

Two Congruence Classes of Extremal Twenty-Point Sets

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Abstract

Alexeev, Mixon and Parshall proved that twenty points in the plane have at most 54 unit-distance pairs and that every maximizing set has the same abstract unit-distance graph. We determine its Euclidean realization classes: there are exactly two, even when reflections are allowed. Both are explicit Minkowski sums of the unit diamond and a five-point triangular fan. Cartesian-product squares force every injective realization to have this form, and one additional edge leaves two relative rotations. A fourth distance moment separates the resulting sets. The proof uses the established abstract-graph enumeration and otherwise consists of elementary geometry and exact algebra. This is a finite congruence classification; it does not resolve the asymptotic question in Erdős Problem 668.

1 The two extremal configurations

For a finite set $P \subset \mathbb{R}^2$, let $e(P)$ count its unordered pairs at distance one, and define

$$u(n) = \max_{|P|=n} e(P).$$

Let $C(n)$ be the cardinality of the family of maximizers modulo all Euclidean isometries, including reflections. Erdős asked whether $C(n)$ tends to infinity, and whether $C(n) > 1$ for every $n > 3$; see [4, 5]. The latter assertion already fails at $n = 4$. A unique extremal abstract graph does not imply a unique extremal point set: the missing information is its realization space.

Identify $\mathbb{R}^2$ with $\mathbb{C}$, and put

$$\zeta = \frac{1 + i\sqrt{3}}{2}, \quad D = \{0, 1, \zeta, 1 + \zeta\}, \quad V = \{0, 1, \zeta, \zeta - 1, 1 - \zeta\}. \quad (1)$$

Thus D is a unit diamond and V is a fan of three unit triangles. The identity $\zeta^2 = \zeta - 1$ will be used throughout.

Theorem 1. *There are exactly two congruence classes of twenty-point sets with 54 unit-distance pairs. Consequently $C(20) = 2$. Representatives are*

$$P_\sigma = D + b_\sigma V, \quad b_\sigma = \zeta \frac{5 + \sigma i\sqrt{11}}{6}, \quad \sigma \in \{+1, -1\}. \quad (2)$$

Their fourth distance moments satisfy

$$\sum_{\{p,q\} \subset P_\sigma} |p - q|^4 = \frac{19070 + 160\sigma\sqrt{33}}{9}. \quad (3)$$

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The upper bound $u(20) = 54$ and uniqueness of the abstract extremal graph are due to Alexeev–Mixon–Parshall (AMP) [1, Theorem 1 and Tables 1–3]. They are the external input to Theorem 1; we do not repeat their graph enumeration. The contribution here is the complete congruence classification of that graph. The rhombus relation used below is also a basic move in AMP’s embeddability algorithm. Minkowski-sum constructions are studied in [3] and in AMP’s Appendix A.



Figure 1: The two congruence classes, each with 54 unit pairs. The highlighted segment is the edge added to the 53-edge Cartesian product. Coordinates in the proof are exact; the drawing is illustrative.

2 The graph and its product subgraph

For a finite planar set X , write H_X for its full unit-distance graph. The graphs H_D and H_V have five and seven edges, respectively. On $D \times V$, take the Cartesian product $H_D \square H_V$ and add the edge

$$\{(1 + \zeta, \zeta - 1), (0, 1)\}. \quad (4)$$

Call this graph G . The Cartesian product joins pairs that agree in one coordinate and form an edge of the corresponding factor in the other. It has $5 \cdot 5 + 4 \cdot 7 = 53$ edges. The pair in (4) differs in both coordinates, so $e(G) = 54$.

Under the explicit labelling in Appendix A, G has graph6 code

S?E@cQHwB?_Q?bDPAcYWKm_BC?OAiw0T[.

This is the unique twenty-vertex entry in AMP’s Table 3 and ancillary file [2]. Thus AMP’s classification identifies every maximizer with an injective unit-distance realization of G . The finite correspondence can be checked directly from the product edge rule and the labelling table; a numerical drawing is not used for this identification.

Lemma 2 (Cartesian squares). *Let A, B be connected graphs, and let $p : V(A) \times V(B) \rightarrow \mathbb{C}$ be an injective unit-distance realization of $A \square B$. Then*

$$p(a, b) = \alpha(a) + \beta(b)$$

for injective unit-distance realizations α of A and β of B .

Proof. For edges $aa' \in E(A)$ and $bb' \in E(B)$, the corresponding product square has four distinct images and four unit sides. The two common unit neighbors of its opposite vertices are symmetric about their midpoint. Hence

$$p(a, b) + p(a', b') = p(a, b') + p(a', b). \quad (5)$$

In particular, $p(a', b) - p(a, b)$ is independent of b , by connectedness of B . Fix vertices a_0, b_0 . Summing the identity along a path in A from a_0 to a gives

$$p(a, b) - p(a_0, b) = p(a, b_0) - p(a_0, b_0).$$

Set $\alpha(a) = p(a, b_0) - p(a_0, b_0)$ and $\beta(b) = p(a_0, b)$. The factor restrictions are injective and preserve unit edges. $\square$

The lemma uses injectivity, not a crossing-free drawing. In particular, it excludes choosing unrelated orientations for individual product squares.

3 Exhaustion of all realizations

Apply Lemma 2 to the Cartesian subgraph of G . Normalize the first factor by a Euclidean isometry. Its points labelled $0, 1, \zeta$ form a unit equilateral triangle. The other common unit neighbor of the points labelled $1, \zeta$, distinct from the origin, is $1 + \zeta$. Thus the first factor is exactly D with these labels, and we may take $\beta(0) = 0$.

Write $\beta(1) = b$. The triangle on the labels $0, 1, \zeta$ forces

$$|b| = 1, \quad \beta(\zeta) = \lambda b, \quad \lambda \in \{\zeta, \bar{\zeta}\}.$$

The other common unit neighbor of $0, \lambda b$, distinct from b , is $(\lambda - 1)b$. Similarly, the other common unit neighbor of $0, b$, distinct from λb , is $(1 - \lambda)b$. Consequently the second factor is

$$bV_\lambda, \quad V_\lambda = \{0, 1, \lambda, \lambda - 1, 1 - \lambda\}.$$

The additional edge (4) imposes

$$|1 + \zeta + (\lambda - 2)b| = 1. \tag{6}$$

Conversely, an injective sum $D + bV_\lambda$ satisfying (6) realizes all 54 edges of G .

The reflection $T(z) = \zeta \bar{z}$ preserves D as a set and sends $D + bV_\lambda$ to

$$D + b'V_{\bar{\lambda}}, \quad b' = \zeta \bar{b}.$$

Since $\zeta \overline{(1 + \zeta)} = 1 + \zeta$, equation (6) is preserved. Thus, up to congruence, we may assume $\lambda = \zeta$. Now

$$\zeta - 2 = -\frac{1 + \zeta}{\zeta}, \quad |1 + \zeta|^2 = 3.$$

Equation (6) becomes

$$\left|1 - \frac{b}{\zeta}\right|^2 = \frac{1}{3}, \quad \left|\frac{b}{\zeta}\right| = 1.$$

It follows that $\operatorname{Re}(b/\zeta) = 5/6$, leaving exactly the two values b_σ in (2). This proves that there are at most two congruence classes.

To establish existence, observe that $D, V \subset \mathbb{Q}(i\sqrt{3})$, whereas $b_\sigma \notin \mathbb{Q}(i\sqrt{3})$: its formula would otherwise put $i\sqrt{11}$ in the distinct quadratic field $\mathbb{Q}(i\sqrt{3})$. A collision

$$d + b_\sigma v = d' + b_\sigma v'$$

with $v \neq v'$ would put $b_\sigma = (d' - d)/(v - v')$ in that field. If $v = v'$, then $d = d'$. Thus each sum has twenty distinct points. It realizes 53 Cartesian edges and the bonus edge, since (6) holds. AMP's upper bound then proves extremality and excludes any further unit pairs. Section 5 also checks the pair count directly.

4 A congruence invariant

For a finite set $X \subset \mathbb{C}$ define

$$M_2(X) = \sum_{\{x,x'\} \subset X} |x - x'|^2, \quad M_4(X) = \sum_{\{x,x'\} \subset X} |x - x'|^4,$$

$$S_2(X) = \sum_{\{x,x'\} \subset X} (x - x')^2.$$

The complex square in S_2 is unaffected by reversing a pair. Direct calculations on the two factors give

X	$ X $	$M_2(X)$	$M_4(X)$	$S_2(X)$
D	4	8	14	4ζ
V	5	17	41	-8ζ

(7)

For example, D has five squared distances equal to one and one equal to three; V has seven equal to one, two equal to three and one equal to four. The complex column also follows from

$$S_2(X) = |X| \sum_{x \in X} x^2 - \left(\sum_{x \in X} x \right)^2.$$

Lemma 3 (Fourth moment of a sum). *If $|b| = 1$ and the sum map $X \times Y \rightarrow X + bY$ is injective, then*

$$M_4(X + bY) = |Y|^2 M_4(X) + |X|^2 M_4(Y) + 8M_2(X)M_2(Y) + 4 \operatorname{Re}(\bar{b}^2 S_2(X) \overline{S_2(Y)}).$$

Proof. For differences $x = x_1 - x_2$, $y = y_1 - y_2$, expand

$$|x + by|^4 = (|x|^2 + |y|^2 + 2 \operatorname{Re}(x\bar{b}\bar{y}))^2.$$

Sum over all ordered pairs in each factor and divide by two. Terms odd in either difference cancel. The identity $4(\operatorname{Re} w)^2 = 2|w|^2 + 2 \operatorname{Re}(w^2)$ gives the displayed expression. $\square$

For $X = D, Y = V$, equation (7) yields

$$M_4(D + bV) = 25 \cdot 14 + 16 \cdot 41 + 8 \cdot 8 \cdot 17 - 128 \operatorname{Re}(b^2) = 2094 - 128 \operatorname{Re}(b^2).$$

Since

$$\operatorname{Re}(b_\sigma^2) = \frac{-7 - 5\sigma\sqrt{33}}{36},$$

we obtain (3). The two values differ by $320\sqrt{33}/9$. The quantity M_4 is invariant under every Euclidean isometry and under relabelling. Hence P_+ and P_- are incongruent, completing the proof of Theorem 1.

5 Verification and scope

The accompanying standard-library Python verifier performs exact arithmetic in

$$\mathbb{Q}(s, t), \quad s^2 = -3, \quad t^2 = -11, \quad \bar{s} = -s, \quad \bar{t} = -t.$$

It checks the product-to-graph6 correspondence, every one of the 190 pair distances in each representative, all four normalized labelled branches, their reflection identifications, the fourth moments and negative controls. A separate C++ program uses integer coordinate numerators with common denominator 12 and an explicit norm formula. Neither calculation is needed to infer that unexamined geometric branches do not exist: that is the content of the argument in Section 3. Neither repeats AMP’s global graph enumeration.

Omitting the bonus-edge condition would be a genuine error. For instance $b = (1+4i\sqrt{3})/7$ gives twenty distinct points but only 53 unit pairs. Omitting injectivity is also invalid: the unrotated sum $D + V$ has collisions. Both are included as negative controls.

The theorem concerns only $n = 20$. It gives no eventual lower bound for $C(n)$ and does not resolve the asymptotic question. Continuous families at other small orders are already discussed in AMP and in the prior working report [6]; they are not asserted here as new results. The finite classification is stated without an absolute priority claim.

AI assistance was used in developing the argument, exact verification code and manuscript. The author is responsible for the claims and source attributions. This preprint has not undergone independent peer review or proof-assistant verification.

A Explicit correspondence with the published graph

The following table labels $(d, v) \in D \times V$ by the vertex order of AMP’s ancillary graph6 record. The extra edge is 415.

Index	d	v	Index	d	v
0	0	$\zeta - 1$	10	ζ	$1 - \zeta$
1	0	$1 - \zeta$	11	1	1
2	$1 + \zeta$	$1 - \zeta$	12	ζ	1
3	$1 + \zeta$	ζ	13	1	ζ
4	$1 + \zeta$	$\zeta - 1$	14	ζ	ζ
5	0	ζ	15	0	1
6	$1 + \zeta$	1	16	$1 + \zeta$	0
7	1	$\zeta - 1$	17	0	0
8	ζ	$\zeta - 1$	18	1	0
9	1	$1 - \zeta$	19	ζ	0

The correspondence is checkable by forming the two factor edge sets, taking their Cartesian product, adding 415, and reading the upper-triangular adjacency bits in graph6 order. It does not depend on graph-isomorphism software.

References

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