for every } n \geq 1. \tag{1}$$

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\*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>. Unrefereed preprint. CC BY 4.0.

The question is Erdős's [Erd65, p. 188]. There he observes that from  $n$  “given numbers”  $a_1, \dots, a_n$  one can always select  $k \geq \lfloor \log n / \log 3 \rfloor$  of them whose sums  $\sum_{r=1}^k \varepsilon_r a_{i_r}$ ,  $\varepsilon_r \in \{0, 1\}$ , are all different, and he writes: “Perhaps this can be improved to  $k \geq \lfloor \log n / \log 2 \rfloor$ .” The example  $a_i = i$  shows that such an improvement would be nearly best possible [Erd65, p. 188]. The problem page also cites item 1.22 of a problem collection of 1999, which we have not seen.

*The reading.* We read the question exactly as it is stated on the problem page:  $A$  is a set of  $n$  distinct real numbers, 0 and negative numbers are allowed, and the empty sum counts. Hence 0 lies in no dissociated set, and no dissociated set contains a pair  $x, -x$  (Lemma 2.1). The problem is record EP-963 of the UnsolvedMath collection [ula26], where its upstream status is “open” and its literature note calls it partially solved, on account of the forum argument described in Section 1.3. On the problem's forum the question is also considered for sets of positive integers [Koi25, BAK26]; in that reading, with  $f^+(n) = \min\{\dim A : A \subset (0, \infty), |A| = n\}$  in place of  $f(n)$ , the bound  $\lfloor \log_2 n \rfloor$  for  $n \leq 31$  already follows from Erdős's greedy argument, and the exact values are different, because  $f(n) = f^+(\lfloor n/2 \rfloor)$  (Remark 2.6).

## 1.2 Results

For  $K \geq 1$  let  $g(K)$  be the largest size of a finite set of real numbers without a dissociated subset of size  $K$ , and let  $g^+(K)$  be the same for sets of positive real numbers. Both are finite by Erdős's greedy argument (Lemma 2.4).

**Proposition 1.1** (Reformulation). *For every  $K \geq 1$  we have  $g(K) = 1 + 2g^+(K)$ , and for every  $n \geq 1$*

$$f(n) = \max\{K \geq 0 : g(K) \leq n - 1\} \quad (g(0) := 0).$$

*The bound (1) holds for all  $n$  if and only if  $g^+(K) \leq 2^{K-1} - 1$  for every  $K \geq 1$ , that is, if and only if every set of  $2^{K-1}$  positive reals contains a dissociated subset of size  $K$ , for every  $K$ . It is also equivalent to the inequality  $|P| \leq 2^{\dim P} - 1$  for all finite sets  $P \subset \mathbb{Z}^r \setminus \{0\}$  ( $r \geq 1$ ) containing no pair  $x, -x$  (Proposition 4.1).*

Elementary arguments give  $g^+(1) = 0$ ,  $g^+(2) = 1$  and  $g^+(3) = 3$  (Lemma 2.5), so the inequality  $g^+(K) \leq 2^{K-1} - 1$  holds with equality for  $K \leq 3$ . Erdős's greedy argument gives  $g^+(K) \leq (3^{K-1} - 1)/2$  (Lemma 2.4); for  $K = 4$  this is  $g^+(4) \leq 13$ , while (1) needs  $g^+(4) \leq 7$ . Our main result is the exact value.

**Theorem 1.2.** *Every set of seven positive real numbers contains a dissociated subset of four elements, and the set  $\{1, 2, 3, 4, 5, 6\}$  contains none. Hence  $g^+(4) = 6$  and  $g(4) = 13$ .*

The proof (Section 3) is computer-assisted, and we give it twice. The first proof is a case analysis over the sign patterns of the possible relations among seven unknown reals  $0 < p_1 < \dots < p_7$ ; each of its 151 terminal cases is closed by an exact Farkas certificate, and the tree of cases is verified by an exact checker which shares no code with the program that found it. The second proof starts from a dissociated triple and enumerates linear forms in exact integer arithmetic. The lower bound is attained by many sets, for instance by  $\{1, 2, 3, a, a + 1, a + 3\}$  for every real  $a > 3$  (Proposition 3.1).

**Corollary 1.3.** *The values of  $f$  for small  $n$  are as follows.*

(a)  $f(1) = 0$ ,  $f(n) = 1$  for  $n = 2, 3$ ,  $f(n) = 2$  for  $4 \leq n \leq 7$ ,  $f(n) = 3$  for  $8 \leq n \leq 13$ , and  $f(n) = 4$  for  $14 \leq n \leq 27$ .

(b)  $f(n) \geq \lfloor \log_2 n \rfloor$  for every  $n \leq 31$ , with equality for  $n \leq 13$  and for  $16 \leq n \leq 27$ .

(c) The first case of (1) not covered by (b), namely  $f(32) \geq 5$ , is equivalent to  $g^+(5) \leq 15$ . It is known that  $13 \leq g^+(5) \leq 40$ .

The upper bound  $f(27) \leq 4$  in (a) and the lower bound  $g^+(5) \geq 13$  in (c) come from the set  $A^* = \{1, \dots, 10, 12, 13, 15\}$ , which has no dissociated subset of five elements [BAK26]; we also show  $g^+(6) \geq 24$  with the set  $\{1, \dots, 21, 24, 25, 27\}$  (Proposition 5.1). Table 1 collects the values.

| $K$ | $g^+(K)$  | $g(K) = 1 + 2g^+(K)$ | needed for (1)<br>$2^{K-1} - 1$ | greedy bound<br>$(3^{K-1} - 1)/2$ | reference       |
|-----|-----------|----------------------|---------------------------------|-----------------------------------|-----------------|
| 1   | 0         | 1                    | 0                               | 0                                 | Lemma 2.5       |
| 2   | 1         | 3                    | 1                               | 1                                 | Lemma 2.5       |
| 3   | 3         | 7                    | 3                               | 4                                 | Lemma 2.5       |
| 4   | 6         | 13                   | 7                               | 13                                | Theorem 1.2     |
| 5   | $\geq 13$ | $\geq 27$            | 15                              | 40                                | Proposition 5.1 |
| 6   | $\geq 24$ | $\geq 49$            | 31                              | 121                               | Proposition 5.1 |

Table 1: Known values. For  $K = 5$  and  $K = 6$  only lower bounds are known. The bound (1) for all  $n$  is equivalent to  $g^+(K) \leq 2^{K-1} - 1$  for all  $K$ .

### 1.3 Related work and status

*Status.* The problem database of the Erdős problems project lists Problem #963 as open, with last update 31 August 2025 [teo26]; this is the state on 15 October 2026.

*The forum.* On 5 December 2025 KoishiChan posted on the problem’s forum an argument for  $f(n) \geq (1 - o(1)) \log_2 n$  [Koi25]: a reduction to sets of positive integers, a random dilation modulo a large prime, a second-moment estimate based on the mean value theorem of Montgomery and Vaughan for character sums, and a recursion. In the discussion of the same day an off-by-one error was pointed out, and the author replied that it can be easily fixed; a formal write-up was requested. On 23 January 2026 the maintainer of the site wrote that he intended to mark the problem as solved; the database entry [teo26] still gives the status open. This argument is unrefereed, it is not used here, and we do not assess it. Even if correct, it does not give (1) for any particular  $n$ . On 3 September 2026 BAKKAOUI observed that  $A^* = \{1, \dots, 10, 12, 13, 15\}$  has no dissociated subset of five elements, so that the interval  $\{1, \dots, 13\}$  is not extremal, and derived  $f(13) \leq 4$ ; the posts also report searches over sets of positive integers in bounded windows and note that 0 lies in no dissociated set, so that with 0 allowed intervals are not extremal already for  $n = 4$  [BAK26]. The bound  $f(13) \leq 4$  is an upper bound for  $f(13)$ . In the reading of this note it is superseded by the elementary value  $f(13) = 3$ : the thirteen numbers  $\{0, \pm 1, \dots, \pm 6\}$  have no dissociated four-element subset (Proposition 3.1(a) and Lemma 2.2), and  $f(13) \geq 3$  by Lemmas 2.3 and 2.5. What  $A^*$  gives here is  $g^+(5) \geq 13$ , hence  $g(5) \geq 27$  and the upper bound  $f(n) \leq 4$  for  $n \leq 27$ .

*Literature.* Dissociated sets are a standard tool in additive combinatorics. Lev and Yuster [LY11] showed that the sizes of the maximal dissociated subsets of a finite subset of an abelian group differ by at most a logarithmic factor. Schoen and Shkredov [SS16], Candela and Helfgott [CH15] and Shkredov [Shk24] bound the additive dimension of structured sets and compare notions of dimension. Dutta [Dut26] studies infinite dissociated sets of positive integers: their counting functions are at most  $\log_2 n + \frac{1}{2} \log_2 \log_2 n + C$  for all  $n$  in a set of positive density, and the sequence produced by the greedy algorithm eventually doubles. Bohman [Boh96, Boh98] studies sets of integers with distinct subset sums and the Conway–Guy sequence, which concern the interval examples of Section 5. None of these works settles the question or determines  $f(n)$  for particular  $n$ . For our searches of the literature, what

was read, and the limits of the novelty statement, see the paragraph “Scope and priority” at the end of the note.

## 2 Reductions

For a finite set  $B \subset \mathbb{R}$ , a *relation* on  $B$  is a vector  $\varepsilon \in \{-1, 0, 1\}^B$ ,  $\varepsilon \neq 0$ , with  $\sum_{b \in B} \varepsilon_b b = 0$ . We put

$$\Sigma^\pm(B) = \left\{ \sum_{b \in B} \varepsilon_b b : \varepsilon \in \{-1, 0, 1\}^B \right\}.$$

**Lemma 2.1.** *Let  $B \subset \mathbb{R}$  be finite.*

- (a)  *$B$  is dissociated if and only if there is no relation on  $B$ .*
- (b) *Every subset of a dissociated set is dissociated. A dissociated set contains neither 0 nor a pair  $x, -x$ .*
- (c) *Let  $\sigma : B \rightarrow \{-1, 1\}$ . Then  $B$  is dissociated if and only if the numbers  $\sigma(b)b$  ( $b \in B$ ) are pairwise distinct and form a dissociated set.*
- (d) *If  $B$  is dissociated and  $x \in \mathbb{R} \setminus B$ , then  $B \cup \{x\}$  is dissociated if and only if  $x \notin \Sigma^\pm(B)$ .*

*Proof.* (a) If  $S \neq T$  are subsets of  $B$  with equal sums, then  $\varepsilon = 1_{S \setminus T} - 1_{T \setminus S}$  is a relation: the sets  $S \setminus T$  and  $T \setminus S$  are disjoint and not both empty. Conversely, a relation  $\varepsilon$  gives the subsets  $S = \{\varepsilon_b = 1\}$  and  $T = \{\varepsilon_b = -1\}$ , which are different and have equal sums.

(b) A relation on a subset, extended by zeros, is a relation on  $B$ . The vectors  $1_{\{0\}}$  and  $1_{\{x, -x\}}$  are relations.

(c) Suppose  $B$  is dissociated. If  $\sigma(b)b = \sigma(b')b'$  with  $b \neq b'$ , then  $b' = -b$ , which (b) excludes; and a relation  $\varepsilon'$  on the set  $\{\sigma(b)b\}$  gives the relation  $\varepsilon_b = \sigma(b)\varepsilon'_{\sigma(b)b}$  on  $B$ , which is impossible. Conversely, if the numbers  $\sigma(b)b$  are pairwise distinct, a relation  $\varepsilon$  on  $B$  gives the relation  $\varepsilon'_{\sigma(b)b} = \sigma(b)\varepsilon_b$  on  $\{\sigma(b)b\}$ .

(d) A relation on  $B \cup \{x\}$  has  $\varepsilon_x \neq 0$  because  $B$  is dissociated, and then  $x = -\varepsilon_x \sum_{b \in B} \varepsilon_b b \in \Sigma^\pm(B)$ . Conversely, if  $x = \sum_{b \in B} \varepsilon_b b$ , then the vector equal to  $\varepsilon$  on  $B$  and to  $-1$  at  $x$  is a relation on  $B \cup \{x\}$ .  $\square$

**Lemma 2.2.**  *$g(K) = 1 + 2g^+(K)$  for every  $K \geq 1$ .*

*Proof.* Let  $P \subset (0, \infty)$  have  $g^+(K)$  elements and no dissociated subset of size  $K$ , and let  $A = \{0\} \cup P \cup (-P)$ , a set of  $1 + 2g^+(K)$  reals. A dissociated subset  $B \subseteq A$  contains neither 0 nor a pair  $x, -x$ , so by Lemma 2.1(c), with  $\sigma(b)$  the sign of  $b$ , the numbers  $|\sigma(b)b|$  ( $b \in B$ ) form a dissociated subset of  $P$  of size  $|B|$ . Hence  $|B| < K$ , and  $g(K) \geq 1 + 2g^+(K)$ .

Conversely, let  $A \subset \mathbb{R}$  have no dissociated subset of size  $K$ , and let  $P = \{|a| : a \in A, a \neq 0\}$ . Then  $|A| \leq 1 + 2|P|$ . If  $Q \subseteq P$  were dissociated with  $|Q| = K$ , choose  $a_q \in A$  with  $|a_q| = q$  for  $q \in Q$ ; by Lemma 2.1(c) the set  $\{a_q : q \in Q\}$  would be a dissociated subset of  $A$  of size  $K$ . Hence  $|P| \leq g^+(K)$  and  $|A| \leq 1 + 2g^+(K)$ .  $\square$

**Lemma 2.3.** *For  $n, K \geq 1$  we have  $f(n) \geq K$  if and only if  $n \geq g(K) + 1$ . Consequently  $f$  is nondecreasing,  $f(n) = \max\{K \geq 0 : g(K) \leq n - 1\}$  with  $g(0) := 0$ , and the following statements are equivalent:*

- (i)  *$f(n) \geq \lfloor \log_2 n \rfloor$  for every  $n \geq 1$ ;*
- (ii)  *$g(K) \leq 2^K - 1$  for every  $K \geq 1$ ;*

(iii)  $g^+(K) \leq 2^{K-1} - 1$  for every  $K \geq 1$ ;

(iv) for every  $K \geq 1$ , every set of  $2^{K-1}$  positive reals contains a dissociated subset of size  $K$ ;

(v)  $|P| \leq 2^{\dim P} - 1$  for every finite set  $P \subset (0, \infty)$ .

*Proof.* If  $n \geq g(K) + 1$ , a set of  $n$  reals without a dissociated  $K$ -subset would contradict the definition of  $g(K)$ ; so  $f(n) \geq K$ . If  $n \leq g(K)$ , every  $n$ -element subset of a set of  $g(K)$  reals without a dissociated  $K$ -subset has none either (Lemma 2.1(b)); so  $f(n) < K$ . Hence the  $K \geq 1$  with  $g(K) \leq n - 1$  are exactly  $1, \dots, f(n)$ , which is the formula for  $f(n)$ , and  $f$  is nondecreasing because the condition  $g(K) \leq n - 1$  is monotone in  $n$ .

(i) $\Rightarrow$ (ii):  $f(2^K) \geq K$  means  $2^K \geq g(K) + 1$ . (ii) $\Rightarrow$ (i): for  $n \geq 2$  let  $K = \lfloor \log_2 n \rfloor \geq 1$ ; then  $n \geq 2^K \geq g(K) + 1$ , so  $f(n) \geq K$ ; and  $f(1) \geq 0$ . (ii) $\Leftrightarrow$ (iii) by Lemma 2.2, and (iii) $\Leftrightarrow$ (iv) by the definition of  $g^+$ . (iii) $\Rightarrow$ (v): if  $\dim P = k$ , then  $P$  has no dissociated subset of size  $k + 1$ , so  $|P| \leq g^+(k + 1) \leq 2^k - 1$ . (v) $\Rightarrow$ (iii): a set  $P$  of  $g^+(K)$  positive reals without a dissociated  $K$ -subset has  $\dim P \leq K - 1$ , so  $g^+(K) = |P| \leq 2^{K-1} - 1$ .  $\square$

**Lemma 2.4** (Erdős's greedy bound). *For  $K \geq 1$  we have  $g^+(K) \leq (3^{K-1} - 1)/2$  and  $g(K) \leq 3^{K-1}$ . Hence  $f(n) \geq \lfloor \log_3(n - 1) \rfloor + 1 \geq \lfloor \log_3 n \rfloor$  for  $n \geq 2$ .*

*Proof.* Let  $P \subset (0, \infty)$  be finite with  $|P| > (3^{K-1} - 1)/2$ . Starting from  $B_0 = \emptyset$  we choose dissociated sets  $B_0 \subset B_1 \subset \dots \subset B_K \subseteq P$  with  $|B_j| = j$ . Given  $B_j$  with  $j \leq K - 1$ , the set  $\Sigma^\pm(B_j)$  has at most  $3^j$  elements, contains 0 and is symmetric, so it contains at most  $(3^j - 1)/2 < |P|$  positive numbers. Choose  $x \in P \setminus \Sigma^\pm(B_j)$ . Then  $x \notin B_j \subseteq \Sigma^\pm(B_j)$ , and  $B_{j+1} = B_j \cup \{x\}$  is dissociated by Lemma 2.1(d). So  $\dim P \geq K$ , which proves the bound for  $g^+$ ; the bound for  $g$  follows from Lemma 2.2. If  $n \geq 2$  and  $K = \lfloor \log_3(n - 1) \rfloor + 1$ , then  $g(K) \leq 3^{K-1} \leq n - 1$ , so  $f(n) \geq K$  by Lemma 2.3; and  $K \geq \lfloor \log_3 n \rfloor$  because  $n \leq 3(n - 1)$ .  $\square$

**Lemma 2.5.** *Three positive reals  $a < b < c$  form a dissociated set unless  $c = a + b$ . Moreover  $g^+(1) = 0$ ,  $g^+(2) = 1$  and  $g^+(3) = 3$ ; hence  $g(1) = 1$ ,  $g(2) = 3$  and  $g(3) = 7$ .*

*Proof.* Let  $\varepsilon$  be a relation on  $\{a, b, c\}$ , normalized by  $\varepsilon_c \in \{0, 1\}$ . If  $\varepsilon_c = 0$ , then  $\varepsilon_a a + \varepsilon_b b$  is one of  $\pm a, \pm b, \pm(a + b), \pm(b - a)$ , all nonzero. If  $\varepsilon_c = 1$ , then  $\varepsilon_a a + \varepsilon_b b = -c$ ; every value of  $\varepsilon_a a + \varepsilon_b b$  other than  $\pm(a + b)$  has absolute value at most  $b < c$ , so  $\varepsilon_a = \varepsilon_b = -1$  and  $c = a + b$ .

A single positive real is dissociated, so  $g^+(1) = 0$ . Two distinct positive reals are dissociated (the four sums  $0, a, b, a + b$  are distinct), so  $g^+(2) \leq 1$ , and  $\{1\}$  shows  $g^+(2) = 1$ . If  $p_1 < p_2 < p_3 < p_4$  are positive reals without a dissociated triple, then  $p_4 = p_1 + p_2$  and  $p_4 = p_1 + p_3$  by the first part, so  $p_2 = p_3$ , which is absurd; hence  $g^+(3) \leq 3$ , and  $\{1, 2, 3\}$ , whose only triple satisfies  $3 = 1 + 2$ , gives  $g^+(3) = 3$ . The values of  $g$  follow from Lemma 2.2; the extremal sets are  $\{0\}$ ,  $\{0, \pm 1\}$  and  $\{0, \pm 1, \pm 2, \pm 3\}$ .  $\square$

*Remark 2.6* (Integers, and the reading with positive numbers). (a) The functions  $f$ ,  $g$  and  $g^+$  do not change if real numbers are replaced by integers (positive integers for  $g^+$ ). Indeed, let  $A \subset \mathbb{R}$  be finite and let  $\varphi$  be a  $\mathbb{Q}$ -linear map from the  $\mathbb{Q}$ -span of  $A$  to  $\mathbb{Q}$  that does not vanish on the finite set  $\Sigma^\pm(A) \setminus \{0\}$ ; such a map exists because the space of these maps, a  $\mathbb{Q}$ -vector space of finite dimension, is not a finite union of proper subspaces. Then  $\sum \varepsilon_a a = 0$  if and only if  $\sum \varepsilon_a \varphi(a) = 0$ , for all  $\varepsilon \in \{-1, 0, 1\}^A$ . So  $\varphi$  is injective on  $A$  and maps dissociated subsets to dissociated subsets and back; after multiplication by a common denominator its values are integers. If  $A \subset (0, \infty)$ , the numbers  $|\varphi(a)|$  are nonzero and pairwise distinct, because  $\varphi(a) \neq 0$  and  $\varphi(a) \pm \varphi(a') = \varphi(a \pm a') \neq 0$  for  $a \neq a'$ , and Lemma 2.1(c) applies.

(b) Let  $f^+(m) = \min\{\dim P : P \subset (0, \infty), |P| = m\}$  for  $m \geq 1$ , and  $f^+(0) = 0$ . As in Lemma 2.3,  $f^+(m) \geq K$  if and only if  $m \geq g^+(K)+1$ . With Lemma 2.2,  $f(n) \geq K$  if and only if  $g^+(K) \leq (n-2)/2$ , that is,  $g^+(K) \leq \lfloor n/2 \rfloor - 1$ ; hence

$$f(n) = f^+(\lfloor n/2 \rfloor) \quad (n \geq 1).$$

Since  $\lfloor \log_2(2m) \rfloor = \lfloor \log_2(2m+1) \rfloor = \lfloor \log_2 m \rfloor + 1$  for  $m \geq 1$ , the bound (1) is equivalent to  $f^+(m) \geq \lfloor \log_2 m \rfloor + 1$  for all  $m \geq 1$ , which is one more than the analogous bound  $f^+(m) \geq \lfloor \log_2 m \rfloor$  for positive numbers. The latter holds for all  $m \leq 31$  by Lemma 2.4 alone, since  $(3^{K-1} - 1)/2 \leq 2^K - 1$  for  $K \leq 4$ . By Lemma 2.5, Theorem 1.2 and Proposition 5.1,  $f^+(m) = 1, 2, 2, 3, 3, 3$  for  $m = 1, \dots, 6$  and  $f^+(m) = 4$  for  $7 \leq m \leq 13$ , whereas  $f(13) = 3$ . In the positive reading the bound  $f(13) \leq 4$  of [BAK26] becomes  $f^+(13) \leq 4$ , which is equivalent to  $g^+(5) \geq 13$ .

### 3 The value $g^+(4) = 6$

#### 3.1 The lower bound

**Proposition 3.1.** *The following sets have no dissociated four-element subset.*

- (a) *The set  $\{1, \dots, 6\}$ ; on the other hand,  $\{1, \dots, 7\}$  has the dissociated subset  $\{3, 5, 6, 7\}$ .*
- (b) *The set  $\{1, 2, 3, a, a+1, a+3\}$ , for every real  $a > 3$ .*

*In particular  $g^+(4) \geq 6$ , and the sets of six positive reals without a dissociated four-element subset are not unique up to scaling.*

*Proof.* (a) Each of the fifteen four-element subsets of  $\{1, \dots, 6\}$  contains one of the triples  $\{1, 2, 3\}$ ,  $\{1, 3, 4\}$ ,  $\{1, 4, 5\}$ ,  $\{2, 3, 5\}$ ,  $\{1, 5, 6\}$ ,  $\{2, 4, 6\}$ , which carry the relations  $3 = 1 + 2$ ,  $4 = 1 + 3$ ,  $5 = 1 + 4$ ,  $5 = 2 + 3$ ,  $6 = 1 + 5$ ,  $6 = 2 + 4$ , or it is  $\{3, 4, 5, 6\}$ , where  $3 + 6 = 4 + 5$ . The sixteen subset sums of  $\{3, 5, 6, 7\}$  are 0, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 21.

(b) Write  $b = a + 1$  and  $c = a + 3$ ; the six numbers are distinct because  $a > 3$ . Every four-element subset contains one of the triples  $\{1, 2, 3\}$ ,  $\{1, a, b\}$ ,  $\{3, a, c\}$ ,  $\{2, b, c\}$ , with the relations  $3 = 1 + 2$ ,  $b = 1 + a$ ,  $c = 3 + a$ ,  $c = 2 + b$ , or it is one of  $\{1, 2, a, c\}$ ,  $\{1, 3, b, c\}$ ,  $\{2, 3, a, b\}$ , with the relations  $c = 1 + 2 + a$ ,  $1 + c = 3 + b$ ,  $2 + b = 3 + a$ . (Indeed, a four-element subset  $T$  that does not contain  $\{1, 2, 3\}$  contains either exactly two of 1, 2, 3 and two of  $a, b, c$ , and each of these nine sets is one of the last three sets or contains one of the triples  $\{1, a, b\}$ ,  $\{3, a, c\}$ ,  $\{2, b, c\}$ ; or it is  $\{x, a, b, c\}$  with  $x \in \{1, 2, 3\}$ , which contains  $\{1, a, b\}$ ,  $\{2, b, c\}$  or  $\{3, a, c\}$ .) For  $a = 4$  the set is  $\{1, 2, 3, 4, 5, 7\}$ , and two members of the family with  $a \neq a'$  are not similar, since 1, 2, 3 are the three smallest elements of both.  $\square$

The fifteen relations of (b) are identities in  $a$ ; part (ii) of `check_note.py` checks them symbolically and checks (a) by enumeration.

#### 3.2 First proof: sign patterns and Farkas certificates

Fix  $N$  and  $K \geq 2$  and consider unknown reals  $0 < p_1 < \dots < p_N$ . By Lemma 2.1(a), a  $K$ -subset  $\{p_j : j \in Q\}$  is not dissociated if and only if some vector  $e \in \{-1, 0, 1\}^N$ ,  $e \neq 0$ , supported in  $Q$ , satisfies  $\sum_j e_j p_j = 0$ . We call such an  $e$  a relation on  $Q$ , and we normalize it so that its last nonzero entry is +1.

**Lemma 3.2.** *Let  $0 < x_1 < \dots < x_K$  and let  $e \in \{-1, 0, 1\}^K$  be a normalized relation,  $\sum_t e_t x_t = 0$ . Then some suffix sum  $s_t = \sum_{u \geq t} e_u$  is negative. For  $K = 4$  and  $a < b < c < d$  the normalized vectors with a negative suffix sum are exactly the six relations*

$$c = a + b, \quad d = a + b, \quad d = a + c, \quad d = b + c, \quad d = a + b + c, \quad a + d = b + c.$$

We call these vectors the *order-feasible patterns*; for  $K = 3, 4, 5, 6, 7$  there are 1, 6, 26, 98, 344 of them.

*Proof.* With  $x_0 = 0$ , summation by parts gives  $\sum_t e_t x_t = \sum_t s_t (x_t - x_{t-1})$ . If  $t^*$  is the position of the last nonzero entry, then  $s_{t^*} = 1$ , and if all  $s_t \geq 0$  the sum is at least  $x_{t^*} - x_{t^*-1} > 0$ . The list for  $K = 4$  is obtained by running through the 40 normalized vectors; part (i) of `check_note.py` does this and also counts the patterns for  $K \leq 7$ .  $\square$

**Lemma 3.3.** *Let  $E \subset \mathbb{Q}^N$  be a finite set of vectors, let  $y \in \mathbb{Q}^N$  with  $y \geq 0$  and  $y \neq 0$ , and put  $v_i = y_i - y_{i+1}$  ( $1 \leq i \leq N$ ,  $y_{N+1} := 0$ ). If  $v = \sum_k \lambda_k r_k$  with  $r_k \in E$  and  $\lambda_k \in \mathbb{Q}$ , then no reals  $0 < p_1 < \dots < p_N$  satisfy  $r \cdot p = 0$  for all  $r \in E$ .*

*Proof.* For such  $p$  we would have  $v \cdot p = \sum_k \lambda_k r_k \cdot p = 0$ , but  $v \cdot p = \sum_i (y_i - y_{i+1}) p_i = \sum_i y_i (p_i - p_{i-1}) > 0$  with  $p_0 := 0$ .  $\square$

By Farkas' lemma such a certificate  $y$  exists whenever the system has no solution; we use only the trivial direction proved above. Passing to the differences  $p_i - p_{i-1}$  of consecutive elements, as in Lemmas 3.2 and 3.3, is also the starting point of Bohman's analysis of sets with distinct subset sums [Boh98, Lemma 1.1].

**Definition 3.4.** A *closed tree* for  $(N, K)$  is a finite rooted tree whose nodes carry lists  $E$  of relations, with the empty list at the root, such that every node carries a  $K$ -subset  $Q \subseteq \{1, \dots, N\}$  and has one case for each order-feasible pattern  $e$  of  $Q$  (the pattern placed on the positions of  $Q$ ); each case is either *refuted*, by a vector  $y$  as in Lemma 3.3 for the list  $E \cup \{e\}$ , or *continued* by a child node whose list is  $E \cup \{e\}$ .

**Proposition 3.5.** *If a closed tree for  $(N, K)$  exists, then every set of  $N$  positive reals has a dissociated subset of size  $K$ ; that is,  $g^+(K) \leq N - 1$ .*

*Proof.* Suppose that  $0 < p_1 < \dots < p_N$  have no dissociated  $K$ -subset. We walk down the tree from the root, keeping the property that  $p$  satisfies all relations of the current node; this holds at the root. At a node with subset  $Q$ , the set  $\{p_j : j \in Q\}$  carries a normalized relation, which by Lemma 3.2 (applied to the elements of  $Q$  in increasing order) is one of the order-feasible patterns  $e$  of  $Q$ . So  $p$  satisfies  $E \cup \{e\}$ , the case  $e$  is not refuted by Lemma 3.3, and we move to its child. This walk never stops, which is impossible in a finite tree.  $\square$

*The tree for  $N = 7$ ,  $K = 4$ .* The file `original/outputs/tree_K4_N7.jsonl` of the reproducibility package is a closed tree for  $(7, 4)$ . It has 30 nodes (1, 6, 9, 12 and 2 at depths 0 to 4), 29 continued and 151 refuted cases; the certificates have denominators at most 13 and between one and six nonzero entries. Table 2 lists the nodes. At the root all six cases are continued: a single relation is never refuted, because for  $E = \{e\}$  the vector  $y$  would be a multiple of the suffix sums of  $e$ , and these have both signs. An example of a refuted case is the first case at node 3. The relations of node 3 are  $p_6 = p_1 + p_2 + p_3$ ,  $p_3 = p_1 + p_2$  and  $p_7 = p_3 + p_5 + p_6$ , and the case is  $p_7 = p_1 + p_2 + p_4$ . The certificate is  $y = (\frac{4}{7}, \frac{2}{7}, 0, 0, \frac{1}{7}, 0, 0)$ , so  $v = (\frac{2}{7}, \frac{2}{7}, 0, -\frac{1}{7}, \frac{1}{7}, 0, 0)$ , and indeed  $v = -\frac{1}{7}r_1 - \frac{2}{7}r_2 - \frac{1}{7}r_3 + \frac{1}{7}e$  for the four relations  $r_1, r_2, r_3, e$  written as  $p_6 - p_1 - p_2 - p_3, \dots, p_7 - p_1 - p_2 - p_4$ . In words: the relations give  $p_7 = 3(p_1 + p_2) + p_5$ , and with the case  $p_4 = 2(p_1 + p_2) + p_5 > p_5$ , contrary to  $p_4 < p_5$ .

| node | par. | relation added          | $Q$              | cont. | ref. | node | par. | relation added          | $Q$              | cont. | ref. |
|------|------|-------------------------|------------------|-------|------|------|------|-------------------------|------------------|-------|------|
| 0    | —    | —                       | $\{1, 2, 3, 6\}$ | 6     | 0    | 15   | 14   | $p_2 + p_7 = p_5 + p_6$ | $\{1, 3, 6, 7\}$ | 1     | 5    |
| 1    | 0    | $p_6 = p_1 + p_2 + p_3$ | $\{1, 2, 3, 7\}$ | 1     | 5    | 16   | 15   | $p_6 = p_1 + p_3$       | $\{1, 2, 4, 5\}$ | 0     | 6    |
| 2    | 1    | $p_3 = p_1 + p_2$       | $\{3, 5, 6, 7\}$ | 5     | 1    | 17   | 10   | $p_1 + p_7 = p_2 + p_4$ | $\{2, 4, 5, 6\}$ | 1     | 5    |
| 3    | 2    | $p_7 = p_3 + p_5 + p_6$ | $\{1, 2, 4, 7\}$ | 0     | 6    | 18   | 17   | $p_2 + p_6 = p_4 + p_5$ | $\{1, 2, 4, 6\}$ | 0     | 6    |
| 4    | 2    | $p_7 = p_3 + p_5$       | $\{1, 2, 4, 7\}$ | 0     | 6    | 19   | 0    | $p_6 = p_1 + p_3$       | $\{1, 4, 5, 6\}$ | 1     | 5    |
| 5    | 2    | $p_7 = p_3 + p_6$       | $\{1, 2, 4, 7\}$ | 0     | 6    | 20   | 19   | $p_1 + p_6 = p_4 + p_5$ | $\{1, 2, 4, 6\}$ | 0     | 6    |
| 6    | 2    | $p_7 = p_5 + p_6$       | $\{1, 2, 4, 7\}$ | 0     | 6    | 21   | 0    | $p_6 = p_2 + p_3$       | $\{2, 3, 4, 5\}$ | 1     | 5    |
| 7    | 2    | $p_3 + p_7 = p_5 + p_6$ | $\{1, 2, 4, 7\}$ | 0     | 6    | 22   | 21   | $p_2 + p_5 = p_3 + p_4$ | $\{2, 3, 5, 7\}$ | 4     | 2    |
| 8    | 0    | $p_6 = p_1 + p_2$       | $\{1, 3, 4, 5\}$ | 1     | 5    | 23   | 22   | $p_7 = p_2 + p_3 + p_5$ | $\{1, 2, 3, 7\}$ | 1     | 5    |
| 9    | 8    | $p_1 + p_5 = p_3 + p_4$ | $\{1, 2, 3, 5\}$ | 0     | 6    | 24   | 23   | $p_3 = p_1 + p_2$       | $\{1, 2, 4, 7\}$ | 0     | 6    |
| 10   | 0    | $p_7 = p_1 + p_2$       | $\{1, 2, 4, 7\}$ | 4     | 2    | 25   | 22   | $p_7 = p_2 + p_5$       | $\{4, 5, 6, 7\}$ | 0     | 6    |
| 11   | 10   | $p_7 = p_1 + p_2 + p_4$ | $\{3, 5, 6, 7\}$ | 1     | 5    | 26   | 22   | $p_7 = p_3 + p_5$       | $\{2, 3, 4, 7\}$ | 0     | 6    |
| 12   | 11   | $p_3 + p_7 = p_5 + p_6$ | $\{4, 5, 6, 7\}$ | 0     | 6    | 27   | 22   | $p_2 + p_7 = p_3 + p_5$ | $\{2, 4, 5, 6\}$ | 0     | 6    |
| 13   | 10   | $p_7 = p_1 + p_4$       | $\{1, 4, 5, 6\}$ | 0     | 6    | 28   | 0    | $p_1 + p_6 = p_2 + p_3$ | $\{2, 4, 5, 6\}$ | 1     | 5    |
| 14   | 10   | $p_7 = p_2 + p_4$       | $\{2, 5, 6, 7\}$ | 1     | 5    | 29   | 28   | $p_2 + p_6 = p_4 + p_5$ | $\{2, 3, 4, 6\}$ | 0     | 6    |

Table 2: The closed tree for  $N = 7$ ,  $K = 4$ . For each node: its parent, the relation added at its parent (the case leading to it), the subset  $Q$  of indices on which it branches, and the numbers of continued and refuted cases. The relations of a node are those added along the path from the root. The certificates are in the file `tree_K4_N7.jsonl`.

*How the tree was found, and how it is checked.* The tree was produced by a search program (Section 6) which uses floating-point linear programming only to choose the subsets  $Q$  and to propose certificates. Each proposed certificate is converted to rationals and verified exactly before it is used, and a case that cannot be certified is continued, so that the program can only make the tree larger. The program need not be trusted: the tree is a finite object, and Proposition 3.5 applies as soon as the conditions of Definition 3.4 are verified. Three exact checkers verify them: the checker written with the first version of the results (`check_tree.py`, which tests  $v \in \text{span}(E \cup \{e\})$  by comparing ranks), the checker of verification run A, and part (i) of `check_note.py`, written for this note. The last two compute explicit rational multipliers  $\lambda_k$  for all 151 certificates and check the identity  $v = \sum_k \lambda_k r_k$  coordinate by coordinate. All three recompute the list of order-feasible patterns, require every case to be listed exactly once at every node, and check that each child assumes no relation other than those of its parent and the case. Eleven deliberately corrupted trees (with a negative, altered or zero certificate, a deleted or duplicated case, a false certificate for a continued case, a changed relation, a case that is not order-feasible, a changed subset  $Q$ , or an open leaf) are rejected by each of the three checkers. The search program regenerates the file byte for byte in a fraction of a second. A second version of the program produces a different closed tree for  $N = 7$  (56 nodes, 281 certificates), and both versions close  $N = 8$ ; for  $N = 6$  the search ends with exact solution points, among them a multiple of  $(1, 2, \dots, 6)$ , as it must since  $g^+(4) \geq 6$ .

### 3.3 Second proof: a dissociated triple and linear forms

This proof was found by verification run A (Section 6). Let  $\ell_1, \ell_2, \ell_3$  be the unit vectors of  $\mathbb{Z}^3$  and let  $L$  be the set of the ten vectors of  $\{-1, 0, 1\}^3$  with at least two nonzero entries and first nonzero entry  $+1$ . For a 4-subset  $Q$  of indices and vectors  $\ell_i$  ( $i \in Q$ ) put

$$W_Q = \left\{ \sum_{i \in Q} \varepsilon_i \ell_i : \varepsilon \in \{-1, 0, 1\}^Q, \varepsilon \neq 0 \right\} \subset \mathbb{Z}^3.$$

**Lemma 3.6.** *Let  $P$  be a set of  $N \geq 4$  positive reals without a dissociated four-element subset. Then  $P$  contains a dissociated triple  $\{a, b, c\}$ , and with  $d = (a, b, c)$  there are distinct  $\ell_4, \dots, \ell_N \in L$  such*

that  $P = \{|\ell_i \cdot d| : 1 \leq i \leq N\}$ . Let  $U$  be the  $\mathbb{Q}$ -linear span of all  $w \in W_Q$ , over all 4-subsets  $Q \subseteq \{1, \dots, N\}$ , with  $w \cdot d = 0$ . Then  $\dim U \leq 2$  and

(C1)  $U$  contains no nonzero vector of  $\{-1, 0, 1\}^3$ ;

(C2)  $U$  contains none of the vectors  $\ell_i \pm \ell_j$ ,  $1 \leq i < j \leq N$ ;

(C3) for every 4-subset  $Q \subseteq \{1, \dots, N\}$ , the set  $W_Q$  contains 0 or meets  $U$ .

*Proof.* The triple exists because  $g^+(3) = 3 < N$  (Lemma 2.5). For  $x \in P \setminus \{a, b, c\}$  the set  $\{a, b, c, x\}$  is not dissociated, so  $x \in \Sigma^\pm(\{a, b, c\})$  by Lemma 2.1(d), that is,  $x = \pm \ell \cdot d$  for a vector  $\ell \in \{-1, 0, 1\}^3$  with first nonzero entry  $+1$ . This  $\ell$  is not a unit vector, since  $x \notin \{a, b, c\}$ , so  $\ell \in L$ , and different  $x$  give different  $\ell$ . Hence  $P = \{|\ell_i \cdot d|\}$ . Every vector of  $U$  is orthogonal to  $d \neq 0$ , so  $\dim U \leq 2$ . (C1) holds because  $\varepsilon \cdot d \neq 0$  for every nonzero  $\varepsilon \in \{-1, 0, 1\}^3$ , as  $\{a, b, c\}$  is dissociated; (C2) holds because  $|\ell_i \cdot d| \neq |\ell_j \cdot d|$  for  $i \neq j$ . For (C3), the four numbers  $|\ell_i \cdot d|$ ,  $i \in Q$ , carry a relation  $\sum_{i \in Q} \eta_i |\ell_i \cdot d| = 0$ ; with  $\sigma_i$  the sign of  $\ell_i \cdot d$  this says  $w \cdot d = 0$  for  $w = \sum_{i \in Q} \eta_i \sigma_i \ell_i \in W_Q$ , and  $w \in U$  unless  $w = 0$ .  $\square$

Conversely, if  $\ell_4, \dots, \ell_N \in L$  are distinct and a subspace  $U \subseteq \mathbb{Q}^3$  of dimension at most 2 satisfies (C1)–(C3), then every  $d \in \mathbb{R}^3$  orthogonal to  $U$  and outside the finitely many planes  $f^\perp$ , for  $f$  a vector excluded from  $U$  by (C1) or (C2), gives a set  $\{|\ell_i \cdot d|\}$  of  $N$  positive reals without a dissociated four-element subset. We do not need this direction for the proof; it explains why the procedure below finds genuine examples for  $N \leq 6$ .

*The enumeration.* Fix distinct  $\ell_4, \dots, \ell_N \in L$ , a *choice of forms*. Call a subspace  $U \subseteq \mathbb{Q}^3$  with  $\dim U \leq 2$  *admissible* if it satisfies (C1)–(C3). Let  $\mathcal{L}$  be the list of the 4-subsets  $Q$  with  $0 \notin W_Q$ , ordered by the number of lines  $\mathbb{Q}w$ ,  $w \in W_Q$ . If  $\mathcal{L}$  is empty,  $U = \{0\}$  is admissible. Otherwise let  $Q_0$  be the first element of  $\mathcal{L}$ . The procedure runs over the lines  $\mathbb{Q}w_0$  with  $w_0 \in W_{Q_0}$ . If the line violates (C1) or (C2), it is discarded. If it satisfies (C3), it is reported as admissible. Otherwise let  $Q_1$  be the first element of  $\mathcal{L}$  for which  $W_{Q_1}$  does not meet  $\mathbb{Q}w_0$ ; the procedure runs over the planes  $\mathbb{Q}w_0 + \mathbb{Q}w_1$  with  $w_1 \in W_{Q_1}$  and reports those that satisfy (C1)–(C3).

**Proposition 3.7.** *If a choice of forms has an admissible subspace, the procedure reports at least one admissible subspace for it.*

*Proof.* Let  $U$  be admissible. If  $\mathcal{L}$  is empty there is nothing to show. Otherwise, by (C3) and  $0 \notin W_{Q_0}$ , the space  $U$  contains a nonzero  $w_0 \in W_{Q_0}$ . Conditions (C1) and (C2) pass from  $U$  to its subspace  $\mathbb{Q}w_0$ , so this line is not discarded. If it satisfies (C3), it is reported. Otherwise  $U$  contains some  $w_1 \in W_{Q_1}$ , by (C3) for  $Q_1$ , and  $w_1 \notin \mathbb{Q}w_0$  by the choice of  $Q_1$ . Since  $\dim U \leq 2$ , we get  $U = \mathbb{Q}w_0 + \mathbb{Q}w_1$ , which is one of the planes tested, and it is reported.  $\square$

By Lemma 3.6 and Proposition 3.7, if the procedure reports nothing for any of the  $\binom{10}{N-3}$  choices of forms, then every set of  $N$  positive reals has a dissociated four-element subset. The procedure is implemented in exact integer arithmetic; lines and planes are represented by primitive integer vectors and normal vectors, so all tests are integer dot products. For  $N = 7$  the 210 choices of forms, and for  $N = 8$  the 252 choices, give no admissible subspace; the computation takes about two seconds. As a control, for  $N = 5$  and  $N = 6$  the procedure reports 588 and 1692 admissible subspaces, and each of them is realized by an integer vector  $d$  and checked directly, which yields for instance  $\{1, \dots, 5\}$ ,  $\{1, \dots, 6\}$  and  $\{1, 2, 3, 4, 5, 7\}$ . Condition (C2) is essential: without it, verification run A found 13,447 spurious configurations for  $N = 7$ , in all of which two of the numbers  $|\ell_i \cdot d|$  coincide. The same run also compared the procedure with exhaustive searches: each of the 134,158 five-element and 38,006 six-element subsets of  $\{1, \dots, 80\}$  without a dissociated four-element subset is explained by a reported configuration. This comparison tests the program; it is not part of the proof.

Verification run B confirmed the theorem by a related method, also starting from a dissociated triple: a case analysis on the linear equations satisfied by  $d$ , in which a branch is closed when the current subspace of possible  $d$  lies in a plane excluded by (C1) or (C2) (for  $N = 7$ : 210 choices of forms, 237,323 nodes, maximal depth 3, no solution), with its own checker.

### 3.4 Conclusion of the proof

*Proof of Theorem 1.2.* By Proposition 3.1(a),  $g^+(4) \geq 6$ . By Proposition 3.5 applied to the closed tree of Section 3.2, or by Lemma 3.6 and Proposition 3.7 with the enumeration of Section 3.3 for  $N = 7$ , every set of seven positive reals has a dissociated four-element subset, so  $g^+(4) \leq 6$ . Finally  $g(4) = 1 + 2g^+(4) = 13$  by Lemma 2.2.  $\square$

Both proofs are computer-assisted. The mathematics on which they rest is Lemmas 3.2, 3.3 and 3.6 and Propositions 3.5 and 3.7; what remains is a finite computation in exact arithmetic, whose data and programs are in the reproducibility package.

## 4 An equivalent form for integer vectors

The definitions of a dissociated set and of  $\dim$  make sense in every abelian group: a finite set is dissociated if its subset sums are pairwise distinct, and  $\dim P$  is the largest size of a dissociated subset of  $P$ . In  $\mathbb{Z}^r$ , as in  $\mathbb{R}$ , a set is dissociated if and only if it carries no relation  $\sum_p \varepsilon_p p = 0$  with  $\varepsilon \in \{-1, 0, 1\}^P \setminus \{0\}$ ; the proof of Lemma 2.1 applies verbatim.

**Proposition 4.1.** *The bound (1) holds for all  $n \geq 1$  if and only if*

( $\star$ ) *for every  $r \geq 1$  and every finite set  $P \subset \mathbb{Z}^r \setminus \{0\}$  that contains no pair  $x, -x$ ,*

$$|P| \leq 2^{\dim P} - 1.$$

*Proof.* By Lemma 2.3, (1) for all  $n$  is equivalent to  $|P| \leq 2^{\dim P} - 1$  for every finite set  $P \subset (0, \infty)$ .

Assume ( $\star$ ) and let  $P \subset (0, \infty)$  be finite. Choose a basis of the  $\mathbb{Q}$ -span  $V$  of  $P$ , let  $c: V \rightarrow \mathbb{Q}^r$  be the coordinate map, and multiply by a common denominator of the coordinates of the elements of  $P$ , so that  $c(P) \subset \mathbb{Z}^r$ . The map  $c$  is  $\mathbb{Q}$ -linear and injective, so a vector  $\varepsilon$  is a relation on a subset of  $P$  if and only if it is a relation on its image; hence  $|c(P)| = |P|$  and  $\dim c(P) = \dim P$ . Moreover  $c(p) \neq 0$ , and  $c(p) = -c(p')$  would give  $p = -p'$ , which is impossible for positive  $p, p'$ . So ( $\star$ ) gives  $|P| \leq 2^{\dim P} - 1$ .

Conversely, assume the inequality for finite sets of positive reals, and let  $P \subset \mathbb{Z}^r \setminus \{0\}$  contain no pair  $x, -x$ . Choose reals  $\alpha_1, \dots, \alpha_r$  that are linearly independent over  $\mathbb{Q}$  and put  $\varphi(x) = \sum_i x_i \alpha_i$ ; then  $\varphi$  is injective on  $\mathbb{Q}^r$ . The numbers  $\varphi(p)$ ,  $p \in P$ , are nonzero and pairwise distinct in absolute value, since  $\varphi(p) = \pm \varphi(p')$  would give  $p = \pm p'$ . A vector  $\varepsilon$  is a relation on a subset of  $P$  if and only if it is a relation on the corresponding numbers  $\varphi(p)$ , and by Lemma 2.1(c) also on the numbers  $|\varphi(p)|$ . So  $P' = \{|\varphi(p)| : p \in P\}$  is a set of  $|P|$  positive reals with  $\dim P' = \dim P$ , and  $|P| = |P'| \leq 2^{\dim P} - 1$ .  $\square$

*Remark 4.2.* (a) In ( $\star$ ) one may assume  $r \leq \dim P$ : a  $\mathbb{Q}$ -basis of the span of  $P$  chosen from  $P$  is linearly independent, hence dissociated, so the rank of  $P$  is at most  $\dim P$ , and the coordinate map of the proof reduces to this case.

(b) The analogue of ( $\star$ ) in the group  $\mathbb{F}_2^r$  is true and sharp, and trivial: there a set is dissociated if and only if it is linearly independent over  $\mathbb{F}_2$ , so  $|P| \leq |\text{span } P \setminus \{0\}| = 2^{\dim P} - 1$  for  $P \subseteq \mathbb{F}_2^r \setminus \{0\}$ , with equality for  $P = \mathbb{F}_2^r \setminus \{0\}$  (the condition on pairs  $x, -x$  is empty because  $x = -x$ ). Erdős's

question asks whether the same bound holds over  $\mathbb{Z}$ , where dependence is measured with coefficients in  $\{-1, 0, 1\}$ .

(c) The argument of Lemma 2.4 reads as follows in this language: if  $B \subseteq P$  is a maximal dissociated subset, every  $p \in P \setminus B$  lies in  $\Sigma^\pm(B)$  by Lemma 2.1(d), and  $P$  contains at most one element of each pair  $x, -x$ ; hence  $|P| \leq (3^{|B|} - 1)/2 \leq (3^{\dim P} - 1)/2$ .

Proposition 4.1 is a reformulation and not progress towards the bound. It shows that (1) is a statement about the  $\{-1, 0, 1\}$ -relations among the elements alone: the order and the size of the numbers play no role. Unlike linearly independent sets, dissociated sets do not form a matroid:  $\{1, 2, 4\}$  and  $\{3, 5, 6, 7\}$  are both maximal dissociated subsets of  $\{1, \dots, 7\}$ .

## 5 Consequences, and the cases $K = 5$ and $K = 6$

**Proposition 5.1.** *The following explicit sets give lower bounds for  $g^+(5)$  and  $g^+(6)$ .*

- (a) *The set  $A^* = \{1, 2, \dots, 10, 12, 13, 15\}$  has no dissociated subset of five elements. Hence  $g^+(5) \geq 13$ ,  $g(5) \geq 27$ , and  $f(n) \leq 4$  for  $n \leq 27$ .*
- (b) *The set  $S_{24} = \{1, 2, \dots, 21, 24, 25, 27\}$  has no dissociated subset of six elements. Hence  $g^+(6) \geq 24$ ,  $g(6) \geq 49$ , and  $f(n) \leq 5$  for  $n \leq 49$ .*

*Proof.* Part (iii) of `check_note.py` runs through the 1287 five-element subsets of  $A^*$  and the 134,596 six-element subsets of  $S_{24}$  and finds a repeated subset sum in each; it also checks directly that the 27 numbers  $\{0\} \cup A^* \cup (-A^*)$  have no dissociated five-element subset. The consequences follow from Lemmas 2.2 and 2.3.  $\square$

These two bounds are proved by exhibition: each is a finite exact check of an explicit set. The set  $A^*$  was posted by BAKKAOUI [BAK26];  $S_{24}$  was found by our computer search in boxes (Remark 5.2).

*Proof of Corollary 1.3.* By Lemmas 2.2 and 2.5 and Theorem 1.2,  $g(1), \dots, g(4)$  are 1, 3, 7, 13, and  $g(5) \geq 27$  by Proposition 5.1. By Lemma 2.3,  $f(n) = \max\{K : g(K) \leq n - 1\}$ , which gives (a): for instance  $f(n) = 4$  for  $14 \leq n \leq 27$  because  $g(4) = 13 \leq n - 1 < 27 \leq g(5)$ . For (b),  $f(n) \geq 4$  for all  $n \geq 14$ , and  $\lfloor \log_2 n \rfloor \leq 4$  for  $n \leq 31$ ; comparing with (a) gives the cases of equality. For (c),  $f(32) \geq 5$  means  $g(5) \leq 31$ , that is,  $g^+(5) \leq 15$  by Lemma 2.2; and  $13 \leq g^+(5) \leq 40$  by Proposition 5.1 and Lemma 2.4.  $\square$

For  $n \geq 28$  the value of  $f(n)$  depends on  $g^+(5)$ : by Lemma 2.3,  $f(n) \geq 5$  if and only if  $n \geq 2g^+(5) + 2$ , and  $f(n) \in \{4, 5\}$  for  $28 \leq n \leq 49$  by Proposition 5.1(b).

*Intervals.* Let  $u_K$  be the smallest possible largest element of a set of  $K$  positive integers with distinct subset sums (for these numbers and the Conway–Guy sequence see [Boh96, Boh98]). The interval  $\{1, \dots, m\}$  has a dissociated subset of size  $K$  if and only if  $m \geq u_K$ , so  $g^+(K) \geq u_K - 1$ . From Lemma 2.5, Proposition 3.1(a) and part (iii) of `check_note.py` (the intervals  $\{1, \dots, 12\}$  and  $\{1, \dots, 23\}$  have no dissociated subsets of five and six elements, respectively, while  $\{3, 6, 11, 12, 13\}$  and  $\{11, 17, 20, 22, 23, 24\}$  are dissociated) we get  $u_1, \dots, u_6 = 1, 2, 4, 7, 13, 24$ . So intervals are extremal for  $K \leq 4$ , while  $A^*$  and  $S_{24}$  beat them for  $K = 5$  and  $K = 6$  ( $13 > 12$  and  $24 > 23$ ). That intervals are not extremal for  $K = 5$  was observed by BAKKAOUI [BAK26].

*Remark 5.2* (Searches in boxes; tested, not used in any proof). Exhaustive computer searches over all subsets of  $\{1, \dots, M\}$  without a dissociated  $K$ -subset gave the following. For  $K = 5$  the largest size is 13; for  $M = 36$  it is attained only by  $A^*$  and  $2A^*$  (two different programs agree), for  $M = 48$  only by  $A^*, 2A^*, 3A^*$ , and for  $M = 60$  only by  $A^*, 2A^*, 3A^*, 4A^*$ . For  $K = 4$  and  $M = 80$  the largest size is 6, attained by 38,006 sets. For  $K = 6$  and  $M = 28$  the searches found  $S_{24}$  and nothing larger,

but they did not finish within their time limits. These are statements about bounded boxes only. By Remark 2.6(a) every finite configuration is realized by positive integers, but not within a prescribed box, so such searches give lower bounds for  $g^+(K)$  and no upper bounds.

*The case  $K = 5$ .* The first open case of (1) is  $g^+(5) \leq 15$ , and the exact value of  $g^+(5)$  would determine  $f(n)$  for all  $n \leq 49$ . The method of Section 3.2 has not decided it: runs of the search program for  $N = 14$  (which would give  $g^+(5) = 13$ ) and  $N = 16$  were stopped by their time limit of 1750 seconds each, without a solution node and without an uncertified case, but unfinished; an unfinished tree proves nothing. The method of Section 3.3 extends as follows. Let  $P \subset (0, \infty)$  have no dissociated five-element subset and  $|P| \geq 7$ . By Theorem 1.2,  $P$  contains a dissociated four-element subset  $D$ ; let  $d \in \mathbb{R}^4$  list its elements. As in Lemma 3.6, every  $x \in P \setminus D$  equals  $|\ell \cdot d|$  for one of the 36 vectors  $\ell \in \{-1, 0, 1\}^4$  with first nonzero entry  $+1$  and at least two nonzero entries. So  $P$  is described by  $d$  and a set  $F$  consisting of the four unit vectors and some of these 36 vectors, subject to three conditions: (C1)  $\varepsilon \cdot d \neq 0$  for all 40 nonzero vectors  $\varepsilon \in \{-1, 0, 1\}^4$  with first nonzero entry  $+1$ , because  $D$  is dissociated; (C2)  $(\ell \pm \ell') \cdot d \neq 0$  for distinct  $\ell, \ell' \in F$ , because the elements of  $P$  are distinct; (C3) every five-element subset of  $F$  has a combination  $w = \sum c_i \ell_i$  with  $c \in \{-1, 0, 1\}^5 \setminus \{0\}$  and  $w \cdot d = 0$ . The subspaces involved have dimension at most 3, but the number of sets  $F$  is far larger than for  $K = 4$ , and we have not carried out this computation.

## 6 Computations and reproducibility

All lemmas and propositions of the note are proved in the text, except that the proof of Proposition 5.1 is a finite exact computation of the package. Theorem 1.2 needs in addition one of two finite computations in exact arithmetic, and Propositions 3.1 and 5.1 are finite exact checks of explicit sets. The reproducibility package contains four groups of programs; their command lines, running times and recorded outputs are listed in its file `README.md`.

*The program written for the note, `check_note.py`,* uses the standard library only, runs in about three seconds on one core of a laptop and writes no files (on request it exports the eleven corrupted trees, so that the other two checkers can be run on them). Its part (i) verifies the closed tree for  $N = 7$  with explicit multipliers for all 151 certificates, the three other closed trees and the control  $N = 6$ , and rejects the eleven corrupted versions of the tree; part (ii) checks Proposition 3.1, with the fifteen relations of (b) as identities in  $a$ , and the set  $\{0, \pm 1, \dots, \pm 6\}$ ; part (iii) checks Proposition 5.1, the set  $\{0\} \cup A^* \cup (-A^*)$  and the intervals; part (iv) computes the values of  $f(n)$  for  $n \leq 31$  (Corollary 1.3); part (v) runs the enumeration of Section 3.3 for  $N = 5, \dots, 8$  and realizes the configurations found for  $N = 5, 6$ .

*The programs with which the results were first obtained (original/)* include the search program in two versions, which need `numpy` and `scipy`. The first version, `dissoc_lp_search_v1_no_nogoods.py`, produced `tree_K4_N7.jsonl`; run with the arguments `--N 7 --K 4 --out newfile --quiet`, it regenerates the file byte for byte in about 0.1 seconds and prints `nodes=30 lps=199 certs=151 uncertified=0 solutions=0`. The second version, `dissoc_lp_search.py`, which also stores infeasible combinations of relations, produced the second tree for  $N = 7$  and the unfinished runs for  $K = 5$ . The folder also contains the checker `check_tree.py` with a corruption test and the box search of Remark 5.2.

*The programs of the two verification runs (verification\_run\_A/, verification\_run\_B/)* include the enumeration of Section 3.3 (`vA_forms_K4.py`, about four seconds), the checker of run A, and the reduction of run B with its checker ( $K = 4$ ,  $N = 7$  in about 16 seconds); the large proof logs of run B are not included and are regenerated by a command given in the `README.md`. The programs of the third verification run (`independent_run_2/`) are described in the next paragraph.

**Verification.** The results were first obtained, with proofs, with the programs in `original/`. Three independent verification runs, all AI-assisted, followed; each wrote its own programs, and none found a mathematical error. The first two examined the first written version of the results, the third examined the final text of this note.

The first run re-derived the lemmas of the first version (Lemma 2.1(a)–(c), Lemmas 2.2 to 2.5, 3.2 and 3.3), Proposition 4.1 and the soundness of the case analysis. It verified the tree for  $N = 7$  with its own checker, which computes explicit multipliers for all 151 certificates, and with the checker of the first version; both checkers rejected five corrupted trees of its own. It found the second proof of Section 3.3 and compared that enumeration with exhaustive searches up to 80. It confirmed the sets  $\{1, \dots, 6\}$ ,  $A^*$  and  $S_{24}$  and the box searches of Remark 5.2, regenerated the tree byte for byte, and compared the statement with [Erd65, p. 188], the problem page, the database entry [teo26] and the forum thread.

The second run confirmed  $g^+(3) \leq 3$  and  $g^+(4) \leq 6$  by the related reduction described at the end of Section 3.3, with its own exact checker and six corruption tests. It regenerated the trees of the first version byte for byte and reproduced the verdicts of its checker and its recorded outputs; it checked the explicit sets and the values of  $f(n)$  for  $n \leq 31$ , and it repeated the search of the literature and checked the bibliographic data.

The two runs required corrections of wording and of sources, all of which were made: the bound  $f(13) \leq 4$  of [BAK26] is an upper bound, and what  $A^*$  gives in our reading is  $g^+(5) \geq 13$ ; the bounds  $g^+(5) \geq 13$  and  $g^+(6) \geq 24$  are proved by exhibition, while the statements about boxes are only tested; the reading with positive integers is explained; the status is taken from the database [teo26]; the forum argument is dated 5 December 2025; negative statements about the literature are given as results of searches; the program version and the command that produce the tree are named; the sets of six positive reals without a dissociated four-element subset are not unique; the conditions of the reduction for  $K = 5$  are stated; and several bibliographic details. After these runs the note added Lemma 2.1(d), the family of Proposition 3.1(b), the identity  $f(n) = f^+(\lfloor n/2 \rfloor)$  of Remark 2.6(b), the bound  $f(n) \leq 5$  for  $n \leq 49$  and the written-out second proof of Section 3.3. The computations of the note were repeated with the exact program of this note, `check_note.py`; its output is recorded in the package.

The third run examined the final text and the package line by line, as if nothing had been checked, and the additions just listed first. It read [Erd65, p. 188], the problem page, the whole forum thread, the database entry [teo26], the corpus record and [Boh98] again, re-derived every proof, and wrote its own programs before reading `check_note.py`: an exact checker of the trees, which computes explicit multipliers for all 151 certificates, confirms Table 2 row by row and rejects twelve corrupted trees of its own; a second implementation of the proof of Section 3.3, which tests every subspace spanned by vectors of the sets  $W_Q$  instead of branching (no admissible subspace for  $N = 7$  and  $N = 8$ ; for  $N = 5$  and  $N = 6$  all 4,104 and 9,268 admissible subspaces are realized and checked, and every five- and six-element subset of  $\{1, \dots, 40\}$  without a dissociated four-element subset is accounted for); exact checks of all explicit sets, of the values of  $f$  and  $f^+$  and of Table 1; and a box search in  $C$ , which reproduces the counts 134,158 and 38,006 of Section 3.3 and, for  $K = 5$  and  $M = 36$ , the maximum 13, attained only by  $A^*$  and  $2A^*$ . It re-ran the package from the archive: the quick programs, the comparison up to 80 and the box searches for  $K = 5$  up to  $M = 60$  reproduced their recorded outputs (the runs stopped by a time limit were not repeated). It also repeated the search of the literature. It found no error and no gap, and it required corrections of wording, credit and record, all made: the bound of [BAK26] is stated as in the forum post; the status in the corpus record is quoted precisely; the abstract speaks of the terminal cases; the proof of Proposition 5.1 is said to be computational; [Boh98, Lemma 1.1] is credited; the set  $F$  of the reduction for  $K = 5$  is defined precisely; and the runs are recorded accurately here and in the package, whose folder `independent_run_2/` contains

the programs and outputs of the third run.

**Scope and priority.** The note determines  $f(n)$  for  $n \leq 27$  and proves (1) for  $n \leq 31$ . The general question, including its first open case  $n = 32$ , remains open; for  $n \geq 32$  nothing is claimed beyond  $f(n) \geq 4$ , Lemma 2.4 and the upper bound  $f(n) \leq 5$  for  $n \leq 49$ . The problem is Erdős’s [Erd65], and so is the greedy bound of Lemma 2.4; the reductions of Section 2 and Proposition 4.1 are elementary reformulations. The set  $A^*$ , the observation that intervals are not extremal for five-element dissociated subsets, and the bound  $f(13) \leq 4$  are due to BAKKAOUI (forum posts of 3 September 2026) [BAK26]. The argument for  $f(n) \geq (1 - o(1)) \log_2 n$  is due to KoishiChan (forum post of 5 December 2025) [Koi25]; it is unrefereed and is not used here. The works on dissociated sets and on distinct subset sums cited in Section 1.3 [Boh96, Boh98, LY11, SS16, CH15, Shk24, Dut26] are related work, and none of them settles the question. To our knowledge Theorem 1.2 and its consequences in Corollary 1.3 are new: no earlier statement of these exact values was found in the sources searched. These were the problem page and its forum thread, the database [teo26], the arXiv (queries for dissociated sets and subsets, distinct subset sums and additive dimension), zbMATH Open, OpenAlex (also the works citing [Erd65]), Crossref, and three web searches; Semantic Scholar did not answer our requests. The searches were made on 15 October 2026, in part by the verification runs. A search that finds nothing is not a proof of novelty, and we make no claim of priority. What was read: [Erd65, p. 188] in the scan of the Erdős archive of the Rényi Institute; the problem page and the whole forum thread; [Dut26] in full; of [Boh98] the introduction with Lemma 1.1; of the other cited papers, the abstracts. The problem collection of 1999 cited on the problem page was not available to us.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

## References

- [BAK26] BAKKAOUI. Comments on Erdős Problem #963. Forum posts, 3 September 2026, <https://www.erdosproblems.com/forum/thread/963>, 2026. Accessed 15 October 2026.
- [Blo26] Thomas F. Bloom. Erdős Problem #963. <https://www.erdosproblems.com/963>, 2026. Page last edited 23 January 2026; accessed 15 October 2026.
- [Boh96] Tom Bohman. A sum packing problem of Erdős and the Conway–Guy sequence. *Proceedings of the American Mathematical Society*, 124(12):3627–3636, 1996. doi:10.1090/S0002-9939-96-03653-2.
- [Boh98] Tom Bohman. A construction for sets of integers with distinct subset sums. *The Electronic Journal of Combinatorics*, 5:Research Paper R3, 1998. doi:10.37236/1341.
- [CH15] P. Candela and H. A. Helfgott. On the dimension of additive sets. *Acta Arithmetica*, 167(1):91–100, 2015. arXiv:1407.4987, doi:10.4064/aa167-1-5.
- [Dut26] Sayan Dutta. The greedy algorithm for dissociated sets, 2026. Preprint, version 4. arXiv:2601.07068.
- [Erd65] Paul Erdős. Extremal problems in number theory. In *Theory of Numbers*, volume 8 of *Proceedings of Symposia in Pure Mathematics*, pages 181–189. American Mathematical Society, Providence, RI, 1965. doi:10.1090/pspum/008/0174539.

- [Koi25] KoishiChan. Comment on Erdős Problem #963. Forum post, 5 December 2025, <https://www.erdosproblems.com/forum/thread/963>, 2025. Accessed 15 October 2026.
- [LY11] Vsevolod F. Lev and Raphael Yuster. On the size of dissociated bases. *The Electronic Journal of Combinatorics*, 18(1):Paper P117, 2011. [arXiv:1005.0155](https://arxiv.org/abs/1005.0155), doi:10.37236/604.
- [Shk24] Ilya D. Shkredov. Additive dimension and the growth of sets. *Discrete Mathematics*, 347(9):Paper 114077, 2024. [arXiv:2205.07296](https://arxiv.org/abs/2205.07296), doi:10.1016/j.disc.2024.114077.
- [SS16] Tomasz Schoen and Ilya D. Shkredov. Additive dimension and a theorem of Sanders. *Journal of the Australian Mathematical Society*, 100(1):124–144, 2016. [arXiv:1404.2044](https://arxiv.org/abs/1404.2044), doi:10.1017/S1446788715000324.
- [teo26] teorth/erdosproblems. Erdős problems database, file data/problems.yaml, entry 963. <https://github.com/teorth/erdosproblems>, 2026. Status “open”, last update 2025-08-31; accessed 15 October 2026.
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Version 1.6.0; record EP-963.