

Explicit Conservation Laws for the Normal Forms of Two-Dimensional gl-Regular Nijenhuis Operators

Alper Ferudun*

October 18, 2026

Abstract

Bolsinov, Konyaev and Matveev asked in 2024 to describe all conservation laws of the normal forms of two-dimensional gl-regular Nijenhuis operators, or at least to find one regular conservation law for each of them. A conservation law of L is a function F with $d(L^*dF) = 0$, regular if dF and L^*dF are linearly independent; everything is local and real analytic. The existence of a regular law was known, by the Cauchy–Kovalevskaya theorem, as were explicit laws at algebraically generic points and the laws $\text{tr } F(L)$ at differentially non-degenerate points. For every normal form and all values of its parameters we give an explicit regular conservation law. For the singular series O , P and S it has the form $f_1(\lambda_1) + f_2(\lambda_2)$ anticipated in the problem, with f_i a finite sum of negative fractional powers, or a logarithm, of the eigenvalue; the lower-order coefficients solve a triangular linear system with unit pivots, and for $k = 1$ there are closed forms, for instance the analytic root Φ of $8y\Phi^3 + 6\Phi = c_0 + 3x$ for S_c^3 . We also describe all real-analytic conservation laws of every normal form; for O , P and S they consist of an explicit finite principal part subject to finitely many linear conditions, at most one logarithm, and arbitrary analytic functions of explicit roots of the eigenvalues. Completeness is proved by complexification, Cauchy estimates on the fibres of the eigenvalue map and a monodromy argument. No printed answer was found in the sources we searched; one 2025 survey could not be consulted. This is an unrefereed note.

1 Introduction

1.1 The problem

Let L be an operator field on a manifold and L^* the dual operator on 1-forms. A function F is a *conservation law* of L if the 1-form L^*dF is closed [BKM24d, §4.1]. On a surface a conservation law is *regular* at a point p if $dF(p)$ and $L^*dF(p)$ are linearly independent, that is, if $dF(p)$ is a cyclic covector of $L^*(p)$ [BKM24d, §4.1], [BKM24c, §1]; for a Nijenhuis operator this says that the hierarchy $f_1 = F$, $df_2 = L^*df_1$ consists of functionally independent functions. As in [BKM24d, §4.1], all objects are local and real analytic.

Problem 4.3 of [BKM24d] (the third problem of its Section 4) refers to the classification of gl-regular Nijenhuis operators in dimension two by explicit normal forms [BKM24b]. It asks to describe all conservation laws of each of these operators, and adds: “Or at least, find one regular conservation law for each of them.” The remark after the problem expects a regular law of the form $f_1(\lambda_1) + f_2(\lambda_2)$, where λ_1, λ_2 are the eigenvalues of L and the f_i are not smooth at the singular point while the sum is; it notes that symmetric choices $f_1 = f_2$ give functions of $\text{tr } L$ and $\det L$, whose differentials vanish at the singular point as a rule. The problem is record AMR-154-0006 of the UnsolvedMath collection

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>. Unrefereed preprint. CC BY 4.0.

[ula26], where it is called Problem 6 (its position among the problems taken from the source) and is listed as open.

1.2 The normal forms

We use coordinates (x, y) and write $L = (l_{ij})$, where i is the row index (upper index) and j the column index, so that $L^*dF = (l_{11}F_x + l_{21}F_y)dx + (l_{12}F_x + l_{22}F_y)dy$. At an algebraically generic point a gl-regular Nijenhuis operator on a surface has one of the following forms [BKM24b, §5]: (G1) $L = \text{diag}(f(x), g(y))$ with $f \neq g$; (G2) $L = \begin{pmatrix} f & -g \\ g & f \end{pmatrix}$ with $f + ig$ holomorphic in $x + iy$ and $g \neq 0$; (G3) $L = \begin{pmatrix} f(y) & 1 \\ 0 & f(y) \end{pmatrix}$. At a point p where the eigenvalues collide, $L(p) = \lambda_0 \text{Id} + J_0$ with $J_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ [BKM24b, §5]. Since $L - \lambda_0 \text{Id}$ is again Nijenhuis and $(L - \lambda_0 \text{Id})^*dF = L^*dF - \lambda_0 dF$, both operators have the same conservation laws and the same regular ones, and we assume $L(p) = J_0$, $p = (0, 0)$. Theorem 5.1 of [BKM24b] (Theorem 6.1 of its arXiv version 1, the only one) states that in suitable coordinates L is one of

$$L_{\text{nil}} = J_0, \quad L_{\text{nd}} = \begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix}, \quad M_{2k-1}, \quad M_{2k}^\varepsilon = \begin{pmatrix} 0 & 1 \\ 0 & g(y) \end{pmatrix}, \quad N_{2k-1}, \quad N_{2k}^\varepsilon = \begin{pmatrix} g(y) & 1 \\ 0 & g(y) \end{pmatrix},$$

with $g(y) = y^{2k-1}$ resp. $g(y) = \varepsilon y^{2k}$ ($k \geq 1$, $\varepsilon = \pm 1$), or a member of one of three series defined by $v = \text{tr } L$ and $u = -\det L$ through the reconstruction formula [BKM24b, (5.1)]

$$L = J^{-1} \begin{pmatrix} v & 1 \\ u & 0 \end{pmatrix} J, \quad J = \begin{pmatrix} v_x & v_y \\ u_x & u_y \end{pmatrix}, \quad (1)$$

in which the denominator $\det J$ cancels. With $k \geq 1$, $\varepsilon = \pm 1$, $c = (c_0, \dots, c_{k-1}) \in \mathbb{R}^k$ and $C(y) = c_0 + c_1 y + \dots + c_{k-1} y^{k-1}$:

$$\begin{aligned} O_{k,c}^{d,\varepsilon} : v &= \alpha x y^{2k-1} + y^k C(y), \quad u = \varepsilon y^d, \quad \alpha = k c_0^2 (1 - \frac{k}{d}) \neq 0, \quad d \geq 2k+1, \quad \varepsilon = 1 \text{ if } d \text{ is odd}; \\ P_{s,c}^{k,\varepsilon} : v &= \alpha x y^s + y^{s-k+1} C(y) + 2\varepsilon y^k, \quad u = -y^{2k}, \quad \alpha = 2\varepsilon k c_0 \neq 0, \quad s \geq 2k; \\ S_c^{2k,\varepsilon} : v &= \alpha x y^{2k-1} + y^k C(y), \quad u = \varepsilon y^{2k}, \quad \alpha = \frac{k}{2}(c_0^2 + 4\varepsilon) \neq 0; \\ S_c^{2k+1} : v &= \alpha x y^{2k} + y^{k+1} C(y), \quad u = y^{2k+1}, \quad \alpha = 2k+1. \end{aligned}$$

Throughout we put

$$\xi := C(y) + \alpha x y^{k-1}, \quad \xi(0, 0) = c_0, \quad (2)$$

so that (v, u) equals $(y^k \xi, \varepsilon y^d)$, $(2\varepsilon y^k + y^{s-k+1} \xi, -y^{2k})$, $(y^k \xi, \varepsilon y^{2k})$ and $(y^{k+1} \xi, y^{2k+1})$ in the four series. The eigenvalues $\lambda_{1,2}$ are the roots of $\lambda^2 - v\lambda - u$.

1.3 Results

Theorem 1.1. *Let L be one of the normal forms of [BKM24b, Theorem 5.1], with any admissible values of the parameters, or of the generic types G1–G3. The function F of Table 1 is real analytic near the origin, is a conservation law of L , and is regular at the origin. If $L(0) = J_0$, regularity at the origin is the inequality $F_x(0, 0) \neq 0$, and the value of $F_x(0, 0)$ is the last entry of the row. For the series O , P and S , on the half-neighbourhood $\{y > 0\}$ the law has the form $f_1(\lambda_1) + f_2(\lambda_2)$, or $2 \text{Re } f(\lambda)$ if the eigenvalues are complex, where the f_i are finite sums of powers of the eigenvalue or a logarithm; the most negative exponents are $-\frac{2k-1}{2k+1}$ for S_c^{2k+1} , $-\frac{k-1}{k}$ for $S_c^{2k,\varepsilon}$ and for λ_1 in $O_{k,c}^{d,\varepsilon}$, $-\frac{k-1}{d-k}$ for λ_2 in $O_{k,c}^{d,\varepsilon}$, and $-\frac{s-1}{2k}$ for $P_{s,c}^{k,\varepsilon}$.*

normal form	regular conservation law F	$F_x(0, 0)$
G1	$x + y \quad (dF \wedge L^* dF = (g - f) dx \wedge dy)$	—
G2	$x \quad (dx \wedge L^* dx = -g dx \wedge dy)$	—
G3, $N_{2k-1}, N_{2k}^\varepsilon$	$\mathcal{E}(x, f'(y))$, where $\mathcal{E}(x, c) := (e^{cx} - 1)/c$ and $L = \begin{pmatrix} f(y) & 1 \\ 0 & f(y) \end{pmatrix}$	1
$L_{\text{nil}}, M_{2k-1}, M_{2k}^\varepsilon, L_{\text{nd}}$	$x \quad (\text{for } L_{\text{nd}}: x = \text{tr } L = \lambda_1 + \lambda_2)$	1
S_c^3	$\Phi = -\frac{1}{2}(\lambda_1^{-1/3} + \lambda_2^{-1/3})$, the analytic root of $8y\Phi^3 + 6\Phi = c_0 + 3x$ with $\Phi(0, 0) = \frac{c_0}{6}$	$\frac{1}{2}$
$S_c^{2k+1}, k \geq 2$	$\lambda_1^\gamma + \lambda_2^\gamma + \sum_{j=1}^{k-1} \frac{b_j}{2} (\lambda_1^{-2j/(2k+1)} + \lambda_2^{-2j/(2k+1)})$, $\gamma = -\frac{2k-1}{2k+1}$	$-(2k-1)$
$S_c^{2,\varepsilon}$, real	$\log \lambda_1/\lambda_2 = 2 \operatorname{arsinh} \frac{\xi}{2} \quad (\varepsilon = 1), = 2 \operatorname{sgn}(c_0) \operatorname{arcosh} \frac{ \xi }{2} \quad (\varepsilon = -1)$	q
$S_c^{2k,\varepsilon}$, real, $k \geq 2$	$(\lambda_1/\rho_+)^{\gamma} - (\lambda_2/\rho_-)^{\gamma} + \sum_{j=1}^{k-1} b_j (\lambda_1/\rho_+)^{-j/k}$, $\gamma = -\frac{k-1}{k}$	$-(k-1)q$
$S_c^{2,-1}$, complex	$\arg \lambda = \arccos \frac{\xi}{2}$	$\frac{q}{2}$
$S_c^{2k,-1}$, complex, $k \geq 2$	$\operatorname{Im}(\lambda/\rho_0)^{\gamma} + \sum_{j=1}^{k-1} b_j \operatorname{Re}(\lambda/\rho_0)^{-j/k}$, $\gamma = -\frac{k-1}{k}$	$-\frac{(k-1)q}{2}$
$P_{s,c}^{1,\varepsilon}$	$\sqrt{1+2\varepsilon x} U_{s-2} \left(\sqrt{1 + \frac{1}{4}\varepsilon c_0 y^{s-1}(1+2\varepsilon x)} \right)$	$\varepsilon(s-1)$
$P_{s,c}^{k,\varepsilon}, k \geq 2$	$\Xi + \sum_{j=1}^{k-1} b_j \Upsilon_j$	$-\varepsilon(s-1)$
$O_{1,c}^{d,\varepsilon}$	$\log(A/c_0) = \frac{1}{d}((d-1)\log \lambda_1 - \log \lambda_2) - \log c_0 $	$\frac{(d-1)c_0}{d}$
$O_{k,c}^{d,\varepsilon}, k \geq 2$	$(\lambda_1/c_0)^{-(k-1)/k} - (-\varepsilon c_0 \lambda_2)^{-(k-1)/(d-k)} + \sum_{j=1}^{k-1} b_j (\lambda_1/c_0)^{-j/k}$	$-(k-1)c_0$

Conventions (the formulas in eigenvalues hold on $\{y > 0\}$; ξ is (2)). S_c^{2k+1} : $\lambda_1 > 0 > \lambda_2$, real $(2k+1)$ -st roots. $S_c^{2k,\varepsilon}$, real case $c_0^2 + 4\varepsilon > 0$: $q = \sqrt{c_0^2 + 4\varepsilon}$, $\rho_{\pm} = \frac{1}{2}(c_0 \pm q)$, $\lambda_{1,2} = \frac{1}{2}y^k(\xi \pm \sqrt{\xi^2 + 4\varepsilon})$; the quotients λ_1/ρ_+ , λ_2/ρ_- are positive and real powers are used. $S_c^{2k,-1}$, complex case $c_0^2 < 4$: $q = \sqrt{4 - c_0^2}$, $\rho_0 = \frac{1}{2}(c_0 + iq)$, λ is the eigenvalue with $\operatorname{Im} \lambda > 0$, $\arg \lambda \in (0, \pi)$, and powers of λ/ρ_0 are principal. $P_{s,c}^{k,\varepsilon}$: U_n is the Chebyshev polynomial of the second kind, $\gamma = -\frac{s-1}{2k}$; if $\varepsilon c_0 > 0$, the eigenvalues are real, $\lambda_{\pm}$ with $|\lambda_+| > |\lambda_-|$, and $\Xi = (|\lambda_+|^{\gamma} - |\lambda_-|^{\gamma})/\sqrt{\varepsilon c_0}$, $\Upsilon_j = \frac{1}{2}(|\lambda_+|^{-j/k} + |\lambda_-|^{-j/k})$; if $\varepsilon c_0 < 0$, they are complex, λ is the eigenvalue with $\operatorname{Im}(\varepsilon \lambda) > 0$, $\Xi = 2 \operatorname{Im}(\varepsilon \lambda)^{\gamma}/\sqrt{-\varepsilon c_0}$ and $\Upsilon_j = \operatorname{Re}(\varepsilon \lambda)^{-j/k}$ with principal powers (for the other eigenvalue Ξ changes sign). $O_{k,c}^{d,\varepsilon}$: $A = \frac{1}{2}\xi(1 + \sqrt{1 + 4\varepsilon y^{d-2k}\xi^{-2}})$, $\lambda_1 = y^k A$, $\lambda_2 = -\varepsilon y^{d-k}/A$, so $\lambda_1/c_0 > 0$ and $-\varepsilon c_0 \lambda_2 > 0$. The constants b_j solve the triangular system (PC) of Section 3.4; for $k = 2$: $b_1 = \frac{3}{5}c_0$ (S_c^5), $b_1 = 0$ ($S_c^{4,\varepsilon}$, $O_{2,c}^{d,\varepsilon}$), $b_1 = \frac{s-1}{2}$ ($P_{s,c}^{2,\varepsilon}$).

Table 1: The regular conservation laws of Theorem 1.1.

For $k \geq 2$ the constants b_j are polynomials in $c_1, \dots, c_{k-2}$ with coefficients depending on c_0 and on the discrete parameters, given by an explicit recursion (Section 3.4); they do not depend on c_{k-1} (Remark 3.1).

Theorem 1.2. *Let V be the space of germs at the origin of real-analytic conservation laws of L , and $V_{\mathbb{C}}$ the space of complex-valued ones; V consists of the real-valued elements of $V_{\mathbb{C}}$.*

- (a) *For the generic types and the forms $L_{\text{nil}}, L_{\text{nd}}, M, N$: (G1) $V = \{F_1(x) + F_2(y)\}$; (G2) V is the space of harmonic functions; (G3, L_{nil}, N ; $L = \begin{pmatrix} f(y) & 1 \\ 0 & f(y) \end{pmatrix}$) $V = \{A(y)\mathcal{E}(x, f'(y)) + B(y)\}$; (M ; $g(y) = \varepsilon y^m$) $V = \{F : F_x + g(y)F_y \text{ depends on } y \text{ only}\} = \{H(\eta) + \int_0^x \varphi(Y(t, \eta)) dt\}$, with the characteristics $Y(t, \eta)$ of Section 4.1; (L_{nd}) $V = \{\operatorname{tr} h(L)\} = \{h(\lambda_1) + h(\lambda_2)\}$. Here $A, B, F_1, F_2, H, \varphi, h$ are arbitrary analytic germs of one variable.*
- (b) *Let L be $O_{k,c}^{d,\varepsilon}$, $P_{s,c}^{k,\varepsilon}$, $S_c^{2k,\varepsilon}$ or S_c^{2k+1} , and let σ, T_1, T_2, N_0 and the type be as in Table 2. Then*

series	σ	T_1	T_2	N_0	type	κ
S_c^{2k+1}	$\sqrt{y}$	$\lambda_1^{1/(2k+1)}$	$\lambda_2^{1/(2k+1)}$	$2k-1$	symmetric	1
$S_c^{2k,\varepsilon}, c_0^2 + 4\varepsilon > 0$	y	$(\lambda_1/\rho_+)^{1/k}$	$(\lambda_2/\rho_-)^{1/k}$	$k-1$	split	q
$S_c^{2k,-1}, c_0^2 < 4$	y	$(\lambda/\rho_0)^{1/k}$	$\overline{T_1}$	$k-1$	split	iq
$O_{k,c}^{d,\varepsilon}$	y	$(\lambda_1/c_0)^{1/k}$	$(-\varepsilon c_0 \lambda_2)^{1/(d-k)}$	$k-1$	split	c_0
$P_{s,c}^{k,\varepsilon}, s \text{ odd}$	y	$(\varepsilon \lambda_+)^{1/k}$	$(\varepsilon \lambda_-)^{1/k}$	$\frac{s-1}{2}$	split	$2\varepsilon w(c_0)$
$P_{s,c}^{k,\varepsilon}, s \text{ even}$	$\sqrt{y}$	$(\varepsilon \lambda_+)^{1/(2k)}$	$-(\varepsilon \lambda_-)^{1/(2k)}$	$s-1$	symmetric	$\varepsilon w(c_0)$

On $\{y > 0\}$, with the conventions of Table 1; for $P_{s,c}^{k,\varepsilon}$ with $\varepsilon c_0 < 0$, $\lambda_+ := \lambda$, $\lambda_- := \bar{\lambda}$, and the roots are principal. Each T_i is an analytic function σr_i of (σ, x) , with $r_1(0,0) = 1$, and $r_2(0,0) = 1$ (split) or -1 (symmetric) (Section 4.2). $M := N_0 + 1$ and κ are the exponent and the constant of the fibre derivative (Section 4.6); $w(c_0) = \sqrt{\varepsilon c_0}$ if $\varepsilon c_0 > 0$ and $i\sqrt{-\varepsilon c_0}$ otherwise.

Table 2: Uniformising functions and fibre constants for Theorem 1.2(b).

$V_{\mathbb{C}}$ consists of the germs which on $\{y > 0\}$ are given by

$$\text{split type: } c \log \frac{T_1}{T_2} + \sum_{n=1}^{N_0} (a_n T_1^{-n} + b_n T_2^{-n}) + g_1(T_1) + g_2(T_2),$$

$$\text{symmetric type: } \sum_{n=1}^{N_0} a_n (T_1^{-n} + T_2^{-n}) + g(T_1) + g(T_2),$$

where g, g_1, g_2 are holomorphic germs at $0 \in \mathbb{C}$ and the constants $c, a_n, b_n \in \mathbb{C}$ satisfy the pole-cancellation condition (PC): the restriction of the finite sum of negative powers to $x = 0$, which is a Laurent series in y , has no principal part. (PC) consists of N_0 (split), $k-1$ (S_c^{2k+1}) or $\frac{s-2}{2}$ ($P_{s,c}^{k,\varepsilon}$, s even) independent linear equations, and each such expression extends to an analytic germ at the origin.

As λ_i is a constant multiple of $T_i^{N_i}$, $N_i \in \{2k+1, k, d-k, 2k\}$, every law of O, P, S is $f_1(\lambda_1) + f_2(\lambda_2)$ on $\{y > 0\}$, with f_i a combination of λ^{-n/N_i} , $1 \leq n \leq N_0$, of $\log \lambda$ (split types) and of an analytic function of λ^{1/N_i} ; the most negative exponents are those of Theorem 1.1, and they decide regularity:

Corollary 1.3. *In Theorem 1.2(b), $F_x(0,0) = -N_0 \kappa a_{N_0}$ if $N_0 \geq 1$, and $F_x(0,0) = \kappa c$ if $N_0 = 0$ (that is, for $S_c^{2,\varepsilon}$ and $O_{1,c}^{d,\varepsilon}$), where $\kappa \neq 0$ is the constant of Table 2. Hence F is regular at the origin if and only if $a_{N_0} \neq 0$, resp. $c \neq 0$.*

1.4 What was known, and what is new

The problem and the normal forms are due to Bolsinov, Konyaev and Matveev [BKM24d, BKM24b]. By [BKM24c, Theorem 1.5], an existence theorem in the real analytic category, a regular conservation law exists near every point, and all laws are M^*df for symmetries M and one regular law df . In dimension two the existence also follows directly from the Cauchy–Kovalevskaya theorem: by (3) the coefficient of F_{xx} in $E(F)$ is l_{12} , which equals 1 at a point where $L = J_0$, so the data $F(0,y) = 0$, $F_x(0,y) = 1$ on the non-characteristic line $\{x = 0\}$ determine a regular law. At algebraically generic points, that is, for the types G1–G3, among them L_{nil} and N , which are Jordan blocks at every point, the laws are described in [BKM24c] (see [BKM24d, §4.1]), and at differentially non-degenerate points, as for L_{nd} , they are the functions $\text{tr } F(L)$ [BKM24d, §4.1]. New, to our knowledge, are the explicit regular laws for the singular normal forms M, O, P, S and Theorem 1.2 for these forms; for G1–G3,

L_{nil} , N and L_{nd} the results are known or elementary and are included for completeness (see “Scope and priority” in Section 5; one survey [Bol25] was not accessible to us). The smooth category, where the existence of regular laws is open [BKM24c], and higher dimensions are not treated.

2 Preliminaries

For a function F we have $d(L^*dF) = E(F) dx \wedge dy$ with

$$E(F) := \partial_x(l_{12}F_x + l_{22}F_y) - \partial_y(l_{11}F_x + l_{21}F_y). \quad (3)$$

If $L(p) = J_0$, then $L^*dF(p) = F_x(p) dy$ and $dF \wedge L^*dF(p) = F_x(p)^2 dx \wedge dy$. So:

Lemma 2.1. *If $L(p) = J_0$, a conservation law F is regular at p if and only if $F_x(p) \neq 0$. Regularity at a point is an open condition.*

If F is analytic on a connected neighbourhood W of the origin and $E(F) = 0$ on a non-empty open subset of W , then $E(F) = 0$ on W , since $E(F)$ is analytic. We use this to pass from the half-neighbourhood $\{y > 0\}$ to a full neighbourhood.

Lemma 2.2 (eigenvalue functions). *Let L be a member of the series O , P , S , or $L = L_{\text{nd}}$ with $(v, u) = (x, y)$. Then $L^*dv = v dv + du$ and $L^*du = u dv$. If λ is a real or complex valued C^1 function on an open set U with $\lambda^2 = v\lambda + u$ and $2\lambda \neq v$ on U , then $L^*d\lambda = \lambda d\lambda$ on U . Hence, if h is C^2 on the range of λ (holomorphic, if λ is complex valued), then $L^*d(h \circ \lambda) = d(\tilde{h} \circ \lambda)$ with $\tilde{h}'(t) = th'(t)$. In particular $h_1(\lambda_1) + h_2(\lambda_2)$ is a conservation law wherever the eigenvalues are real and distinct, and $2 \operatorname{Re} h(\lambda)$ is one wherever they are complex conjugate.*

Proof. Formula (1) says $JL = \begin{pmatrix} v & 1 \\ u & 0 \end{pmatrix} J$. The rows of J are dv and du and the rows of JL are L^*dv and L^*du , so the two identities hold where $\det J \neq 0$, hence everywhere, as both sides are polynomial. For L_{nd} they are read off the matrix. Differentiating $\lambda^2 = v\lambda + u$ gives $(2\lambda - v) d\lambda = \lambda dv + du$; applying L^* , which is linear over functions,

$$(2\lambda - v)L^*d\lambda = \lambda(v dv + du) + u dv = \lambda^2 dv + \lambda du = \lambda(2\lambda - v) d\lambda.$$

Then $L^*d(h \circ \lambda) = h'(\lambda)\lambda d\lambda = d(\tilde{h} \circ \lambda)$, a closed form. $\square$

Lemma 2.3 (Hadamard). *Let $H(\xi, y)$ be analytic near $(c_0, 0)$, let $m \geq 0$ be an integer such that $\partial_\xi H$ is divisible by y^m , and let $j \leq k - 1 + m$ be an integer. Then, with ξ as in (2),*

$$y^{-j}H(\xi, y) = y^{-j}H(C(y), y) + \alpha x y^{k-1+m-j}K(x, y)$$

with K analytic near the origin and $K(0, 0) = (y^{-m}\partial_\xi H)(c_0, 0)$. Consequently, for finitely many such terms (with m_t, j_t) put

$$F := \sum_t a_t y^{-j_t} H_t(\xi, y) \quad (y > 0), \quad \Lambda(y) := \sum_t a_t y^{-j_t} H_t(C(y), y).$$

Then Λ is a convergent Laurent series, $F - \Lambda$ is analytic near the origin, F extends to an analytic function near the origin if and only if Λ has no principal part, and in that case $F_x(0, 0) = \alpha \sum_{j_t=k-1+m_t} a_t (y^{-m_t}\partial_\xi H_t)(c_0, 0)$.

Proof. By (2), $\xi - C(y) = \alpha x y^{k-1}$, and $H(\xi, y) - H(C(y), y) = (\xi - C(y)) \int_0^1 \partial_\xi H(C(y) + \tau(\xi - C(y)), y) d\tau$; the integral is y^m times an analytic function K with the stated value at the origin. The x -derivative of $\alpha x y^{k-1+m-j}K$ at the origin is $\alpha K(0, 0)$ if $j = k - 1 + m$ and 0 if $j < k - 1 + m$. The function $y^J \Lambda(y)$, $J = \max_t j_t$, is analytic at 0, so Λ is a convergent Laurent series, and it is analytic at 0 if and only if $F = \Lambda + (F - \Lambda)$ extends analytically. $\square$

We shall use the even functions of z

$$\mathcal{S}_g(Z) := \frac{\sinh(g \operatorname{arsinh} z)}{z}, \quad \mathcal{C}_g(Z) := \cosh(g \operatorname{arsinh} z), \quad Z = z^2,$$

which are analytic for $|Z| < 1$, with $\mathcal{S}_g(0) = g$, $\mathcal{C}_g(0) = 1$; the Taylor coefficients of $f = z\mathcal{S}_g(z^2)$ and $f = \mathcal{C}_g(z^2)$ in z follow from $(1+z^2)f'' + zf' - g^2f = 0$, that is, $f_{n+2} = (g^2 - n^2)f_n/((n+1)(n+2))$. Since $\operatorname{arsinh}(it) = i \arcsin t$, we have $\sin(g \arcsin t) = t\mathcal{S}_g(-t^2)$ and $\cos(g \arcsin t) = \mathcal{C}_g(-t^2)$; and $\sinh(n\theta) = \sinh \theta U_{n-1}(\cosh \theta)$ gives $\mathcal{S}_{-n}(Z) = -U_{n-1}(\sqrt{1+Z})$ for integers $n \geq 1$.

3 Proof of Theorem 1.1

3.1 Generic types and elementary forms

For $L = \begin{pmatrix} f(y) & 1 \\ 0 & f(y) \end{pmatrix}$ (G3, L_{nil} with $f = 0$, N) we have $L^*dF = fF_x dx + (F_x + fF_y) dy$ and

$$E(F) = F_{xx} - f'(y)F_x. \quad (4)$$

The function $F = \mathcal{E}(x, f'(y))$ is analytic, as $\mathcal{E}(x, c) = \sum_{n \geq 1} c^{n-1} x^n / n!$ is entire, and $F_x = e^{f'(y)x}$, $F_{xx} = f'(y)F_x$; so $E(F) = 0$ and $F_x \neq 0$ everywhere, which is regularity at every point by Lemma 2.1 applied to $L - f(y)\text{Id}$. For L_{nil} and M the first row of L is $(0, 1)$, so $L^*dx = dy$; for L_{nd} , $L^*dx = x dx + dy = d(\frac{1}{2}x^2 + y)$. For G1, $L^*dF = fF_x dx + gF_y dy$, so $E(F) = (g - f)F_{xy}$ and $E(x + y) = 0$, with $dF \wedge L^*dF = (g - f) dx \wedge dy \neq 0$. For G2, $L^*dx = f dx - g dy$ is closed because $f_y = -g_x$, and $dx \wedge L^*dx = -g dx \wedge dy \neq 0$.

3.2 The eigenvalues on $\{y > 0\}$

In each series we write the eigenvalues as functions of (ξ, y) ; that their sum is v and their product $-u$ is checked directly from the expressions of (v, u) after (2). They are distinct and non-zero for small $y > 0$. (S_c^{2k+1}) With $s = \sqrt{y}$ and $\mu := \operatorname{arsinh}(s\xi/2)$: $\lambda_1 = s^{2k+1}e^\mu$, $\lambda_2 = -s^{2k+1}e^{-\mu}$. ($S_c^{2k, \varepsilon}$, $c_0^2 + 4\varepsilon > 0$) $\lambda_{1,2} = y^k \rho_\pm(\xi)$ with $\rho_\pm(\xi) = \frac{1}{2}(\xi \pm \sqrt{\xi^2 + 4\varepsilon})$, $\rho_\pm(c_0) = \rho_\pm$ and $\rho'_\pm/\rho_\pm = \pm(\xi^2 + 4\varepsilon)^{-1/2}$. ($S_c^{2k, -1}$, $c_0^2 < 4$) $\lambda = y^k e^{i\theta(\xi)}$, $\theta(\xi) = \arccos \frac{\xi}{2} \in (0, \pi)$, $\theta' = -(4 - \xi^2)^{-1/2}$. ($P_{s, \varepsilon}^{k, \varepsilon}$) Put $e := s - 2k + 1 \geq 1$ and $Z := \frac{1}{4}\varepsilon y^e \xi$; then $v = 2\varepsilon y^k(1 + 2Z)$ and $-u = y^{2k}$. If $\varepsilon c_0 > 0$, then $Z > 0$ and $\lambda_\pm = \varepsilon y^k e^{\pm \varphi}$ with $\cosh \varphi = 1 + 2Z$, $\varphi = 2 \operatorname{arsinh} \sqrt{Z}$; if $\varepsilon c_0 < 0$, then $Z < 0$ and $\varepsilon \lambda = y^k e^{i\psi}$ with $\cos \psi = 1 + 2Z$, $\psi = 2 \arcsin \sqrt{-Z} \in (0, \pi)$. ($O_{k, \varepsilon}^{d, \varepsilon}$) A is the root of $A^2 - \xi A - \varepsilon y^{d-2k} = 0$ with $A(\xi, 0) = \xi$, analytic near $(c_0, 0)$ as $c_0 \neq 0$ and $d - 2k \geq 1$; $\lambda_1 = y^k A$, $\lambda_2 = -\varepsilon y^{d-k}/A$.

3.3 The terms of the laws

On $\{y > 0\}$ every term of a law in Table 1 is of the form $y^{-j}H(\xi, y)$ with H analytic near $(c_0, 0)$. (S_c^{2k+1}) As $\sinh(g \operatorname{arsinh} \frac{s\xi}{2}) = \frac{s\xi}{2}\mathcal{S}_g(\frac{y\xi^2}{4})$, for odd n we get $\lambda_1^{n/(2k+1)} + \lambda_2^{n/(2k+1)} = 2s^n \sinh \frac{n\mu}{2k+1} = y^{(n+1)/2} \xi \mathcal{S}_{n/(2k+1)}(\frac{y\xi^2}{4})$, and for even n the sum is $2y^{n/2} \mathcal{C}_{n/(2k+1)}(\frac{y\xi^2}{4})$. The leading term ($n = 1 - 2k$) is $y^{-(k-1)}H_0$ with $H_0 = \xi \mathcal{S}_\gamma(\frac{y\xi^2}{4})$, and the j -th correction term (with its factor $\frac{1}{2}$) is $y^{-j} \mathcal{C}_{-2j/(2k+1)}(\frac{y\xi^2}{4})$. ($S_c^{2k, \varepsilon}$) In the real case $(\lambda_{1,2}/\rho_\pm)^a = y^{ka}(\rho_\pm(\xi)/\rho_\pm)^a$, so $H_0 = (\rho_+(\xi)/\rho_+)^a - (\rho_-(\xi)/\rho_-)^a$, $H_j = (\rho_+(\xi)/\rho_+)^{-j/k}$, and $\log|\lambda_1/\lambda_2| = \log|\rho_+(\xi)/\rho_-(\xi)|$. In the complex case $(\lambda/\rho_0)^a = y^{ka} e^{ia(\theta(\xi) - \theta(c_0))}$, so $H_0 = \sin(\gamma(\theta(\xi) - \theta(c_0)))$, $H_j = \cos(\frac{j}{k}(\theta(\xi) - \theta(c_0)))$, and $\arg \lambda = \theta(\xi)$. ($P_{s, \varepsilon}^{k, \varepsilon}$) If $\varepsilon c_0 > 0$, then $\frac{1}{2}(|\lambda_+|^\gamma - |\lambda_-|^\gamma) = y^{k\gamma} \sinh(2\gamma \operatorname{arsinh} \sqrt{Z}) = y^{k\gamma} \sqrt{Z} \mathcal{S}_{2\gamma}(Z)$ and $\frac{1}{2}(|\lambda_+|^a + |\lambda_-|^a) = y^{ka} \mathcal{C}_{2a}(Z)$. Since $2\sqrt{Z} = y^{e/2} \sqrt{\varepsilon c_0} \omega$ with $\omega := \sqrt{\xi/c_0}$, and $k\gamma + \frac{e}{2} = -(k-1)$, we obtain

$$\Xi = y^{-(k-1)} \omega \mathcal{S}_{-(s-1)/k}(Z), \quad \Upsilon_j = y^{-j} \mathcal{C}_{-2j/k}(Z). \quad (5)$$

If $\varepsilon c_0 < 0$, then $(\varepsilon\lambda)^a = y^{ka} e^{ia\psi}$, $\text{Im}(\varepsilon\lambda)^\gamma = y^{k\gamma} \sin(2\gamma \arcsin \sqrt{-Z}) = y^{k\gamma} \sqrt{-Z} \mathcal{S}_{2\gamma}(Z)$, $\text{Re}(\varepsilon\lambda)^a = y^{ka} \mathcal{C}_{2a}(Z)$ and $2\sqrt{-Z} = y^{e/2} \sqrt{-\varepsilon c_0} \omega$, which gives (5) again. $(O_{k,c}^{d,\varepsilon}) (\lambda_1/c_0)^a = y^{ka} (A/c_0)^a$ and $(-\varepsilon c_0 \lambda_2)^b = y^{(d-k)b} (A/c_0)^{-b}$, so $H_0 = (A/c_0)^{-(k-1)/k} - (A/c_0)^{(k-1)/(d-k)}$, $H_j = (A/c_0)^{-j/k}$, and $\log(A/c_0) = \frac{1}{d}((d-k) \log|\lambda_1| - k \log|\lambda_2|) - \log|c_0|$.

In all cases the leading term has $j = k-1$, the j -th correction term has $H_j(c_0, 0) = 1$, and (i) in S_c^{2k+1} and $P_{s,c}^{k,\varepsilon}$ the correction terms depend on ξ only through $y\xi^2$ resp. $y^e \xi$, so that $\partial_\xi H_j$ is divisible by y ; (ii) in $S_c^{2k,\varepsilon}$ and $O_{k,c}^{d,\varepsilon}$, $H_0(c_0, 0) = 0$.

3.4 End of the proof

Each term is $f_1(\lambda_1) + f_2(\lambda_2)$ or $2 \text{Re } f(\lambda)$ with f_i smooth on the range of the eigenvalue, which avoids 0, so F is a conservation law on $\{y > 0\}$ by Lemma 2.2. We apply Lemma 2.3 with $m = 0$ to all terms. Let $h_{t,i}$ be the coefficient of y^i in $H_t(C(y), y)$ ($t = 0$ for the leading term, $t = j$ for the corrections). As $h_{j,0} = 1$, the coefficient of y^{-i} , $1 \leq i \leq k-1$, in $\Lambda(y) = y^{-(k-1)} H_0(C(y), y) + \sum_j b_j y^{-j} H_j(C(y), y)$ is $b_i + \sum_{j>i} b_j h_{j,j-i} + h_{0,k-1-i}$, and the pole-cancellation system

$$b_i = -h_{0,k-1-i} - \sum_{j=i+1}^{k-1} b_j h_{j,j-i}, \quad i = k-1, k-2, \dots, 1, \quad (\text{PC})$$

is triangular with unit pivots and has a unique solution. With these b_j , F extends analytically to a neighbourhood of the origin, and $E(F) = 0$ there because it vanishes on $\{y > 0\}$. By Lemma 2.3, $F_x(0, 0) = \alpha \sum_{j_t=k-1} a_t \partial_\xi H_t(c_0, 0)$. The correction with $j = k-1$ does not contribute: by (i) in S_c^{2k+1} and P , and in $S_c^{2k,\varepsilon}$ and O because $b_{k-1} = -h_{0,0} = -H_0(c_0, 0) = 0$ by (ii). So $F_x(0, 0) = \alpha \partial_\xi H_0(c_0, 0)$, which is $(2k+1)\mathcal{S}_\gamma(0) = -(2k-1)$ for S_c^{2k+1} ; $\frac{k}{2}q^2 \cdot \gamma(\rho'_+/\rho_+ - \rho'_-/\rho_-)(c_0) = \frac{k}{2}q^2 \cdot \frac{2\gamma}{q} = -(k-1)q$ for $S_c^{2k,\varepsilon}$ (real) and $-\frac{k}{2}q^2 \cdot \gamma\theta'(c_0) = -\frac{(k-1)q}{2}$ (complex); $2\varepsilon k c_0 \cdot \omega'(c_0) \mathcal{S}_{2\gamma}(0) = 2\varepsilon k c_0 \cdot \frac{\gamma}{c_0} = -\varepsilon(s-1)$ for P ; and $k c_0^2 \frac{d-k}{d} \cdot \left(-\frac{k-1}{k} - \frac{k-1}{d-k}\right) \frac{1}{c_0} = -(k-1)c_0$ for O , as $\partial_\xi A(c_0, 0) = 1$. For $k = 1$ there are no corrections and $F = H_0$ with $j = 0$: for $S_c^{2,\varepsilon}$, $F = \log|\rho_+(\xi)/\rho_-(\xi)|$ and $F_x(0, 0) = \alpha \cdot \frac{2}{q} = q$, or $F = \theta(\xi)$ and $F_x(0, 0) = \alpha\theta'(c_0) = \frac{q}{2}$; for $O_{1,c}^{d,\varepsilon}$, $F_x(0, 0) = \alpha/c_0 = \frac{d-1}{d}c_0$; S_c^3 and $P_{s,c}^{1,\varepsilon}$ are treated in Section 3.5. All values are non-zero, since $q \neq 0$, $c_0 \neq 0$ for O and P , and $s \geq 2$; so F is regular by Lemma 2.1. The exponents in the statement are read off Table 1. $\square$

Remark 3.1. Let $k \geq 2$. The coefficients $h_{t,i}$ entering (PC) have $i \leq k-2$, and the coefficient of y^i in $H_t(C(y), y)$ is a polynomial in $c_1, \dots, c_i$; so the b_j are polynomials in $c_1, \dots, c_{k-2}$ and do not depend on c_{k-1} . This agrees with the fact that moving the base point along $\{y = 0\}$ to $(x_0, 0)$, that is, putting $x' = x - x_0$, turns L into the same normal form with c_{k-1} replaced by $c_{k-1} + \alpha x_0$ and the other parameters unchanged [BKM24b, §5, the paragraph between Remark 5.5 and Proposition 5.6] (in the arXiv version, the corresponding paragraph of Section 6): ξ , every $H_t(\xi, y)$ and the b_j are unchanged, so the law of Table 1 is the same function in both coordinate systems.

3.5 Closed forms for $k = 1$

Here $\xi = c_0 + \alpha x$. (S_c^3 , $\alpha = 3$.) The function $F = \lambda_1^{-1/3} + \lambda_2^{-1/3} = \xi \mathcal{S}_{-1/3}(\frac{y\xi^2}{4})$ is analytic. With $S = \sinh \frac{\mu}{3}$ we have $F = s^{-1}(e^{-\mu/3} - e^{\mu/3}) = -2S/s$, so $\Phi := -\frac{1}{2}F = S/s$, and $\sinh \mu = 3S + 4S^3$ gives $\frac{s\xi}{2} = 3s\Phi + 4s^3\Phi^3$, that is, $8y\Phi^3 + 6\Phi = c_0 + 3x$ on $\{y > 0\}$, hence everywhere. As $\Phi(0, 0) = c_0/6$ and the derivative of the cubic in Φ there is 6, Φ is the analytic root given by the implicit function theorem, and $\Phi_x(0, 0) = \frac{3}{6} = \frac{1}{2}$. ($S_c^{2,\varepsilon}$.) For $\varepsilon = 1$, $\rho_+(\xi) = e^{\text{arsinh}(\xi/2)}$ and $\rho_-(\xi) = -1/\rho_+(\xi)$; for $\varepsilon = -1$, $\rho_+(\xi)\rho_-(\xi) = 1$ and $\rho_\pm(\xi) = \text{sgn}(c_0)e^{\pm \text{sgn}(c_0) \text{arcosh}(|\xi|/2)}$. This gives the entries of Table 1. ($P_{s,c}^{1,\varepsilon}$.) Here $\xi = c_0(1 + 2\varepsilon x)$, $\omega = \sqrt{1 + 2\varepsilon x}$, $Z = \frac{1}{4}\varepsilon c_0 y^{s-1}(1 + 2\varepsilon x)$, and (5) with $\mathcal{S}_{-(s-1)}(Z) = -U_{s-2}(\sqrt{1+Z})$

(Section 2) shows that the entry of Table 1 is $-\Xi$; so it is a regular law with $F_x(0,0) = \varepsilon(s-1)$. ($O_{1,c}^{d,\varepsilon}$) $A/c_0 = \frac{1}{2}(1 + \alpha x/c_0)(1 + (1 + 4\varepsilon y^{d-2}(c_0 + \alpha x)^{-2})^{1/2})$ with $\alpha = c_0^2(d-1)/d$.

4 Proof of Theorem 1.2

4.1 Part (a)

If λ_1, λ_2 are functions with $L^*d\lambda_i = \lambda_i d\lambda_i$ and $d\lambda_1 \wedge d\lambda_2 \neq 0$, and $F = G(\lambda_1, \lambda_2)$, then $L^*dF = \lambda_1 G_1 d\lambda_1 + \lambda_2 G_2 d\lambda_2$ and

$$d(L^*dF) = (\lambda_2 - \lambda_1) G_{12} d\lambda_1 \wedge d\lambda_2. \quad (6)$$

(G1) $E(F) = (g-f)F_{xy}$, so $F_{xy} = 0$. (G2) By the Cauchy–Riemann equations $E(F) = -g(F_{xx} + F_{yy})$. (G3, L_{nil}, N) By (4), $F_x = A(y)e^{f'(y)x}$, that is, $F = A(y)\mathcal{E}(x, f'(y)) + B(y)$. (M) Here $E(F) = \partial_x(F_x + gF_y)$, so F is a law if and only if $\varphi := F_x + g(y)F_y$ depends on y only. The solution of $dy/dx = g(y)$ through $(0, \eta)$ is $Y(x, \eta) = \eta e^x$ if $m = 1$ and $Y(x, \eta) = \eta(1 - \varepsilon(m-1)x\eta^{m-1})^{-1/(m-1)}$ if $m \geq 2$, and $\eta(x, y) = ye^{-x}$ resp. $y(1 + \varepsilon(m-1)xy^{m-1})^{-1/(m-1)}$; along it $\frac{d}{dx}F(x, Y) = \varphi(Y)$. Hence $F = H(\eta) + \int_0^x \varphi(Y(t, \eta)) dt$ with $H(\eta) = F(0, \eta)$, and conversely every such function, with H, φ analytic, is an analytic law. (L_{nd}) The function $G(l_1, l_2) := F(l_1 + l_2, -l_1 l_2)$ is analytic and symmetric. Where $x^2 + 4y > 0$ the eigenvalues are real, distinct and coordinates, $F = G(\lambda_1, \lambda_2)$, and Lemma 2.2 and (6) show that $E(F) = 0$ forces $G_{12} = 0$ on an open set, hence everywhere. So $G = h_1(l_1) + h_2(l_2)$, and by symmetry $h_1 - h_2$ is constant; thus $F = h(\lambda_1) + h(\lambda_2) + \text{const} = \text{tr } h(L) + \text{const}$. Conversely $\text{tr } h(L)$ is analytic and a law by Lemma 2.2. This recovers the description of [BKM24d, §4.1] at differentially non-degenerate points.

4.2 The uniformising functions

Put $N := 1$, $\sigma := y$ in the split types and $N := 2$, $\sigma := s = \sqrt{y}$ in the symmetric types, and let $w(\xi) := \sqrt{\varepsilon\xi}$ be the branch near c_0 with $w(c_0) = \sqrt{\varepsilon c_0}$ if $\varepsilon c_0 > 0$ and $w(c_0) = i\sqrt{-\varepsilon c_0}$ if $\varepsilon c_0 < 0$. By Section 3.2, on $\{y > 0\}$ the functions of Table 2 are: $T_1 = s e^{\mu/(2k+1)}$, $T_2 = -s e^{-\mu/(2k+1)}$ with $\mu = \text{arsinh } \frac{s\xi}{2} (S_c^{2k+1})$; $T_{1,2} = y(\rho_{\pm}(\xi)/\rho_{\pm})^{1/k}$, resp. $T_{1,2} = y e^{\pm i(\theta(\xi) - \theta(c_0))/k} (S_c^{2k,\varepsilon})$, real resp. complex case); $T_1 = y(A/c_0)^{1/k}$, $T_2 = y(A/c_0)^{-1/(d-k)} (O_{k,c}^{d,\varepsilon})$; and for $P_{s,c}^{k,\varepsilon}$

$$T_{\pm} = y \exp\left(\pm \frac{2}{k} \text{arsinh } \frac{y^{e/2} w(\xi)}{2}\right) \quad (s \text{ odd}), \quad T_{\pm} = \pm s \exp\left(\pm \frac{1}{k} \text{arsinh } \frac{s^e w(\xi)}{2}\right) \quad (s \text{ even}),$$

where for P we used $\varphi = 2 \text{arsinh}(y^{e/2} w(\xi)/2)$ in the real case and $\text{arsinh}(it) = i \arcsin t$ in the complex case. We use these formulas as definitions on a complex polydisc around $(\sigma, x) = (0, 0)$, where $\xi = C(\sigma^N) + \alpha x \sigma^{N(k-1)}$. Thus $T_i = \sigma r_i$ with r_i analytic near $(\sigma, x) = (0, 0)$ (an analytic function of (ξ, σ) near $(c_0, 0)$), $r_1(0, 0) = 1$, and $r_2(0, 0) = 1$ in the split, -1 in the symmetric types; $\lambda_i = \kappa_i T_i^{N_i}$ with non-zero constants κ_i and $N_i \in \{2k+1, k, d-k, 2k\}$. In the symmetric types $T_1(-s, x) = T_2(s, x)$, because ξ is even and μ , resp. $\text{arsinh}(s^e w/2)$ with e odd, is odd in s .

4.3 The inclusion $\supseteq$

Conservation laws. Each term is $h_1(\lambda_1) + h_2(\lambda_2)$ with h_i holomorphic on the range of the eigenvalue, which avoids 0; by Lemma 2.2, which holds verbatim for complex valued h_i and for $\lambda_2 = \bar{\lambda}_1$, it is a conservation law on $\{y > 0\}$, hence near the origin once it is analytic there.

Analytic parts. In the split types $g_i(T_i) = g_i(yr_i)$ and $\log(T_1/T_2) = \log(r_1/r_2)$ are analytic at the origin. In the symmetric types $g(T_1) + g(T_2)$ is analytic in (s, x) and, by the swap, even in s , hence analytic in (x, y) .

Principal parts. Every negative power, or the combinations below, is $y^{-j}H(\xi, y)$ as in Lemma 2.3. $(S_c^{2k, \varepsilon}, O)$ $T_i^{-n} = y^{-n}r_i^{-n}$, with $j = n \leq k-1$, $m = 0$. $(P, s \text{ odd})$ $T_+^{-n} + T_-^{-n} = 2y^{-n}\mathcal{C}_{2n/k}(Z)$ ($j = n$, $m = e$) and $T_+^{-n} - T_-^{-n} = -y^{-n+e/2}w(\xi)\mathcal{S}_{2n/k}(Z)$ ($j = n - \frac{e}{2}$, $m = 0$), where $n \leq \frac{s-1}{2} = k-1 + \frac{e}{2}$. (S_c^{2k+1}) $T_1^{-n} + T_2^{-n} = -y^{(1-n)/2}\xi\mathcal{S}_{n/(2k+1)}(\frac{y\xi^2}{4})$ for odd n ($j = \frac{n-1}{2} \leq k-1$, $m = 0$), and $2y^{-n/2}\mathcal{C}_{n/(2k+1)}(\frac{y\xi^2}{4})$ for even n ($j = \frac{n}{2} \leq k-1$, $m = 1$). $(P, s \text{ even})$ $T_+^{-n} + T_-^{-n} = -y^{(e-n)/2}w(\xi)\mathcal{S}_{n/k}(Z)$ for odd n ($j = \frac{n-e}{2} \leq k-1$, $m = 0$), and $2y^{-n/2}\mathcal{C}_{n/k}(Z)$ for even n ($j = \frac{n}{2} \leq \frac{s-2}{2} \leq k-1 + e$, $m = e$). By Lemma 2.3 the finite sum of negative powers differs from its restriction to $x = 0$, a Laurent series in y , by an analytic function, and (PC) makes it analytic. At $x = 0$ the term of index n has a pole of order exactly n with leading coefficient 1 in the split types. In the symmetric types the term of even index n has a pole of order exactly $\frac{n}{2}$ with leading coefficient 2, and the term of odd index n has a pole of order at most $\frac{n-1}{2}$ (S_c^{2k+1}) resp. $\frac{n-e}{2}$ ($P, s \text{ even}$; no pole if $n < e$); its order can be smaller, for instance for S_c^{2k+1} with $c_0 = 0$. So the equation for the coefficient of y^{-i} involves $a_i + b_i$ (split), resp. a_{2i} (symmetric), and otherwise only terms of larger index: (PC) is a triangular system of $N_0, k-1$, resp. $\frac{s-2}{2}$ independent equations.

4.4 The inclusion $\subseteq$

Lemma 4.1. *Let f be holomorphic on a disc $D(z_0, R) \subset \mathbb{C}$ and $a \neq 0$ with $|f' - a| \leq \frac{1}{2}|a|$ on $D(z_0, R)$. Then f is injective on $D(z_0, R)$ and $f(D(z_0, R)) \supseteq D(f(z_0), \frac{1}{2}|a|R)$.*

Proof. For $z \neq z'$ in the disc, $|f(z) - f(z') - a(z - z')| \leq \frac{1}{2}|a||z - z'|$, so $f(z) \neq f(z')$. If $|w - f(z_0)| < \frac{1}{2}|a|R'$ with $R' < R$, then on $|z - z_0| = R'$ we have $|(f(z) - w) - (a(z - z_0) + f(z_0) - w)| \leq \frac{1}{2}|a|R' < |a(z - z_0) + f(z_0) - w|$, and by Rouché's theorem $f - w$ has a zero in $D(z_0, R')$, as $a(z - z_0) + f(z_0) - w$ has one there. $\square$

We use four properties of T_1, T_2 , proved in Sections 4.5 and 4.6: for all sufficiently small $\rho, r > 0$, on $\Omega_0 := \{|\sigma| < \rho, |x| < r\} \subset \mathbb{C}^2$,

- (I) $\Psi := (T_1, T_2)$ is injective on $\Omega := \Omega_0 \cap \{\sigma \neq 0\}$;
- (II) $|\partial_\sigma T_i - r_i(0, 0)| \leq \frac{1}{2}$;
- (III) $D_1 := \partial_x T_1 - \partial_\sigma T_1 \partial_x T_2 / \partial_\sigma T_2 = \kappa \sigma^M (1 + \epsilon_1)$ and $D_2 := \partial_x T_2 - \partial_\sigma T_2 \partial_x T_1 / \partial_\sigma T_1 = -r_2(0, 0) \kappa \sigma^M (1 + \epsilon_2)$ with $|\epsilon_i| \leq \frac{1}{4}$, where $M := N_0 + 1$ and $\kappa \neq 0$ are given in Table 2;
- (IV) $|\partial_x T_i| \leq C_0 |\sigma| |\partial_\sigma T_i|$ for a constant C_0 .

By (II), $|T_i/\sigma - r_i(0, 0)| \leq \frac{1}{2}$, so $\frac{1}{2}|\sigma| \leq |T_i| \leq \frac{3}{2}|\sigma|$ on Ω_0 .

Let $F \in V_{\mathbb{C}}$. Its complexification is holomorphic on a polydisc in $\mathbb{C}^2$, bounded there by some K , and satisfies $E(F) = 0$. Choose ρ, r as above and so small that (x, σ^N) stays in that polydisc, and put $\hat{F}(\sigma, x) := F(x, \sigma^N)$ on Ω_0 .

Step 1. By (I), Ψ is a biholomorphism of Ω onto the open set $D := \Psi(\Omega)$ (an injective holomorphic map between open subsets of $\mathbb{C}^2$ has a holomorphic inverse), and D is connected. The function $G := \hat{F} \circ \Psi^{-1}$ is holomorphic on D and $|G| \leq K$.

Step 2. Put $\lambda_i := \kappa_i T_i^{N_i}$ on Ω . On the real part $\{0 < \sigma < \rho, |x| < r\}$ these are the eigenvalue functions, so $\lambda_1 + \lambda_2 = v$ and $\lambda_1 \lambda_2 = -u$ there, hence on Ω . On Ω the discriminant $v^2 + 4u$, which is $y^{2k}(\xi^2 + 4\varepsilon y^{d-2k})$, $16y^{2k}Z(1+Z)$, $y^{2k}(\xi^2 + 4\varepsilon)$, $y^{2k+1}(4 + y\xi^2)$ for O, P, S^{2k}, S^{2k+1} , does not vanish, nor do the λ_i . Near each point of Ω , $(x, y) = (x, \sigma^N)$ are coordinates, Lemma 2.2 gives $L^* d\lambda_i = \lambda_i d\lambda_i$, and $d\lambda_1 \wedge d\lambda_2 \neq 0$ since $T_i \neq 0$ and Ψ is biholomorphic. By (6), $E(F) = 0$ gives $\partial^2 G / \partial T_1 \partial T_2 = 0$ on D .

Step 3 (fibre lemma). For fixed x , the map $\sigma \mapsto T_1(\sigma, x)$ is injective on $|\sigma| < \rho$ and its image contains $D(0, \frac{\rho}{2})$, by (II) and Lemma 4.1 with $R = \rho$, $|a| = 1$. So for $0 < |t| < \frac{\rho}{2}$ the set $\Psi^{-1}(\{T_1 = t\})$ is the graph $\{(\sigma_1(t, x), x) : |x| < r\}$ of a holomorphic function, hence connected, and $\partial G/\partial T_1$, whose T_2 -derivative vanishes, is constant on $\{T_1 = t\} \cap D$. This defines a holomorphic function χ_1 on $0 < |t| < \frac{\rho}{2}$ with $\partial G/\partial T_1 = \chi_1(T_1)$. We bound it for $0 < |t| < \frac{\rho}{6}$. Let $\sigma_0 := \sigma_1(t, 0)$ and $T_2^* := T_2(\sigma_0, 0)$; then $|\sigma_0| \leq 2|t|$ and $|T_2^*| \leq \frac{3}{2}|\sigma_0| \leq 3|t| < \frac{\rho}{2}$. Applied to T_2 , the same argument shows that $\{T_2 = T_2^*\} \cap \Omega$ is the graph of $\sigma(x) := \sigma_2(T_2^*, x)$, $|x| < r$, and $|\sigma(x)| \leq 2|T_2^*| \leq 6|t| < \rho$: the level set stays inside Ω . Along it, $\tau(x) := T_1(\sigma(x), x)$ satisfies $\tau(0) = t$, $(\tau(x), T_2^*) \in D$ and $|G(\tau(x), T_2^*)| \leq K$, and $\tau' = D_1(\sigma(x), x)$ because $\sigma' = -\partial_x T_2/\partial_\sigma T_2$. By (IV), $|\sigma'/\sigma| \leq C_0$, so $|\sigma(x)/\sigma_0 - 1| \leq e^{C_0|x|} - 1 \leq \frac{1}{8M}$ for $|x| \leq R := \min\{r, C_0^{-1} \log(1 + \frac{1}{8M})\}$. With $a := \kappa\sigma_0^M$, (III) gives on $D(0, R)$

$$|\tau'/a - 1| \leq \frac{5}{4}((1 + \frac{1}{8M})^M - 1) + \frac{1}{4} \leq \frac{5}{4}(e^{1/8} - 1) + \frac{1}{4} < \frac{1}{2}.$$

By Lemma 4.1, $\tau(D(0, R)) \supseteq D(t, \frac{1}{2}|\kappa||\sigma_0|^M R)$, so $G(\cdot, T_2^*)$ is holomorphic and bounded by K on that disc, and Cauchy's estimate gives

$$|\chi_1(t)| = |\partial_{T_1} G(t, T_2^*)| \leq \frac{2K}{|\kappa|R|\sigma_0|^M} \leq \frac{2K(3/2)^M}{|\kappa|R} |t|^{-M}.$$

The same argument with T_1 and T_2 exchanged, using D_2 , gives the bound for χ_2 . By Riemann's theorem $t^M \chi_i(t)$ extends holomorphically to $t = 0$: χ_i has at most a pole of order M at 0.

Step 4 (monodromy). On $\Omega' := \{0 < |\sigma| < \frac{\rho}{3}, |x| < r\}$ we have $|T_i| < \frac{\rho}{2}$, so on the connected set $D' := \Psi(\Omega')$, $dG = \chi_1(T_1) dT_1 + \chi_2(T_2) dT_2$. Along the loop $\theta \mapsto (\sigma_* e^{i\theta}, x_*)$ in Ω' both $T_i = \sigma r_i$ wind once around 0, since $|\arg(\pm r_i)| < \frac{\pi}{6}$ by (II). As G is single valued, $0 = \oint dG = 2\pi i(\text{res}_0 \chi_1 + \text{res}_0 \chi_2)$.

Split type: let $c := \text{res}_0 \chi_1 = -\text{res}_0 \chi_2$ and let H_i be the Laurent series with $H'_i = \chi_i \mp c/t$, a pole of order at most $M - 1 = N_0$ and no constant term. The function $\log(T_1/T_2) := \log(r_1/r_2)$ (principal branch) is holomorphic on Ω' with differential $dT_1/T_1 - dT_2/T_2$. So $G - c \log(T_1/T_2) - H_1(T_1) - H_2(T_2)$ has zero differential on D' and is constant. *Symmetric type:* $\hat{F}$ is even in s , Ω' is invariant under $s \mapsto -s$, and this map exchanges T_1 and T_2 ; so $G(w_1, w_2) = G(w_2, w_1)$ on D' , $\chi_1 = \chi_2 =: \chi$, the residue of χ vanishes, and $G = H(T_1) + H(T_2) + \text{const}$ with $H' = \chi$, a pole of order at most N_0 .

Step 5. Writing H_i (resp. H) as its principal part plus a holomorphic germ, we obtain the form of Theorem 1.2(b) on Ω' , in particular on $\{y > 0\}$. The logarithm and the holomorphic parts are analytic at the origin (Section 4.3), hence so is the sum of negative powers, and by Lemma 2.3 it satisfies (PC). This proves Theorem 1.2(b). $\square$

4.5 Injectivity

$S_c^{2k, \varepsilon}$ and O . As functions of (ξ, y) near $(c_0, 0)$, the map $(\xi, y) \mapsto (R, T_2)$, $R := r_1/r_2$, has Jacobian determinant $\partial_\xi R(c_0, 0) = (\partial_\xi r_1 - \partial_\xi r_2)(c_0, 0)$ at $(c_0, 0)$, which is $\frac{2}{kq}$, $-\frac{2i}{kq}$ and $\frac{d}{k(d-k)c_0}$ for $S_c^{2k, \varepsilon}$ (real and complex) and O , by Section 3.2 and $\partial_\xi A(c_0, 0) = 1$. So it is injective near $(c_0, 0)$; $(T_1, T_2) = (RT_2, T_2)$ determines (R, T_2) as $T_2 \neq 0$; and $(x, y) \mapsto (\xi, y)$ is injective on $\{y \neq 0\}$. P , s odd. $T_+ T_- = y^2$ determines y up to sign, and the sign by (II), since T_-/y is close to 1. Then $T_+/T_- = \exp(\frac{4}{k} \text{arsinh} \frac{y^{e/2} w(\xi)}{2})$ with a small exponent determines $y^{e/2} w(\xi)$, hence $w(\xi)^2 = \varepsilon \xi$ and x . *Symmetric types.* $T_1 T_2 = -s^2$ determines s up to sign, and the sign by (II). Then $-T_1/T_2 = e^{2\mu/(2k+1)}$, resp. $e^{\frac{2}{k} \text{arsinh}(s^e w(\xi)/2)}$, with a small exponent determines $s\xi$, resp. $s^e w(\xi)$, hence ξ and x .

4.6 The fibre derivative

Write $T_i = \sigma r_i(\xi, \sigma)$ with $\xi = C(\sigma^N) + \alpha x \sigma^{N(k-1)}$ and r_i analytic near $(c_0, 0)$. Then

$$\partial_x T_i = \alpha \sigma^{1+N(k-1)} \partial_\xi r_i, \quad \partial_\sigma T_i = r_i + \sigma(\partial_\xi r_i \partial_\sigma \xi + \partial_\sigma r_i).$$

At $(\sigma, x) = (0, 0)$, $\partial_\sigma T_i = r_i(0, 0) = \pm 1$; this gives (II) by continuity, and (IV) since $|\partial_x T_i| \leq C|\sigma|$. As $\partial_\sigma T_2 \neq 0$ near $(0, 0)$, the function

$$D_1 = \alpha \sigma^{1+N(k-1)} \left[\partial_\xi r_1 - \partial_\xi r_2 \frac{\partial_\sigma T_1}{\partial_\sigma T_2} \right] \quad (7)$$

is analytic on a polydisc around $(\sigma, x) = (0, 0)$, including $\sigma = 0$. We show $D_1 = \sigma^M h(\sigma, x)$ with h analytic and $h(0, 0) = \kappa \neq 0$; then $|h/\kappa - 1| \leq \frac{1}{4}$ on a smaller polydisc by continuity, and the same holds for $D_2 = -(\partial_\sigma T_2 / \partial_\sigma T_1) D_1$, whose factor tends to $-r_2(0, 0)$; this is (III). ($S_c^{2k, \varepsilon}$, O ; $N = 1$) The bracket in (7) is analytic with value $\partial_\xi R(c_0, 0)$ at the origin, as $\partial_\sigma T_1 / \partial_\sigma T_2 \rightarrow r_1/r_2 = 1$; so $M = k$ and $\kappa = \alpha \partial_\xi R(c_0, 0)$, that is q , iq and c_0 . (S_c^{2k+1} ; $N = 2$) Here $r_1 = e^{\mu'}$, $r_2 = -e^{-\mu'}$ with $\mu' = \frac{1}{2k+1} \operatorname{arsinh} \frac{s\xi}{2}$, so $\partial_\xi r_1 = e^{\mu'} \partial_\xi \mu'$, $\partial_\xi r_2 = e^{-\mu'} \partial_\xi \mu'$, $\partial_\xi \mu' = \frac{s}{2(2k+1)} (1 + \frac{y\xi^2}{4})^{-1/2}$, and the bracket is $\partial_\xi \mu' [e^{\mu'} - e^{-\mu'} \partial_s T_1 / \partial_s T_2]$, where the last factor tends to $1 - (-1) = 2$. So $D_1 = (2k+1) s^{2k-1} \cdot \frac{s}{2(2k+1)} \cdot 2(1 + O(|s| + |x|))$: $M = 2k$, $\kappa = 1$. (P , s odd; $N = 1$) Here $r_\pm = e^{\pm \nu'}$, $\nu' = \frac{2}{k} \operatorname{arsinh} \frac{y^{e/2} w(\xi)}{2}$, $\partial_\xi r_\pm = \pm r_\pm \partial_\xi \nu'$, $\partial_\xi \nu' = \frac{1}{k} y^{e/2} w'(\xi) (1 + Z)^{-1/2}$, and the bracket is $\partial_\xi \nu' [r_+ + r_- \partial_y T_+ / \partial_y T_-]$, the last factor tending to 2. So $M = k + \frac{e}{2} = \frac{s+1}{2}$ and $\kappa = \frac{2\alpha}{k} w'(c_0) = 2\varepsilon w(c_0)$, using $w' = \varepsilon/(2w)$ and $c_0/w(c_0) = \varepsilon w(c_0)$. (P , s even; $N = 2$) Here $r_\pm = \pm e^{\pm \nu}$, $\nu = \frac{1}{k} \operatorname{arsinh} \frac{s^e w(\xi)}{2}$, $\partial_\xi r_\pm = e^{\pm \nu} \partial_\xi \nu$, $\partial_\xi \nu = \frac{1}{2k} s^e w'(\xi) (1 + Z)^{-1/2}$, and the bracket is $\partial_\xi \nu [e^\nu - e^{-\nu} \partial_s T_+ / \partial_s T_-]$, the last factor tending to 2. So $M = 2k - 1 + e = s$ and $\kappa = \frac{\alpha}{k} w'(c_0) = \varepsilon w(c_0)$. In every case $M = N_0 + 1$ (Table 2). For $k = 1$ the function $h(0, x)$ depends on x , which is harmless.

4.7 Proof of Corollary 1.3

We compute $F_x(0, 0)$ with Lemma 2.3, using the representations of Section 4.3. The holomorphic parts do not contribute: in the split types $g_i(T_i) = g_i(yr_i)$ has $j = 0$ and ∂_ξ divisible by y , and in the symmetric types $\partial_x T_i = O(\sigma)$. The logarithm $\log(r_1/r_2)$ has $j = 0$, $m = 0$ in $S_c^{2k, \varepsilon}$ and O , so it contributes $c \alpha \partial_\xi R(c_0, 0) = c\kappa$ when $k = 1$ and nothing when $k \geq 2$; in P , s odd, ∂_ξ of $\log(T_+/T_-) = \frac{4}{k} \operatorname{arsinh} \frac{y^{e/2} w}{2}$ is divisible by $y^{e/2}$ and there is no contribution. Among the negative powers only the index $n = N_0$ contributes: in $S_c^{2k, \varepsilon}$ and O , (PC) gives $a_{N_0} + b_{N_0} = 0$, and the contribution is $-N_0 \alpha (a_{N_0} \partial_\xi r_1 + b_{N_0} \partial_\xi r_2)(c_0, 0) = -N_0 \kappa a_{N_0}$; in P , s odd, only $-\frac{1}{2}(a_{N_0} - b_{N_0}) y^{-(k-1)} w(\xi) \mathcal{S}_{2N_0/k}(Z)$ contributes, with $b_{N_0} = -a_{N_0}$ by (PC), giving $-\alpha a_{N_0} w'(c_0) \frac{2N_0}{k} = -N_0 \kappa a_{N_0}$; in S_c^{2k+1} the odd term $n = N_0 = 2k - 1$ gives $-\alpha a_{N_0} \mathcal{S}_{N_0/(2k+1)}(0) = -N_0 a_{N_0}$; in P , s even, the odd term $n = N_0 = s - 1$ gives $-\alpha a_{N_0} w'(c_0) \frac{N_0}{k} = -N_0 \kappa a_{N_0}$. The even terms of the symmetric types and the sums $T_+^{-n} + T_-^{-n}$ in P , s odd, have $m \geq 1$ and $j < k - 1 + m$. $\square$

For the laws of Table 1 one finds $a_{N_0} = 1$ (S_c^{2k+1} , $S_c^{2k, \varepsilon}$ real and O , $k \geq 2$), $-\frac{1}{2}$ (S_c^3), $\frac{1}{2i}$ ($S_c^{2k, -1}$, $k \geq 2$), $\pm 1/w(c_0)$ (P ; $-$ for $k = 1$), and $c = 1, \frac{1}{2i}, \frac{d-1}{d}$ for $S_c^{2, \varepsilon}$ (real, complex) and $O_{1, c}^{d, \varepsilon}$, in agreement with the last column of Table 1.

5 Computations and reproducibility

No theorem depends on a computation. The program `check_note.py` of this note (standard library, exact arithmetic over $\mathbb{Q}$ and $\mathbb{Q}(i)$, about 20 seconds; 355 checks, all passed) checks in the notation of the note: the series O , P , S rebuilt by (1) (112 parameter sets with $k \leq 4$) and L_{nil} , L_{nd} , M , N ; Theorem 1.1 for these 112 sets as exact series in the box $x \leq^6 y \leq^9$ ((PC), no poles, $E(F) = 0$, $F_x(0, 0)$), the quoted b_j , Remark 3.1 and three negative controls; the closed forms of Section 3.5 (38 instances); the factorisation of D_1 with the constants of Table 2 (25 instances); Theorem 1.2(b) and Corollary 1.3 on 5-jets (16 instances: the 5-jets of formal solutions form a space of dimension $11 = 2N + 1$, spanned

by the generators of Theorem 1.2(b), but not without their negative powers); and Theorem 1.2(a). The package also contains the programs with which the results were first obtained (`original/`, with SymPy and mpmath; among them Theorem 1.1 as exact series to total degree 9 in 124 instances, and the jet count in 23), run again when the note was written with outputs identical to the recorded ones (about three minutes in all), and the programs and outputs of the three verification runs described next (`verification_run_A/`, `verification_run_B/`, `independent_run_2/`).

Verification. The results were first obtained with written proofs and the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each re-derived the proofs and wrote its own programs, and neither read the other’s report. The first compared the statement and the normal forms with [BKM24d], both versions of [BKM24b], and [BKM24c], re-derived all proofs, and checked 20 normal forms, the closed forms (26 instances), Theorem 1.1 as exact series (29 instances, $k \leq 4$), the formulas in eigenvalues to 40 digits (34), the exponents and constants of Section 4.6 (25), and the jet count (18, $N = 6$). The second checked Theorem 1.1 with its own exact series package in 160 instances with $k \leq 6$ (boxes up to $x^{\leq 10}y^{\leq 16}$, with negative controls), the closed forms and Lemma 2.2 symbolically, the factorisation of D_1 (32 instances) and the jet count (39, $N = 8$); it ran the programs of `original/` again with byte-identical outputs, and repeated the literature search and the Crossref check of the bibliography. Neither run found a mathematical error. Both required the general computation of the fibre derivative (Section 4.6); their other corrections, all made, concerned the number of instances (124, not 125), the reference for the shift of c_{k-1} (Remark 3.1), the branch conventions of Table 1, the description of [Akp22], journal versions, the regularity criterion (Corollary 1.3, optional) and the survey [Bol25]. Section 4.6, Corollary 1.3, Remark 3.1 and `check_note.py` were written with the note from the computations of the two runs.

A third independent verification run, also AI-assisted, then examined the note as written. It compared the statement, the normal forms and the cited theorems with [BKM24d], both versions of [BKM24b] and [BKM24c] once more, re-derived every proof line by line, including the radii and constants of the fibre lemma and of Section 4.6, and wrote its own programs (`independent_run_2/`). These compute the eigenvalues as exact Puiseux series from the characteristic equation with the conventions of Table 1 and form the laws from them, without the representations of Section 3.3 and without the recursion (PC); they check Theorem 1.1 in 113 instances with $k \leq 4$ (no poles, $E(F) = 0$ in boxes from $x^{\leq 2}y^{< 14}$ to $x^{\leq 2}y^{< 32}$, $F_x(0,0)$, the quoted b_j and Remark 3.1), the closed forms (38 instances), and Table 2, the factorisation of D_1 , the rank of (PC), Corollary 1.3 and the jet count in 36 instances with $k \leq 3$, as well as Theorem 1.2(a). The run also re-ran `check_note.py` and the programs of `original/`, with outputs identical to the recorded ones, and repeated the literature search. It found no mathematical error. It required three corrections of the text, which were made: the statement of novelty is restricted to M , O , P and S (the forms L_{nil} and N are algebraically generic), a sentence on pole orders in Section 4.3 now says “at most” for the terms of odd index, and the description of [Akp22] was corrected.

Scope and priority. The note answers both parts of Problem 4.3 of [BKM24d] in the real analytic category; the smooth category and higher dimensions are not treated. The problem, the normal forms and (1) are due to Bolsinov, Konyaev and Matveev [BKM24d, BKM24b]; the existence of regular laws is [BKM24c, Theorem 1.5]; the laws at algebraically generic points are described in [BKM24c] (see [BKM24d, §4.1]), and at differentially non-degenerate points, such as the origin for L_{nd} , they are the functions $\text{tr } F(L)$ [BKM24d, §4.1]. So Theorem 1.2(a) and the laws of Table 1 for G1–G3, L_{nil} , N and L_{nd} are known or elementary and are included for completeness: for a Jordan block, [BKM24c, Theorem 1.4] treats an eigenvalue that is constant or has non-zero differential, which covers L_{nil} and N_1 , and for the other forms N and G3 the description in Theorem 1.2(a) is a direct

computation. What the note adds, to our knowledge, are the explicit regular laws for the singular normal forms M , O , P , S and the description of all analytic laws of these forms; no priority is claimed. No printed answer to the question was found in the sources searched: the L^AT_EX source of [BKM24d] (version 1; the published chapter was not accessible to us); [BKM24b] in both versions, [BKM24c], [BKM24a]; the full texts of [AB26, Akp26, AM26] and of [Akp22], which studies the singular points with $\text{d tr } L \neq 0$ (among the normal forms above: L_{nd} , M_1 , O and P with $k = 1$, and $S_c^{2,\varepsilon}$ with $c_0 \neq 0$), but contains no conservation laws; two papers of the same authors on Stäckel problems (arXiv:2512.17609, arXiv:2604.03195), where conservation laws occur only in general form; the abstracts of further recent papers of these authors and of the two works found to cite [BKM24d] (arXiv:2411.02290, arXiv:2512.22064); and searches in arXiv, Crossref, OpenAlex, zbMATH Open and Semantic Scholar, and six web searches, on 18 October 2026, partly repeated by the third verification run. The survey [Bol25] was not accessible to us: its full text could not be retrieved (the host of the journal’s full texts was unreachable, again at a last attempt on 18 October 2026), and only its abstract, keywords and references were read. Its keywords include conservation laws, and a full-text index (OpenAlex) reports the phrases “regular conservation law”, “normal forms” and “singular points” in it; an overlap with this note cannot be excluded. A search that finds nothing is not a proof of novelty. What was read: Section 4 of [BKM24d], Section 5 of [BKM24b] in both versions (we use the numbering of the published version), and Section 1 of [BKM24c] (arXiv numbering). From [BKM24b] we use only the list of normal forms and (1); the proofs of cited results were not checked, and those of [BKM24c] are not used.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [AB26] Dinmukhammed Akpan and Alexey Bolsinov. Beltrami problem in dimension two: local normal forms, 2026. [arXiv:2606.10751](#).
- [Akp22] Dinmukhammed Akpan. Singularities of two-dimensional Nijenhuis operators. *European Journal of Mathematics*, 8(4):1328–1340, 2022. [arXiv:2503.11852](#), [doi:10.1007/s40879-022-00585-x](#).
- [Akp26] Dinmukhammed Akpan. From quadratic integrals to Nijenhuis operators: two-dimensional dictionary, 2026. [arXiv:2609.05103](#).
- [AM26] Dinmukhammed Akpan and Matteo Manenti. On three-dimensional gl-regular Nijenhuis operators, 2026. [arXiv:2607.04913](#).
- [BKM24a] Alexey V. Bolsinov, Andrey Yu. Konyaev, and Vladimir S. Matveev. Applications of Nijenhuis geometry V: geodesic equivalence and finite-dimensional reductions of integrable quasilinear systems. *Journal of Nonlinear Science*, 34(2):33, 2024. [arXiv:2306.13238](#), [doi:10.1007/s00332-023-10008-0](#).
- [BKM24b] Alexey V. Bolsinov, Andrey Yu. Konyaev, and Vladimir S. Matveev. Nijenhuis geometry III: gl-regular Nijenhuis operators. *Revista Matemática Iberoamericana*, 40(1):155–188, 2024. [arXiv:2007.09506](#), [doi:10.4171/rmi/1416](#).
- [BKM24c] Alexey V. Bolsinov, Andrey Yu. Konyaev, and Vladimir S. Matveev. Nijenhuis geometry IV: conservation laws, symmetries and integration of certain non-diagonalisable systems

of hydrodynamic type in quadratures. *Nonlinearity*, 37(10):105003, 2024.
arXiv:2304.10626, doi:10.1088/1361-6544/ad6acc.

- [BKM24d] Alexey V. Bolsinov, Andrey Yu. Konyaev, and Vladimir S. Matveev. Research problems on relations between Nijenhuis geometry and integrable systems, 2024. Version 1, cited here. Published in *2024 MATRIX Annals, Part I*, MATRIX Book Series, Springer, 2026, pp. 229–268. arXiv:2410.04276, doi:10.1007/978-3-032-16202-1_8.
- [Bol25] Alexey V. Bolsinov. Geodesically equivalent metrics and Nijenhuis geometry. *Sbornik: Mathematics*, 216(5):583–607, 2025. Russian original: Mat. Sb. 216:5 (2025), 5–32. doi:10.4213/sm10220e.
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Version 1.8.0; record AMR-154-0006.