

The Functions ψ_b of Bownik and Weber Are Frame Wavelets for $1/8 < b \leq 1/6$

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October 18, 2026

Abstract

For $0 < b < 1$ let $\psi_b \in L^2(\mathbb{R})$ be given by $\widehat{\psi_b} = \mathbf{1}_{(-1, -b) \cup (b, 1)}$. Bownik and Weber (Studia Math. 159 (2003)) determined for which b the dyadic affine system of ψ_b is a frame, except for $1/8 < b \leq 1/6$, where they believed it to be a frame and called $b = 1/6$ the hardest case; Bownik and Rzesotnik list this question as open in their survey of open problems in wavelet theory. We prove that ψ_b is a frame wavelet for every $b \in (1/8, 1/6]$, with frame bounds $1/54$ and 6 (not sharp). With the results of Bownik and Weber, ψ_b is a frame wavelet exactly for $b \in (0, 1/6] \cup [1/3, 1/2]$. The proof uses the equivalence of affine and quasi-affine frames (Ron and Shen; Chui, Shi and Stöckler) and the fiberization of shift-invariant systems (Ron and Shen; Bownik). Every fiber of the dual Gramian is, after a change of signs, the Laplacian of a forest plus a nonnegative diagonal term; for $1/8 < b \leq 1/6$ no vertex with one child has a child with one child, a flow gives an isoperimetric inequality, and the discrete Cheeger inequality gives the lower bound. A computer-assisted computation from one of the verification runs improves it to $1/16$. No earlier answer was found in the sources we searched. This is an unrefereed note.

1 Introduction

1.1 The problem

We use the conventions of the survey of Bownik and Rzesotnik [BR20]: $\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx$, $\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k)$, $\mathcal{X}(\psi) = \{\psi_{j,k} : j, k \in \mathbb{Z}\}$, and $\psi \in L^2(\mathbb{R})$ is a *frame wavelet* if there are *frame bounds* $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{j,k} \rangle|^2 \leq B\|f\|^2 \quad \text{for all } f \in L^2(\mathbb{R}). \quad (1)$$

For $0 \leq b < 1$ let $E_b = (-1, -b) \cup (b, 1)$ and $\widehat{\psi_b} = \mathbf{1}_{E_b}$. The following question of Bownik and Weber [BW03] is restated as open in [BR20] (Problem 2.7 of the arXiv version); it is record AMR-155-0008 of the UnsolvedMath collection [ula26], where it is called Problem 8 of [BR20] and listed as open.

Problem 1.1 ([BW03]; [BR20, Problem 2.7]). *For what values of $1/8 < b \leq 1/6$ is ψ_b a frame wavelet?*

Bownik and Weber [BW03, Example 2, pp. 473–477] treat $\widehat{\psi} = \mathbf{1}_{[-2\pi, -b] \cup [b, 2\pi]}$, $0 \leq b \leq 2\pi$, with the thresholds $\pi/4$, $\pi/3$, $2\pi/3$, π . Their Fourier transform is $\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx$ [BW03, p. 456]

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and their affine system is $\{2^{j/2}\psi(2^jx - k) : j, k \in \mathbb{Z}\}$, so their function with parameter b is the function $\psi_{b/(2\pi)}$ of this note (whether the endpoints of the intervals are included does not matter). Hence their thresholds become $1/8, 1/6, 1/3, 1/2$, and their results, as tabulated in [BR20], are: ψ_b is not a frame wavelet for $b = 0$, for $1/6 < b < 1/3$ and for $1/2 < b < 1$; it is a frame wavelet (not a Riesz wavelet) for $0 < b \leq 1/8$, a biorthogonal Riesz wavelet for $1/3 \leq b < 1/2$, and the Shannon orthonormal wavelet for $b = 1/2$. Their perturbation argument for $b \leq 1/8$ fails for $1/8 < b \leq 1/6$; they believe, on heuristic evidence, that ψ_b is still a frame wavelet there, and write that the borderline case “ $b = \pi/3$ is the hardest” [BW03, p. 474], i.e. $b = 1/6$ here.

1.2 Results

Theorem 1.2. *Let $1/8 < b \leq 1/6$. Then*

$$\frac{1}{54}\|f\|^2 \leq \sum_{j,k \in \mathbb{Z}} |\langle f, (\psi_b)_{j,k} \rangle|^2 \leq 6\|f\|^2 \quad \text{for all } f \in L^2(\mathbb{R}).$$

In particular ψ_b is a frame wavelet for every $b \in (1/8, 1/6]$.

Corollary 1.3. *Let $0 \leq b < 1$. Then ψ_b is a frame wavelet if and only if $b \in (0, 1/6] \cup [1/3, 1/2]$.*

Corollary 1.3 follows from Theorem 1.2 and the results of [BW03] just listed, since Riesz bases are frames. Neither constant in Theorem 1.2 is sharp: a computer-assisted computation gives the lower bound $1/16$ (Remark 6.1), and numerically the optimal lower bound for $b = 1/6$ is about 0.07 (Remark 6.2).

1.3 Method, and what is new

By theorems of Ron and Shen, of Chui, Shi and Stöckler, and of Bownik, the frame bounds of ψ_b are the essential infimum and supremum of the spectra of the dual Gramian fibers $G(\xi)$ of the quasi-affine system (Section 2). For an indicator function, $G(\xi)$ is, after a change of signs, the Laplacian of a graph on $\xi + \mathbb{Z}$ plus a nonnegative diagonal term (Section 3). For $1/8 < b \leq 1/6$ the graph is a forest in which no vertex with one child has a child with one child, the only use of $b \leq 1/6$ (Section 4; Remark 4.5 treats $b > 1/6$). A flow on each tree gives $|F| \leq 3|\partial F|$, the discrete Cheeger inequality gives the lower bound $1/54$, and the Schur test the upper bound 6 (Section 5). The reduction and the two inequalities are known; the forest structure and the flow argument are, as far as we know, new. Section 7 discusses frame wavelet sets; the searches are listed under “Scope and priority” in Section 8.

2 Reduction to the fibers of the quasi-affine system

For $\psi \in L^2(\mathbb{R})$ the *quasi-affine system* of Ron and Shen [RS97] is the system, invariant under integer translations,

$$\mathcal{X}^q(\psi) = \{\psi_{j,k} : j \geq 0, k \in \mathbb{Z}\} \cup \{2^j\psi(2^j(\cdot - k)) : j < 0, k \in \mathbb{Z}\};$$

this is the normalisation of [RS97, Section 1], of [Bow00a, p. 204] and of the standard quasi-affine system of [BL11, Section 3.1].

Proposition 2.1. *Let $\psi \in L^2(\mathbb{R})$ and $0 < A \leq B < \infty$. The affine system $\mathcal{X}(\psi)$ is a frame for $L^2(\mathbb{R})$ with bounds A, B if and only if $\mathcal{X}^q(\psi)$ is a frame for $L^2(\mathbb{R})$ with bounds A, B .*

This is [RS97, Theorem 5.5(b)] (numbering of the authors’ preprint), proved there under the decay condition (4.6) of [RS97]. In the normalisation of [RS97] (Fourier transform with $e^{-i\omega x}$) the transform

of ψ_b is $\mathbf{1}_{2\pi E_b}$, which is bounded and vanishes outside $[-2\pi, 2\pi]$; hence the quantities $c(\psi_b, k)$ of (4.6), the suprema over $[-\pi, \pi]$ of $\sum_{\alpha \in 2\pi\mathbb{Z}, |\alpha| > 2^k} \mathbf{1}_{2\pi E_b}(\cdot + \alpha)$, vanish for $2^k \geq 3\pi$, and (4.6) holds. Without decay conditions the proposition is due to Chui, Shi and Stöckler [CSS98], as stated in [Bow00a, p. 205 and Theorem 1.3(ii)] and [BW03, Remark 1]; it is also contained in [BL11, Theorem 1.1 and Proposition 3.10].

Let $\Phi \subset L^2(\mathbb{R})$ be countable and $E(\Phi) = \{\varphi(\cdot - k) : \varphi \in \Phi, k \in \mathbb{Z}\}$. The *dual Gramian* of Φ at $\xi \in \mathbb{R}$ is the matrix $G(\xi)_{m,n} = \sum_{\varphi \in \Phi} \hat{\varphi}(\xi + m) \overline{\hat{\varphi}(\xi + n)}$, $m, n \in \mathbb{Z}$.

Proposition 2.2. *Let $\Phi \subset L^2(\mathbb{R})$ be countable and $0 < A \leq B < \infty$. Then $E(\Phi)$ is a frame for $L^2(\mathbb{R})$ with bounds A, B if and only if, for almost every ξ , the matrix $G(\xi)$ defines a bounded operator on $\ell^2(\mathbb{Z})$ whose spectrum is contained in $[A, B]$.*

This is [Bow00b, Theorem 2.5(ii), last statement] (see also [Bow00b, Theorem 2.3(i)]); Bownik attributes it to Ron and Shen, in whose paper it is [RS95, Theorem 3.3.5]. For a bounded self-adjoint operator it suffices to estimate the quadratic form on finitely supported vectors. We apply Proposition 2.2 to

$$\Phi = \{\psi_{j,k} : j \geq 0, 0 \leq k < 2^j\} \cup \{\varphi_j : j < 0\}, \quad \varphi_j(x) = 2^j \psi(2^j x), \quad (2)$$

for which $E(\Phi) = \mathcal{X}^q(\psi)$, since $\psi_{j,k}(x - l) = \psi_{j,k+2^j l}$ for $j \geq 0$ and $(k, l) \mapsto k + 2^j l$ maps $\{0, \dots, 2^j - 1\} \times \mathbb{Z}$ bijectively onto $\mathbb{Z}$.

3 The dual Gramian of the quasi-affine system of ψ_b

Throughout this section $0 < b < 1$, $\psi = \psi_b$, $E = E_b$, and $G(\xi)$ is the dual Gramian of the family Φ of (2). For $\xi \in \mathbb{R}$ and $m, n \in \mathbb{Z}$ we write $y = \xi + m$ and $y' = \xi + n$, and we index rows and columns by the points of $\xi + \mathbb{Z}$ instead of $\mathbb{Z}$.

Lemma 3.1. *For $\xi \in \mathbb{R}$ and $m, n \in \mathbb{Z}$,*

$$G(\xi)_{m,n} = \#\{j \in \mathbb{Z} : 2^{-j}y \in E, 2^{-j}y' \in E, \text{ and } j \leq 0 \text{ or } 2^j \mid (m - n)\}.$$

In particular $G(\xi)_{m,m} = d(y) := \#\{j \in \mathbb{Z} : 2^{-j}|y| \in (b, 1)\}$. If $1/8 < b < 1/4$ and $y \neq 0$, then $d(y) \in \{2, 3\}$.

Proof. A substitution gives $\widehat{\psi_{j,k}}(\zeta) = 2^{-j/2} e^{-2\pi i k 2^{-j} \zeta} \hat{\psi}(2^{-j} \zeta)$ and $\widehat{\varphi_j}(\zeta) = \hat{\psi}(2^{-j} \zeta)$. For $j < 0$ the generator φ_j contributes $\hat{\psi}(2^{-j}y) \overline{\hat{\psi}(2^{-j}y')}$ to $G(\xi)_{m,n}$ ($\hat{\psi}$ is real). For $j \geq 0$ the 2^j generators $\psi_{j,k}$ contribute $2^{-j} \hat{\psi}(2^{-j}y) \overline{\hat{\psi}(2^{-j}y')} \sum_{k=0}^{2^j-1} e^{-2\pi i k(m-n)/2^j}$, and the sum equals 2^j if $2^j \mid (m - n)$ and 0 otherwise. Since $\hat{\psi} = \mathbf{1}_E$, each contribution is 0 or 1, and only the finitely many j with $2^{-j}|y| \in (b, 1)$ can contribute. Since $E = -E$, the diagonal entry counts the j with $2^{-j}|y| \in (b, 1)$, that is, the integers in the open interval $(\log_2|y|, \log_2|y| + \log_2(1/b))$. For $1/8 < b < 1/4$ its length $\log_2(1/b)$ lies in $(2, 3)$, so it contains two or three integers. $\square$

Lemma 3.2. *Let $\xi \in \mathbb{R}$ and $m \neq n$. Then $G(\xi)_{m,n} \in \{0, 1\}$, and $G(\xi)_{m,n} = 1$ if and only if y and y' have opposite signs and there is $j \geq 0$ with*

$$|y| + |y'| = 2^j \quad \text{and} \quad |y|, |y'| \in (2^j b, 2^j(1 - b)). \quad (3)$$

Proof. Let $p = n - m \neq 0$ and suppose that the index j contributes, i.e. $2^{-j}y, 2^{-j}y' \in E \subset (-1, 1)$ and $j \leq 0$ or $2^j \mid p$. Then $|2^{-j}p| < 2$, and $2^{-j}p$ is a nonzero integer (for $j \leq 0$ because $p \in \mathbb{Z}$, for $j > 0$ by divisibility). Hence $2^{-j}p = \pm 1$, so $|p| = 2^j$ with $j \geq 0$, and j is unique; thus $G(\xi)_{m,n} \in \{0, 1\}$. Put

$u = 2^{-j}y$ and $u' = 2^{-j}y' = u \pm 1$. If $u \in (b, 1)$, then $u+1 \notin E$, so $u' = u-1$, and $u-1 \in (-1, -b)$ means $u < 1-b$; if $u \in (-1, -b)$, then similarly $u' = u+1$ and $u > b-1$. In both cases u, u' have opposite signs, $|u| + |u'| = 1$ and $|u|, |u'| \in (b, 1-b)$, which is (3) after multiplication by 2^j . Conversely, if (3) holds and y, y' have opposite signs, then $|p| = |y - y'| = 2^j$, so $2^j \mid p$, and $2^{-j}y, 2^{-j}y' \in \pm(b, 1-b) \subset E$; hence the index j contributes. $\square$

Definition 3.3. For $\xi \in \mathbb{R} \setminus \mathbb{Z}$ the *fiber graph* $\Gamma(\xi)$ has the vertex set $V(\xi) = \xi + \mathbb{Z}$, and $y, y' \in V(\xi)$ are adjacent if they have opposite signs and satisfy (3) for some $j \geq 0$, the *level* of the edge. We write $\deg(y)$ for the degree of y and $S(y) = d(y) - \deg(y)$ for its *surplus*.

By Lemmas 3.1 and 3.2,

$$G(\xi) = D(\xi) + A(\xi), \quad (4)$$

where $D(\xi) = \text{diag}(d(y))$ and $A(\xi)$ is the adjacency matrix of $\Gamma(\xi)$. The level of an edge at y determines its other endpoint, $y' = -\text{sgn}(y)(2^j - |y|)$, and each edge at y of level j satisfies $2^{-j}|y| \in (b, 1-b) \subset (b, 1)$, so that j is counted in $d(y)$. Hence $\deg(y) \leq d(y)$, i.e. $S(y) \geq 0$.

Lemma 3.4. Let $1/8 < b < 1/4$ and $\xi \in \mathbb{R} \setminus \mathbb{Z}$. Then $G(\xi)$ is a bounded operator on $\ell^2(V(\xi))$ with $\|G(\xi)\| \leq 6$. For every finitely supported w , the vector v with $v_y = \text{sgn}(y)w_y$ satisfies $\|v\| = \|w\|$ and

$$\langle G(\xi)w, w \rangle = Q_\xi(v) := \sum_{\{y, y'\} \in \Gamma(\xi)} |v_y - v_{y'}|^2 + \sum_{y \in V(\xi)} S(y)|v_y|^2, \quad (5)$$

the first sum running over the edges of $\Gamma(\xi)$.

Proof. All entries of $G(\xi)$ are nonnegative and the sum of the entries of the row of y is $d(y) + \deg(y) \leq 2d(y) \leq 6$. By the Schur test [Sch11] with constant weights and the symmetry of $G(\xi)$, $|\langle G(\xi)w, w' \rangle| \leq \sum_{y, y'} G(\xi)_{y, y'} (|w_y|^2 + |w_{y'}|^2)/2 \leq 3(\|w\|^2 + \|w'\|^2)$ for finitely supported w, w' ; hence $\|G(\xi)\| \leq 6$. Every edge joins vertices of opposite signs, so $w_y \overline{w_{y'}} = -v_y \overline{v_{y'}}$ on edges, and $\langle G(\xi)w, w \rangle = \sum_y d(y)|v_y|^2 - \sum_{\{y, y'\}} 2\text{Re}(v_y \overline{v_{y'}}) = Q_\xi(v)$, because $d(y) = S(y) + \deg(y)$ and $\sum_y \deg(y)|v_y|^2 = \sum_{\{y, y'\}} (|v_y|^2 + |v_{y'}|^2)$. $\square$

4 The fiber forest for $1/8 < b \leq 1/6$

In this section $1/8 < b \leq 1/6$. We use $1/8 < b < 1/4$, $4b < 1-b$ and, in Lemma 4.4 only, $1-2b \geq 4b$. For a statement P let $[P]$ be 1 if P holds and 0 otherwise. The *exceptional set*

$$\Xi_0 = \{\xi \in \mathbb{R} : \xi + m = 0 \text{ or } 2^{-j}|\xi + m| \in \{b, 1-b, 1\} \text{ for some } m, j \in \mathbb{Z}\}$$

is countable. Let $\xi \notin \Xi_0$. Then $0 \notin V(\xi)$, and $2^{-j}|y| \notin \{b, 1-b, 1\}$ for all $y \in V(\xi)$ and $j \in \mathbb{Z}$. Also $\xi \notin \frac{1}{2}\mathbb{Z}$ (otherwise 0 or $\frac{1}{2}$ would be a vertex, and $2 \cdot \frac{1}{2} = 1$), so exactly one vertex y_c , the *central vertex*, lies in $(-\frac{1}{2}, \frac{1}{2})$, and all other vertices satisfy $|y| > \frac{1}{2}$. Adjacent vertices have different moduli: $|y| = |y'|$ and $|y| + |y'| = 2^j$ would give $2^{-(j-1)}|y| = 1$.

For $y \in V(\xi)$ let $n(y) \in \mathbb{Z}$ be defined by $2^{n(y)-1} \leq |y| < 2^{n(y)}$, and let $\tau(y) = 2^{-n(y)}|y|$ be the *position* of y . For $\xi \notin \Xi_0$ we have $\tau(y) \in (\frac{1}{2}, 1)$, $\tau(y) \neq 1-b$ and $\tau(y) \neq 4b$ (since $2^{-n(y)-2}|y| \neq b$). By Lemma 3.2, for $y \in V(\xi)$ and $j \geq 0$ there is an edge of level j at y if and only if $2^{-j}|y| \in (b, 1-b)$, and then its other endpoint is $-\text{sgn}(y)(2^j - |y|)$.

Lemma 4.1. Let $\xi \notin \Xi_0$ and $y \in V(\xi)$ with $|y| > \frac{1}{2}$; put $n = n(y) \geq 0$ and $\tau = \tau(y)$. (a) $d(y) = 2 + [\tau > 4b]$. (b) The edges at y are: an edge of level n to $y^- = -\text{sgn}(y)(2^n - |y|)$, present if and only if $\tau < 1-b$, with $|y^-| < 2^{n-1} < |y|$; an edge of level $n+1$ to $c_1(y) = -\text{sgn}(y)(2^{n+1} - |y|)$, always present, with $n(c_1(y)) = n+1$ and $\tau(c_1(y)) = 1 - \tau/2 \in (\frac{1}{2}, \frac{3}{4})$; and an edge of level $n+2$ to $c_2(y) = -\text{sgn}(y)(2^{n+2} - |y|)$, present if and only if $\tau > 4b$, with $n(c_2(y)) = n+2$ and $\tau(c_2(y)) = 1 - \tau/4$. (c) $S(y) = 1$ if $\tau > 1-b$, and $S(y) = 0$ if $\tau < 1-b$.

Proof. We have $2^{-j}|y| = 2^{n-j}\tau$. For $j \leq n-1$ this is at least $2\tau > 1$; for $j = n$ it is $\tau \in (b, 1)$; for $j = n+1$ it is $\tau/2 \in (\frac{1}{4}, \frac{1}{2}) \subset (b, 1-b)$; for $j = n+2$ it is $\tau/4 \in (\frac{1}{8}, \frac{1}{4})$, which lies in $(b, 1)$, or in $(b, 1-b)$, if and only if $\tau > 4b$; for $j \geq n+3$ it is less than $\frac{1}{8} < b$. This gives (a), and (b) follows from the criterion before the lemma: the level n requires $\tau < 1-b$, and the moduli are $|y^-| = 2^n(1-\tau)$ with $1-\tau < \frac{1}{2}$, $|c_1(y)| = 2^{n+1}(1-\tau/2)$ with $1-\tau/2 \in (\frac{1}{2}, \frac{3}{4})$, and $|c_2(y)| = 2^{n+2}(1-\tau/4)$ with $1-\tau/4 \in (\frac{3}{4}, \frac{7}{8})$. Hence $\deg(y) = [\tau < 1-b] + 1 + [\tau > 4b]$, and (c) follows from (a). $\square$

Lemma 4.2. *Let $\xi \notin \Xi_0$ and $u = |y_c| \in (0, \frac{1}{2})$. The neighbours of y_c are a vertex z_0 with $|z_0| = 1-u$ (level 0), present if and only if $u > b$, and a vertex z_1 with $|z_1| = 2-u$ (level 1), present if and only if $u > 2b$. Both have larger modulus than y_c , and $S(y_c) \geq 1$.*

Proof. An edge of level $j \geq 0$ at y_c needs $2^{-j}u \in (b, 1-b)$. Since $u < \frac{1}{2} < 1-b$, this means $u > b$ for $j = 0$ and $u > 2b$ for $j = 1$, and it is impossible for $j \geq 2$ because $u/4 < \frac{1}{8} < b$. The other endpoints have moduli $1-u > u$ and $2-u > u$. The index $n(y_c) \leq -1$ is counted in $d(y_c)$, because $\tau(y_c) \in (b, 1)$, but it is not the level of an edge, while every level of an edge at y_c is counted in $d(y_c)$; hence $S(y_c) \geq 1$. $\square$

In the following, the *parent* of a vertex is its unique neighbour of smaller modulus, if there is one; the other neighbours are its *children*. A vertex without parent is a *root*.

Lemma 4.3. *Let $\xi \notin \Xi_0$. (a) Every vertex has at most one parent. The roots are y_c and the vertices y with $|y| > \frac{1}{2}$ and $\tau(y) > 1-b$. (b) $\Gamma(\xi)$ is a forest, and each of its components, a tree, contains exactly one root. From every vertex the root of its tree is reached by finitely many steps to the parent; their number is the depth of the vertex. (c) $S(r) \geq 1$ for every root r , and $S(y) = 0$ for every other vertex y . (d) A vertex $y \neq y_c$ has the children $c_1(y)$, and also $c_2(y)$ if and only if $\tau(y) > 4b$; in particular every root $y \neq y_c$ has two children. The central vertex has at most two children.*

Proof. (a) and (d) follow from Lemma 4.1(b) and Lemma 4.2: for $|y| > \frac{1}{2}$ the vertices $c_1(y)$ and $c_2(y)$ have modulus at least $2^n > |y|$, so y^- is the only possible parent; a root $y \neq y_c$ has $\tau(y) > 1-b > 4b$. (b) Following parents, the modulus decreases strictly, and $V(\xi) \cap [-R, R]$ is finite for every R ; hence a root is reached after finitely many steps. On a cycle, a vertex of maximal modulus would have two neighbours of smaller modulus on the cycle, i.e. two parents. If a tree contained two roots, a vertex of maximal modulus on the path joining them would either be an inner vertex of the path, with two parents, or one of the two roots, with a parent. (c) follows from Lemma 4.1(c) and Lemma 4.2. $\square$

Call a vertex $y \neq y_c$ *simple* if $\tau(y) < 4b$ and *branching* if $\tau(y) > 4b$. By Lemma 4.3(d), a simple vertex has one child and a branching vertex has two; roots other than y_c are branching.

Lemma 4.4. *Let $\xi \notin \Xi_0$. If y is simple, then $c_1(y)$ is branching. Consequently, the parent of a simple vertex is a root or a branching vertex.*

Proof. $\tau(c_1(y)) = 1-\tau(y)/2 > 1-2b \geq 4b$, because $\tau(y) < 4b$ and $b \leq 1/6$ (the only use of $b \leq 1/6$; at $b = 1/6$ the strict inequality comes from $\tau(y) \neq 4b$, i.e. from $\xi \notin \Xi_0$). If the parent q of a simple vertex y were simple, then y , the only child of q , would be $c_1(q)$ and hence branching. The central vertex behaves in the same way, although this is not used below: if y_c has only one child, then $b < u < 2b$ by Lemma 4.2, and the child is z_0 , with $n(z_0) = 0$ and $\tau(z_0) = 1-u > 1-2b \geq 4b$, so z_0 is branching. $\square$

Remark 4.5 ($b > 1/6$). Let $1/6 < b < 1/5$. Lemmas 4.1 to 4.3 remain valid (their proofs use only $1/8 < b < 1/5$), but Lemma 4.4 fails: the map $\tau \mapsto 1-\tau/2$ sends $I_b = [2-8b, 4b]$ into $[1-2b, 4b] \subset I_b$ and the interior of I_b into itself, and positions below $4b$ are simple. So a vertex $y \neq y_c$ with $\tau(y)$ in the interior of I_b starts an infinite chain $y, c_1(y), c_1(c_1(y)), \dots$ of simple vertices, each with $d = 2$ and two neighbours. For $v_i = \min(i, L-1-i)$ on L consecutive vertices of the chain the form (5) is at most

L , while $\|v\|^2$ is of order L^3 . Hence the bottom of the spectrum of $G(\xi)$ is 0 whenever such a vertex exists, e.g. $y = \xi$ for $\xi \in (\max(\frac{1}{2}, 2 - 8b), 4b) \setminus \Xi_0$, a set of positive measure, and by Propositions 2.1 and 2.2 ψ_b is not a frame wavelet. This agrees with [BW03, Example 2, Case II], which proves it for $1/6 < b < 1/3$ with test functions concentrated near $\pm \frac{2}{3}2^k$; $\frac{2}{3}$ is the fixed point of $\tau \mapsto 1 - \tau/2$. At $b = 1/6$ the interval $I_b = \{\frac{2}{3}\} = \{4b\}$ is the single excluded position.

5 A flow, an isoperimetric inequality, and the lower bound

Let $1/8 < b \leq 1/6$, $\xi \notin \Xi_0$, and let T be a tree of $\Gamma(\xi)$ with root r . Let Γ'_T be the graph obtained from T by adding a new vertex g and the *ghost edge* $\{r, g\}$. By Lemma 4.3, every vertex of Γ'_T other than g has degree at most 3: a vertex other than the root has one parent and one or two children, a root $r \neq y_c$ has two children and the ghost edge, and y_c has at most two children and the ghost edge.

Lemma 5.1 (flow). *For the vertices $y \neq r$ of T define $p(y) \geq 0$ by induction on the depth: $p(y) = 0$ if the parent of y is r , and otherwise $p(y) = (p(q) + 1)/k(q)$, where q is the parent of y and $k(q) \in \{1, 2\}$ is its number of children. Then $p(y) \leq 3$ for every $y \neq r$, and $p(y) \leq 2$ if y is simple.*

Proof. For depth 1, $p(y) = 0$. Let y have depth at least 2, so that its parent q is not a root and is simple or branching, and assume the claim for q . If q is branching, then $p(y) = (p(q) + 1)/2 \leq (3 + 1)/2 = 2$. If q is simple, then $y = c_1(q)$ is branching by Lemma 4.4, and $p(y) = p(q) + 1 \leq 2 + 1 = 3$. In both cases $p(y) \leq 3$, and $p(y) \leq 2$ unless q is simple, in which case y is not simple. $\square$

Lemma 5.2 (isoperimetric inequality). *For every finite nonempty set F of vertices of T ,*

$$|F| \leq 3|\partial F|,$$

where ∂F is the set of edges of Γ'_T with exactly one endpoint in F ; the ghost edge belongs to ∂F if and only if $r \in F$.

Proof. Send $p(y)$ units along the edge from the parent of y to y , for every $y \neq r$, and one unit along the ghost edge from r to g . The net outflow (outgoing minus incoming) at every vertex x of T is 1: at r it is $1 + \sum_y p(y) = 1$, the sum running over the children of r ; at $x \neq r$ with k children it is $k \cdot (p(x) + 1)/k - p(x) = 1$. Summing over $x \in F$, the edges with both endpoints in F cancel, and

$$|F| = \sum_{e \in \partial F} (\text{flow along } e \text{ out of } F) \leq 3|\partial F|,$$

because the flow out of F along an edge of ∂F is at most 3 (Lemma 5.1) on edges to children, 1 on the ghost edge, and $-p \leq 0$ on an edge from the parent of a vertex of F into F . $\square$

Lemma 5.3 (Cheeger inequality). *For every finitely supported $v : V(T) \rightarrow \mathbb{C}$,*

$$Q_T(v) := \sum_{\{y, y'\} \in T} |v_y - v_{y'}|^2 + |v_r|^2 \geq \frac{1}{54} \|v\|^2.$$

Proof. Since $||a| - |a'|| \leq |a - a'|$, we have $Q_T(|v|) \leq Q_T(v)$, so we may assume $v \geq 0$. Put $v_g = 0$; then $Q_T(v) = \sum_{\{x, x'\}} (v_x - v_{x'})^2$, summed over the edges of Γ'_T . For $t \geq 0$ let $F_t = \{x \in V(T) : v_x^2 > t\}$, a finite set. An edge $\{x, x'\}$ lies in ∂F_t exactly for t between v_x^2 and $v_{x'}^2$, so by Lemma 5.2

$$\frac{1}{3} \|v\|^2 = \frac{1}{3} \int_0^\infty |F_t| dt \leq \int_0^\infty |\partial F_t| dt = \sum_{\{x, x'\}} |v_x^2 - v_{x'}^2|.$$

By the Cauchy–Schwarz inequality, $(v_x + v_{x'})^2 \leq 2(v_x^2 + v_{x'}^2)$ and the degree bound 3, the last sum, $\sum |v_x - v_{x'}|(v_x + v_{x'})$, is at most $Q_T(v)^{1/2}(2 \sum_x \deg_{\Gamma'_T}(x)v_x^2)^{1/2} \leq Q_T(v)^{1/2}(6\|v\|^2)^{1/2}$. Hence $\frac{1}{3}\|v\|^2 \leq (6Q_T(v))^{1/2}\|v\|$, i.e. $Q_T(v) \geq \|v\|^2/54$. $\square$

The discrete Cheeger inequality for infinite graphs of bounded degree goes back to Dodziuk [Dod84], which we could not access. Lemma 5.3 is the special case of [BKW15, Lemma 3.5] (numbering of arXiv:1209.4911v1) for the graph Γ'_T (T infinite) with unit weights, the counting measure, the path metric in which every edge has length $3^{-1/2}$ (an intrinsic metric, because all degrees in Γ'_T are at most 3) and $U = V(T)$: by Lemma 5.2 their isoperimetric constant is $\alpha(U) \geq 3^{-1/2}/3$, so $\alpha(U)^2/2 \geq 1/54$; the proof above is theirs.

Proof of Theorem 1.2. Let $\xi \notin \Xi_0$, let w be finitely supported, and let $v_y = \text{sgn}(y)w_y$. Every edge of $\Gamma(\xi)$ lies in exactly one tree, every tree has one root r_T with $S(r_T) \geq 1$, and $S \geq 0$ (Lemma 4.3). By Lemmas 3.4 and 5.3, $\langle G(\xi)w, w \rangle = Q_\xi(v) \geq \sum_T Q_T(v|_T) \geq \frac{1}{54} \sum_T \|v|_T\|^2 = \frac{1}{54}\|w\|^2$, and $\langle G(\xi)w, w \rangle \leq 6\|w\|^2$. Hence the spectrum of $G(\xi)$ lies in $[1/54, 6]$ for every ξ outside the null set Ξ_0 . By Proposition 2.2, $\mathcal{X}^q(\psi_b) = E(\Phi)$ is a frame for $L^2(\mathbb{R})$ with bounds $1/54$ and 6 , and by Proposition 2.1 so is $\mathcal{X}(\psi_b)$. $\square$

6 Remarks on the constants

Remark 6.1 (a computer-assisted lower bound). The second independent verification run found another route, which uses only Lemmas 4.1 and 4.2 (valid also for $b = 1/8$, with $\frac{1}{8} \leq b$ in the proofs) and gives, with computer assistance: *for every $b \in [1/8, 1/6]$ and $\xi \notin \Xi_0$, $G(\xi) \geq \frac{1}{16}$* , so the lower frame bound of ψ_b is at least $1/16$. For finite $W \subset V(\xi)$, $G_W - \lambda I$ is eliminated from the leaves of the forest induced on W , with pivots $\pi(x) = d(x) - \lambda - \sum 1/\pi(y)$ over the children $y \in W$ of x ; it is positive definite if all pivots are positive. By Lemma 4.1, $d(x)$ and the positions of the children of $x \neq y_c$ depend only on $\tau(x)$ and the sign of $\tau(x) - 4b$. The program divides $[\frac{1}{2}, 1]$ into 3072 closed cells and finds rationals $P^*(i) > 0$ with $d - \lambda - \sum_{\text{children}} 1/\min_k P^*(k) \geq P^*(i)$ for every cell i and admissible type, the minimum running over the cells k meeting the image of cell i ; by induction from the leaves, $\pi(x) \geq P^*(i)$ if $\tau(x)$ lies in cell i , and y_c is treated with Lemma 4.2. For $\lambda = 1/16$ all inequalities were verified in exact rational arithmetic on 128 closed b -intervals covering $[1/8, 1/6]$ (seven minutes; repeated with identical output). Theorem 1.2 does not depend on this result.

	$N = 8$	$N = 12$	$N = 16$	$N = 18$	status of ψ_b
$b = 1/7$	0.3910	0.2951	0.2506	0.2369	frame (Theorem 1.2)
$b = 3/20$	0.3116	0.2280	0.1893	0.1774	frame (Theorem 1.2)
$b = 1/6$	0.2307	0.1611	0.1284	0.1183	frame (Theorem 1.2)
$b = 1/5$	0.0810	0.0437	0.0273	0.0223	not a frame [BW03]

Table 1: Smallest eigenvalue of the compression of $G(\xi)$, $\xi = 67/100$, to the 2^{N+1} vertices with $|y| < 2^N$ (floating point; numerical illustration only).

Remark 6.2 (numerical evidence). A compression of $G(\xi)$ to a finite set depends on ξ only through finitely many strict inequalities, so its smallest (largest) eigenvalue bounds the optimal lower (upper) frame bound of ψ_b from above (below). For $b = 1/6$, $\xi = 67/100$ and $N = 16, 20$ the smallest eigenvalue comes from the tree of y_c ; the compression to the 3,145,727 vertices of this tree of modulus below 2^{40} , computed by the first and the third verification runs, has smallest eigenvalue 0.0819. So the optimal lower frame bound for $b = 1/6$ lies in $[1/16, 0.082]$ (the lower end is the computer-assisted bound of

Remark 6.1, the upper end is computed in floating point). An extrapolation in the depth gives about 0.075 for this fiber, and the floating-point version of the programme of Remark 6.1 converges up to $\lambda \approx 0.0697$ for b near $1/6$; we expect the optimal lower bound near $b = 1/6$ to be about 0.07 (tested, not proved). The largest eigenvalue for $b = 1/7$, $N = 18$ is 5.76.

7 Relation to frame wavelet sets

In the terminology of Dai, Diao and Gu [DDG01], ψ_b is a frame wavelet if and only if $2\pi E_b$ is a *frame wavelet set*. A characterization of frame wavelet sets is open [DD05, Problem 2.12], and the sufficient conditions of the review [DD05] do not decide $2\pi E_b$. In our normalisation (τ -equivalent points differ by an integer; the dilation orbits $\{\pm 2^k t : k \in \mathbb{Z}\}$ have positions $t \in [\frac{1}{2}, 1)$), $E_b(\tau, 1) = \pm[1 - b, 1)$ up to endpoints, so the condition $\bigcup_k 2^k E_b(\tau, 1) = \mathbb{R}$ of [DD05, Theorem 2.5] (from [DDG01]) fails. The condition of [DD05, Theorem 2.11] (from [DDG04]) is $\mathbb{R} = \Omega_1 \cup \dots \cup \Omega_n$, where $\Omega_1 = \bigcup_k 2^k E_b(\tau, 1)$ and Ω_{j+1} consists of the dilation orbits of the points of $E_b \setminus (\Omega_1 \cup \dots \cup \Omega_j)$ whose τ -equivalent point lies in $\Omega_1 \cup \dots \cup \Omega_j$. A point $x \in E_b$ with $|x| \in (b, 1 - b)$ has position $t \in \{|x|, 2|x|, 4|x|\}$, and its τ -equivalent point has position $P(t)$, $1 - t/2$ or $1 - t/4$, respectively, $P(t)$ being the position of $1 - t$. As $P(1 - t/2) = P(1 - t/4) = t$ and $1 - t/2, 1 - t/4 < 1 - b$, induction shows that the orbit of position t lies in $\Omega_1 \cup \dots \cup \Omega_n$ if and only if $P^i(t) \in [1 - b, 1)$ for some $i < n$. Since $P(t) = 2 - 2t$ on $[\frac{7}{12}, \frac{3}{4}]$, positions within $2^{-n}/12$ of the fixed point $\frac{2}{3}$ need more than n steps, and the condition fails for every n . (The necessary condition [DD05, Theorem 2.7] holds.) We could not read Li and Cao [LC03], Guo [Guo13] and Yadav and Kumar [YK23] beyond titles, abstracts and the summaries in zbMATH Open. According to these summaries, [LC03] and [YK23] give necessary and sufficient conditions for frame wavelet sets (in [LC03] for reducing subspaces); we do not know whether these conditions decide $2\pi E_b$.

8 Computations and reproducibility

The reproducibility package (folder `reproducibility/` of the source archive; command lines and running times in its `README.md`) contains the program of this note, the programs with which the result was first obtained, and those of the three verification runs; no step of the proof of Theorem 1.2 depends on them. *The program of this note*, `check_note.py` (standard library, about 20 seconds), compares Lemma 3.1 with the sum over the generators; computes compressions of $G(\xi)$ to the windows $0 < |y| \leq 128$ from the entry formula, in exact arithmetic for all pairs, and checks on them Lemma 3.2 and Lemmas 4.1 to 4.4, the bounds of Lemma 5.1, Lemma 5.2 on random sets F , and the positive definiteness of $G_W - \frac{1}{54}I$ and $6I - G_W$ (88 random and near-exceptional fibers; $b = 1/8$ and nine values in $(1/8, 1/6]$); and checks the facts used in Remark 4.5 and Section 7. *The programs with which the result was first obtained* (`original/`) build $G(\xi)$ from the entry formula, but only for index differences that are powers of two unless the option `--allpairs` is given (`fiber_gramian.py`, whose structural checks use child counts from the rules of Section 4), or from the edge rule of Lemma 3.2 (`fiber_sparse.py`; its field `n_roots` counts all vertices with positive surplus in the window, including those whose children lie outside it). The structural lemmas are checked against the definition by `check_note.py` and by the second verification run, whose `window_seq.py` produced Table 1 (the original programs give the same values).

Verification. The result was first obtained with proofs and with the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each examined the first written version of the result, re-derived the proofs, wrote its own programs and examined the sources. The first compared Propositions 2.1 and 2.2 with the cited texts and found the decay condition of [RS97, Theorem 5.5];

its programs checked Lemma 3.1 against the generator sums and the structural lemmas on 442 fibers, computed the affine quadratic form directly, without quasi-affine systems or fibers (exactly 1 for the Shannon wavelet, $(3 \mp \sqrt{5})/2$ for $b = 2/5$, agreement with fiber compressions to 12 digits for $b = 1/6$), and computed the compressions of Remark 6.2. The second checked the structural claims against the definition in exact arithmetic (470 fibers; above $b = 1/6$, Lemma 4.4 fails, as it must), tested its checks on wrong inputs, re-ran the original programs with identical outputs, checked the bibliography against Crossref, searched the literature, and built the programme of Remark 6.1. Neither found a gap. The corrections they required were made: the decay condition and the unconditional sources for Proposition 2.1, a misprint in the exceptional configuration of the central vertex, two formulations in Lemmas 3.4 and 5.1, the description of the original programs, the bibliographic data, the statement on the optimal lower bound, and Section 7.

A third independent verification run, also AI-assisted, examined the text of this note. It read again [BR20] (arXiv version 1, L^AT_EX source, and the Crossref record of the published chapter), [BW03, pp. 453–456, 468 and 473–477], [RS97] (the authors’ preprint: Section 1, condition (4.6), Theorem 5.5), [Bow00b, Proposition 1.2, Definition 2.1, Theorems 2.3 and 2.5], [Bow00a, pp. 204–205], [BL11, Theorem 1.1, Definitions 2.2 and 3.1, Proposition 3.10], [BKW15, Sections 2 and 3] and [DD05, Section 2], and re-derived every proof line by line. Its own programs (in the package, `independent_run_2/`) checked Lemma 3.1 against the generator sums with complex exponentials (16 fibers, deviation below $3 \cdot 10^{-14}$); on 132 fibers (14 values of b , random fibers and fibers within 10^{-9} of the exceptional configurations), with all entries of windows of 512 vertices computed from the entry formula, Lemma 3.2 in its printed form, Lemmas 3.4, 4.1 to 4.4 and 5.1, Lemma 5.2 for all finite sets F in the inner part of the window (by a dynamic programme), Lemma 5.3 and the spectral bounds, with exact elimination of $G_W - \frac{1}{54}I$ and $6I - G_W$ on 12 further fibers; the facts used in Remark 4.5 and Section 7; and, with two eigenvalue methods, Table 1, the largest eigenvalue quoted in Remark 6.2, and the tree of y_c of Remark 6.2 (3,145,727 vertices, smallest eigenvalue 0.08190), on which it also confirmed Lemmas 5.1 to 5.3. It ran every program of the package again from the archive; all recorded outputs were reproduced, up to timing fields and the last digits of iterative eigenvalue computations, among them the certificate of Remark 6.1. It repeated the literature search. It found no mathematical error and no gap. Its corrections were made: the normalisation and the page range of [BW03], the problem number used by the record, the central vertex in the proof of Lemma 4.4, the metric in the citation of [BKW15], the description of `check_note.py`, the authors of [LC03], the statements on [LC03, YK23] and on the scope, and Remark 6.2.

Scope and priority. The note answers Problem 1.1. The bounds $1/54$ and 6 are not optimal, and the optimal bounds are not determined; the bound $1/16$ of Remark 6.1 is computer-assisted. Dual frame wavelets of ψ_b (the third column of the table of [BR20]) are not considered for $1/8 < b \leq 1/6$. The question, its conventions and the known ranges of b are due to Bownik and Weber [BW03], who believed the remaining range to consist of frame wavelets; Bownik and Rzeszotnik [BR20] restated it as open. Section 2 uses theorems of Ron and Shen [RS95, RS97], Chui, Shi and Stöckler [CSS98], Bownik [Bow00b, Bow00a] and Bownik and Lemvig [BL11]; the discrete Cheeger inequality is due to Dodziuk [Dod84], in the form of [BKW15], and the Schur test to Schur [Sch11]. Only the forest structure of Section 4 and the flow argument of Section 5 are, to our knowledge, not in print.

What was read: [BR20] (arXiv v1, L^AT_EX source); [BW03] (publisher’s open-access file), the conventions on p. 456, Remark 1 and Example 2; of [RS97], in the authors’ preprint (not compared with the journal version), Theorems 1.1 and 5.5 and condition (4.6); [RS95, Theorem 3.3.5]; [Bow00b, Definition 2.1, Theorems 2.3, 2.5] and [BL11, Theorem 1.1, Section 3.1, Proposition 3.10] in the authors’ copies; [Bow00a, Section 1]; [BKW15, Sections 1–3] (arXiv v1); [DD05, Section 2]. The proofs of the cited theorems were not checked. [CSS98, Dod84, DDG01, DDG04, Sch11] were not read; what

we say about them comes from [Bow00a, BW03, BKW15, DD05]. Of [LC03, Guo13, YK23] only titles, available abstracts and the summaries in zbMATH Open were seen. The searches (18 October 2026) were: arXiv API queries (the authors of the problem, frame wavelets given by characteristic functions, frame wavelet sets, quasi-affine systems), Crossref, zbMATH Open, the works citing [BW03] (28 in OpenAlex, 31 in Semantic Scholar) and [BR20] (2 and 7), whose titles and available abstracts were read, the arXiv listing of M. Bownik, the works citing [DDG01] in OpenAlex, and a few web searches. The third verification run repeated arXiv API queries of the same kinds and the query for M. Bownik, searched zbMATH Open for frame wavelet sets (titles, and the summaries of [LC03, Guo13, YK23] and of three further works on frame wavelet sets), checked all DOIs with Crossref and made one web search; OpenAlex refused its requests (rate limit), so the counts of citing works were not checked again. No printed answer to the question was found in these sources; a search that finds nothing is not a proof of novelty, and no priority is claimed.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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