

Quadratic Hénon Maps over $\mathbb{Q}$ with Rational Cycles of Length Three and Four for Every Jacobian

Alper Ferudun*

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Abstract

In a list of open problems on Hénon maps (Arnold Math. J. 10 (2024)), P. Ingram conjectured that for all but finitely many $\delta \in \mathbb{Q}$ every $\mathbb{Q}$ -rational periodic point of every map $f(x, y) = (y, y^2 + c - \delta x)$ with $c \in \mathbb{Q}$ has period dividing 2. We show that the statement, as printed, is false for every $\delta \in \mathbb{Q}$, in each reading the text supports (finitely many exceptional δ ; for fixed δ , finitely many exceptional c ; finitely many exceptional pairs). For $\delta \neq 1$ and $c = -(29\delta^2 - 38\delta + 29)/16$ the point $(\frac{5-7\delta}{4}, -\frac{1+\delta}{4})$ has period 3; for $\delta = 1$ and $c = -1 - s^2$, $s \in \mathbb{Q}$, there is a rational 3-cycle; for every $\delta \neq 0$ an explicit c gives a rational 4-cycle. Such cycles are essentially implicit in the source: the interpolation construction that opens the same section yields rational 4-cycles for every $\delta \notin \{0, 1\}$, and our 4-cycles are its normal form; so the printed conjecture contradicts its own section and is probably not the intended statement. We also describe all rational 3-cycles: shifted by $\frac{1}{2}$, their x -coordinates are the rational points, off the line $u = v = w$, of the cone over the nodal cubic $u^3 + v^3 + w^3 - u^2v - v^2w - w^2u = 0$. Hence every $\delta \neq 1$ has infinitely many c with a rational 3-cycle, and for $\delta = 1$ one exists if and only if $-(1 + c)$ is a rational square. Computations, which are not theorems, find for $\delta = -1$ exactly Ingram's periods 1, 2, 3, 4, 6, 8, and a rational 5-cycle for $\delta = -4$ and, by a duality $\delta \leftrightarrow 1/\delta$, for $\delta = -\frac{1}{4}$. The case $\delta = 0$ is the classical map $z^2 + c$ (Morton; Morton and Raianu), the case $\delta = -1$ is Ingram's, and the novelty of the note is modest. This is an unrefereed note.

1 Introduction

1.1 The conjecture

For $c, \delta \in \mathbb{Q}$ let

$$f_{c,\delta}(x, y) = (y, y^2 + c - \delta x). \quad (1)$$

For $\delta \neq 0$ this is a quadratic Hénon map: a polynomial automorphism of the plane with constant Jacobian determinant δ and inverse $f_{c,\delta}^{-1}(x, y) = ((x^2 + c - y)/\delta, x)$. Every map $(x, y) \mapsto (y, Ay^2 + By + E - \delta x)$ with $A, B, E \in \mathbb{Q}$ and $A \neq 0$ is conjugate over $\mathbb{Q}$ to some $f_{c,\delta}$ with the same δ (Lemma 2.1(b)). A point $P \in \mathbb{Q}^2$ is *periodic* if $f^n(P) = P$ for some $n \geq 1$; the least such n is its *period*, and the orbit of a point of period n is an *n-cycle*.

The list of open problems on Hénon maps [dH24] contains a section on Hénon maps over number fields by P. Ingram [dH24, Section 11]. Its Conjecture 4 states, in the notation (1): *for all but finitely many $\delta \in \mathbb{Q}$, every $\mathbb{Q}$ -rational periodic point of every map $f_{c,\delta}$ with $c \in \mathbb{Q}$ has period dividing 2*, that is, period 1 or 2. It has the same number and wording in the arXiv version and in the journal's HTML edition. It follows Conjecture 3, Ingram's conjecture [Ing14] that for $\delta = -1$ the only periods

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
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are 1, 2, 3, 4, 6, 8, and it is followed by the remark that infinite families of examples would show both conjectures to be sharp, if true; no evidence for Conjecture 4 is given. The problem is record AMR-152-0049 of the UnsolvedMath dataset [ula26].

The text supports three readings. (R1) There is a finite set $S \subset \mathbb{Q}$ such that for all $\delta \in \mathbb{Q} \setminus S$ and all $c \in \mathbb{Q}$ every rational periodic point of $f_{c,\delta}$ has period 1 or 2 (the literal reading). (R2) For every δ , or for all but finitely many δ , the same holds for all but finitely many $c \in \mathbb{Q}$. (R3) The same holds for all but finitely many pairs $(c, \delta) \in \mathbb{Q}^2$. Variants which exclude $\delta = 0$ (where $f_{c,0}$ is not invertible) or restrict c and δ to $\mathbb{Z}$ are also treated (Corollary 1.4).

1.2 Results

Theorem 1.1 (rational cycles for every Jacobian). *Let $\delta \in \mathbb{Q}$.*

- (a) *If $\delta \neq 1$, let $t = \frac{1-\delta}{4}$ and $c = -\frac{29\delta^2-38\delta+29}{16}$. Then $(-\frac{1}{2} + 7t, -\frac{1}{2} + t)$ is a point of period 3 of $f_{c,\delta}$; the x -coordinates of its orbit are $-\frac{1}{2} + 7t$, $-\frac{1}{2} + t$, $-\frac{1}{2} - 5t$.*
- (b) *If $\delta = 1$ and $\mu \in \mathbb{Q} \setminus \{0\}$, then $(-\frac{1}{2} - \mu, -\frac{1}{2} + \mu)$ is a point of period 3 of $f_{c,1}$ with $c = -1 - (\mu - \frac{1}{2})^2$; the x -coordinates of its orbit are $-\frac{1}{2} - \mu$, $-\frac{1}{2} + \mu$, $-\frac{1}{2} + \mu$.*
- (c) *If $\delta \neq 0$, let*

$$p = \frac{1 - 2\delta - \delta^3}{2\delta}, \quad q = \frac{\delta^3 - 2\delta^2 - 1}{2\delta}, \quad r = \frac{\delta^3 + 1}{2\delta},$$

$$c = -\frac{\delta^6 - 2\delta^5 + 2\delta^4 + 2\delta^3 + 2\delta^2 - 2\delta + 1}{4\delta^2}.$$

Then (p, q) is a point of period 4 of $f_{c,\delta}$, with orbit $(p, q) \mapsto (q, r) \mapsto (r, r) \mapsto (r, p) \mapsto (p, q)$.

- (d) *If $\delta = 1$ and $\nu \in \mathbb{Q} \setminus \{0\}$, then $(-\nu, -\nu)$ is a point of period 4 of $f_{-\nu^2, 1}$, with orbit $(-\nu, -\nu) \mapsto (-\nu, \nu) \mapsto (\nu, \nu) \mapsto (\nu, -\nu) \mapsto (-\nu, -\nu)$.*

For example, $\delta = 3$ and $c = -11$ give the integral 3-cycle $(-4, -1) \mapsto (-1, 2) \mapsto (2, -4)$, and for $\delta = 0$ part (a) gives the 3-cycle $\frac{5}{4} \mapsto -\frac{1}{4} \mapsto -\frac{7}{4}$ of $z \mapsto z^2 - \frac{29}{16}$.

Theorem 1.2 (all rational 3-cycles). *Let $F(u, v, w) = u^3 + v^3 + w^3 - u^2v - v^2w - w^2u$.*

- (a) *Let $a, b, d \in \mathbb{Q}$ and $(u, v, w) = (a + \frac{1}{2}, b + \frac{1}{2}, d + \frac{1}{2})$. There are $c, \delta \in \mathbb{Q}$ such that (a, b) is a point of period 3 of $f_{c,\delta}$ with orbit $(a, b) \mapsto (b, d) \mapsto (d, a)$ if and only if $F(u, v, w) = 0$ and u, v, w are not all equal. Then c and δ are unique: $\delta = 1 + \frac{w^2 - v^2}{v - u}$ if $u \neq v$, $\delta = 1$ if $u = v$, and $c = d - b^2 + \delta a$.*
- (b) *The plane cubic $C = \{F = 0\} \subset \mathbb{P}^2$ is irreducible over $\overline{\mathbb{Q}}$. Its only singular point is $(1 : 1 : 1)$, which is a node whose two tangents are defined over $\mathbb{Q}(\sqrt{-3})$ and conjugate; C has geometric genus 0.*
- (c) *The map*

$$(h, k) \mapsto (w_0 + h, w_0 + k, w_0), \quad w_0 = -\frac{h^3 - h^2k + k^3}{2(h^2 - hk + k^2)},$$

is a bijection from $\mathbb{Q}^2 \setminus \{(0, 0)\}$ onto the set of $(u, v, w) \in \mathbb{Q}^3$ with $F(u, v, w) = 0$ which are not all equal; its inverse is $(u, v, w) \mapsto (u - w, v - w)$. In these coordinates $\delta = 1 - \frac{hk(h-k)}{h^2 - hk + k^2}$.

Thus, after the shift by $\frac{1}{2}$, the x -coordinates of the rational 3-cycles are the rational points, off the line $u = v = w$, of the affine cone over the nodal cubic C , and (c) is the inverse of the projection from the node. Theorem 1.1(a) is the ruling $(h, k) = (12t, 6t)$ of this cone and Theorem 1.1(b) the ruling $(h, k) = (-2\mu, 0)$.

Corollary 1.3 (the fibres). (a) Let $\delta \in \mathbb{Q} \setminus \{1\}$. For $\tau \in \mathbb{Q} \setminus \{0, 1\}$ put $\varepsilon = \frac{\delta-1}{2\tau(\tau-1)}$,

$$a = -\frac{1}{2} - \varepsilon(\tau^3 - \tau^2 + 2\tau - 1), \quad b = -\frac{1}{2} + \varepsilon(\tau^3 - 3\tau^2 + 2\tau - 1), \quad d = -\frac{1}{2} + \varepsilon(\tau^3 - \tau^2 + 1),$$

and $c_\delta(\tau) = -N_\delta(\tau)/(4\tau^2(\tau-1)^2)$ with

$$\begin{aligned} N_\delta(\tau) = & (\delta-1)^2\tau^6 - 2(2\delta-1)(\delta-1)\tau^5 + (9\delta^2-8\delta+4)\tau^4 \\ & - (8\delta^2-6\delta+8)\tau^3 + (4\delta^2-8\delta+9)\tau^2 - 2(\delta-1)(\delta-2)\tau + (\delta-1)^2. \end{aligned}$$

Then (a, b) is a point of period 3 of $f_{c_\delta(\tau), \delta}$ with orbit $(a, b) \mapsto (b, d) \mapsto (d, a)$. Every rational point of period 3 of a map $f_{c, \delta}$ with $c \in \mathbb{Q}$ arises in this way from exactly one τ , and the three points of a cycle arise from τ , $1 - \frac{1}{\tau}$ and $\frac{1}{1-\tau}$. The rational function c_δ has degree 6. Hence infinitely many $c \in \mathbb{Q}$ give a map $f_{c, \delta}$ with a rational 3-cycle, and each map $f_{c, \delta}$ has at most two rational 3-cycles.

(b) The map $f_{c, 1}$ has a rational point of period 3 if and only if $-(1+c)$ is the square of a rational number.

Corollary 1.4 (the readings). For every $\delta \in \mathbb{Q}$ there are infinitely many $c \in \mathbb{Q}$ such that $f_{c, \delta}$ has a rational point of period 3, and for every $\delta \neq 0$ there is a $c \in \mathbb{Q}$ such that $f_{c, \delta}$ has a rational point of period 4. Consequently [dH24, Section 11, Conjecture 4] is false in each of the readings (R1), (R2) and (R3), also when $\delta = 0$ is excluded. For integral c and δ and integral points, (R1) and (R3) remain false, and (R2) remains false at $\delta = 1$; for a fixed integer $\delta \neq 1$ only finitely many integers c carry a rational 3-cycle (Remark 3.2).

Proposition 1.5 (duality). Let $\delta \in \mathbb{Q} \setminus \{0\}$ and $T(x, y) = (\delta y, \delta x)$. Then $T^{-1} \circ f_{c, \delta}^{-1} \circ T = f_{c/\delta^2, 1/\delta}$. Hence $f_{c, \delta}$ and $f_{c/\delta^2, 1/\delta}$ have the same numbers of rational n -cycles for every n ; if $(x_n)_{n \in \mathbb{Z}}$ is the sequence of x -coordinates of a cycle of $f_{c, \delta}$, then (x_{-n}/δ) is that of a cycle of $f_{c/\delta^2, 1/\delta}$.

For $\delta = 0$ the restriction in Theorem 1.1(c) cannot be removed: the periodic points of $f_{c, 0}$ are the points $(z, z^2 + c)$ with z periodic for $z \mapsto z^2 + c$, with the same period, and $z^2 + c$ has no rational point of period 4 by a theorem of Morton [Mor98].

1.3 Relation to the source and to known results

Section 11 of [dH24] opens with an interpolation argument: prescribing a cycle of length $\deg F + 2$ for $(x, y) \mapsto (y, F(y) - \delta x)$ is a system of linear equations for the coefficients of F and for δ , with determinant not identically zero. For $\deg F = 2$ over $\mathbb{Q}$, with the source's own substitution $y_3 = y_2$, it gives rational 4-cycles for every $\delta \notin \{0, 1\}$, whose normal form is Theorem 1.1(c), and the same linear algebra gives 3-cycles for every $\delta \neq 1$ (Remark 3.3). So the printed Conjecture 4 contradicts the construction with which its own section begins. We believe that it is not the intended statement, which cannot be recovered from the text (the remark on sharpness does not help, since 2-cycles exist for every δ , Lemma 2.1(c)); the counterexamples are essentially implicit in the source.

For $\delta = 0$ everything reduces to $z \mapsto z^2 + c$. Its rational 3-cycles were parametrised by Morton [Mor92] and by Walde and Russo [WR94], and among the c with a rational 3-cycle, $c = -\frac{29}{16}$ has the smallest numerator and denominator in absolute value [MR22, Theorems 1 and 4] (both as stated in the zbMATH review of [MR22]); there is no rational point of period 4 [Mor98] or 5 [FPS97], nor, conditionally on the Birch and Swinnerton-Dyer conjecture for a certain abelian variety, of period 6 [Sto08]; Poonen conjectured that there is none of period greater than 3 [Poo98]. The case $\delta = -1$ is Ingram's family $(y, x + y^2 + b)$ [Ing14]: he conjectured that its rational periods lie in $\{1, 2, 3, 4, 6, 8\}$ and verified this for all b of height at most 1000 (Conjecture 1.5 and Proposition 1.6 of arXiv:1111.3609v1;

we did not check the numbering of the journal version). This is Conjecture 3 of [dH24, Section 11], which the present note does not affect.

New here are the explicit families for every Jacobian, in particular for $\delta = 1$, where the interpolation gives no 3-cycles, and the complete description of the rational 3-cycles; the novelty is modest. The computations of Section 5 suggest a corrected statement, which we only ask (Question 5.2); integral parameters (Remark 3.2) or “period dividing 4” (Theorem 1.1(a), (b)) do not repair the printed statement.

2 Preliminaries

Lemma 2.1. *Let $c, \delta \in \mathbb{Q}$.*

- (a) *For $x_0, x_1 \in \mathbb{Q}$ let $x_{n+2} = x_{n+1}^2 + c - \delta x_n$ for $n \geq 0$. Then $f_{c,\delta}^n(x_0, x_1) = (x_n, x_{n+1})$ for all $n \geq 0$, and (x_0, x_1) is a periodic point of period N if and only if the sequence (x_n) is periodic with least period N . In particular, if $x_0, \dots, x_{N-1} \in \mathbb{Q}$ satisfy $x_{i+2} = x_{i+1}^2 + c - \delta x_i$ for all i , with indices modulo N , then (x_0, x_1) is periodic, and its period is the least period of the cyclic sequence $(x_0, \dots, x_{N-1})$.*
- (b) *Let $g(x, y) = (y, Ay^2 + By + E - \delta x)$ with $A, B, E \in \mathbb{Q}$ and $A \neq 0$, and let $L(x, y) = (x/A + m, y/A + m)$ with $m = -B/(2A)$. Then $L^{-1} \circ g \circ L = f_{c,\delta}$ with $c = A(Am^2 + Bm + E - \delta m - m)$. In particular L maps the rational periodic points of $f_{c,\delta}$ bijectively onto those of g , preserving periods.*
- (c) *For $a \in \mathbb{Q}$ let $b = -1 - \delta - a$ and $c = a(1 + \delta) - b^2$. If $a \neq b$, then (a, b) is a point of period 2 of $f_{c,\delta}$.*

Proof. (a) The formula for f^n follows by induction, and since the sequence is determined by two consecutive terms, $f^N(x_0, x_1) = (x_0, x_1)$ if and only if $x_{n+N} = x_n$ for all n . A cyclic sequence as in the last statement, continued periodically, is the sequence of (x_0, x_1) . (b) With $L^{-1}(X, Y) = (A(X - m), A(Y - m))$, the first coordinate of $L^{-1}(g(L(x, y)))$ is y and the second is $y^2 + (2Am + B)y + A(Am^2 + Bm + E - \delta m - m) - \delta x$, where $2Am + B = 0$. (c) By (a) we need $a = b^2 + c - \delta a$ and $b = a^2 + c - \delta b$. The first is the definition of c ; the difference of the two equations is $(a - b)(a + b + 1 + \delta) = 0$, which holds because $a + b = -1 - \delta$. The period is 2 because $a \neq b$. $\square$

Proof of Proposition 1.5. We have $f_{c,\delta}^{-1}(T(x, y)) = ((\delta^2 y^2 + c - \delta x)/\delta, \delta y)$, and $T^{-1}(X, Y) = (Y/\delta, X/\delta)$ maps this point to $(y, y^2 + c/\delta^2 - x/\delta) = f_{c/\delta^2, 1/\delta}(x, y)$. A bijection and its inverse have the same periodic points, with the same periods, and conjugation by the bijection T of $\mathbb{Q}^2$ preserves rationality and periods. If $P = (x_n, x_{n+1})$ lies on a cycle of $f = f_{c,\delta}$, then $g = f_{c/\delta^2, 1/\delta}$ maps $T^{-1}(P) = (x_{n+1}/\delta, x_n/\delta)$ to $T^{-1}(f^{-1}(P)) = T^{-1}(x_{n-1}, x_n) = (x_n/\delta, x_{n-1}/\delta)$; so the x -coordinates along the orbit of $T^{-1}(P)$ are $x_{n+1}/\delta, x_n/\delta, x_{n-1}/\delta, \dots$. $\square$

The constants c of Theorem 1.1(a) and (c) satisfy $\delta^2 c(1/\delta) = c(\delta)$, so the duality maps the map of each of these families at δ to the map of the same family at $1/\delta$.

3 Proof of Theorem 1.1

By Lemma 2.1(a) it suffices in each case to check that the given cyclic sequence of x -coordinates satisfies $x_{i+2} = x_{i+1}^2 + c - \delta x_i$, with indices modulo its length, and to determine its least period. For a cyclic sequence (a, b, d) of length 3 these are the three equations

$$d = b^2 + c - \delta a, \quad a = d^2 + c - \delta b, \quad b = a^2 + c - \delta d, \quad (2)$$

which we call (E1), (E2), (E3); the least period is 3 unless $a = b = d$.

Proof of Theorem 1.1. (a) Write $\delta = 1 - 4t$. Then $29\delta^2 - 38\delta + 29 = 20 - 80t + 464t^2$, so $c = -\frac{5}{4} + 5t - 29t^2$. With $a = -\frac{1}{2} + 7t$, $b = -\frac{1}{2} + t$, $d = -\frac{1}{2} - 5t$ we get

$$\begin{aligned} b^2 + c - \delta a &= (t^2 - t + \frac{1}{4}) + (-\frac{5}{4} + 5t - 29t^2) + (28t^2 - 9t + \frac{1}{2}) = -\frac{1}{2} - 5t = d, \\ d^2 + c - \delta b &= (25t^2 + 5t + \frac{1}{4}) + (-\frac{5}{4} + 5t - 29t^2) + (4t^2 - 3t + \frac{1}{2}) = -\frac{1}{2} + 7t = a, \\ a^2 + c - \delta d &= (49t^2 - 7t + \frac{1}{4}) + (-\frac{5}{4} + 5t - 29t^2) + (-20t^2 + 3t + \frac{1}{2}) = -\frac{1}{2} + t = b, \end{aligned}$$

where the last bracket in each line is $-\delta a = (1 - 4t)(\frac{1}{2} - 7t)$, $-\delta b = (1 - 4t)(\frac{1}{2} - t)$ and $-\delta d = (1 - 4t)(\frac{1}{2} + 5t)$. Since $a - b = 6t \neq 0$, the period is 3.

(b) Put $s = \mu - \frac{1}{2}$, so that $a = -1 - s$, $b = d = s$ and $c = -1 - s^2$. Then (E1)–(E3) with $\delta = 1$ read $s^2 - 1 - s^2 + 1 + s = s$, $s^2 - 1 - s^2 - s = -1 - s$ and $(1 + s)^2 - 1 - s^2 - s = s$, which hold. Since $a - b = -2\mu \neq 0$, the period is 3.

(c) Here the equations are

$$r = q^2 + c - \delta p, \quad r = r^2 + c - \delta q, \quad p = r^2 + c - \delta r, \quad q = p^2 + c - \delta r,$$

which we number (1)–(4). From the definitions,

$$r - p = \delta^2 + 1, \quad r - q = \frac{\delta^2 + 1}{\delta}, \quad q + r = \delta(\delta - 1), \quad p + r = \frac{1 - \delta}{\delta}.$$

First, $p - r^2 + \delta r = r(1 + \delta) - r^2 - (\delta^2 + 1)$, and multiplying by $4\delta^2$,

$$2\delta(\delta^3 + 1)(\delta + 1) - (\delta^3 + 1)^2 - 4\delta^2(\delta^2 + 1) = -(\delta^6 - 2\delta^5 + 2\delta^4 + 2\delta^3 + 2\delta^2 - 2\delta + 1);$$

so $c = p - r^2 + \delta r$, which is (3). Equation (2) minus (3) is $r - p = \delta(r - q)$, (1) minus (2) is $(q - r)(q + r) = \delta(p - q)$, and (4) minus (3) is $q - p = (p - r)(p + r)$; by the four relations above both sides of each are $\delta^2 + 1$, $-(\delta^2 + 1)(\delta - 1)$ and $(\delta^2 + 1)(\delta - 1)/\delta$ respectively. Finally $f^2(p, q) = (r, r) \neq (p, q)$ because $r - p = \delta^2 + 1 \neq 0$, so the period, which divides 4, is 4.

(d) Since $x_{i+1}^2 = \nu^2$ for every term of the cyclic sequence $(-\nu, -\nu, \nu, \nu)$, the recurrence with $c = -\nu^2$ and $\delta = 1$ reads $x_{i+2} = -x_i$, which holds. Again $f^2(-\nu, -\nu) = (\nu, \nu) \neq (-\nu, -\nu)$, so the period is 4. $\square$

Remark 3.1. For $\delta \notin \{0, 1\}$, Theorem 1.1(c) is the only rational 4-cycle through a point (r, r) of the diagonal: for x -coordinates (p, q, r, r) with $q \neq r$, (2) minus (3) and (1) minus (2) give $p = r + \delta(q - r)$ and $q + r = \delta(\delta - 1)$, then (4) minus (3) becomes $(\delta - 1)(2r - \delta^2 + \delta)(2\delta r - \delta^3 - 1) = 0$, and $r = \delta(\delta - 1)/2$ would give $q = r$. For $\delta = 1$ the same steps give $p = q$, $r = -q$ and nothing else: Theorem 1.1(d).

Remark 3.2 (integral examples). For $\delta \in \mathbb{Z}$ the constant c of Theorem 1.1(a) is an integer if and only if $\delta \equiv 3 \pmod{4}$, and then the 3-cycle is integral: then $t \in \frac{1}{2} + \mathbb{Z}$ and $c = -6 - 24n - 29n^2$ for $t = n + \frac{1}{2}$, while $t \in \mathbb{Z}$ gives $c \in -\frac{5}{4} + \mathbb{Z}$ and $t \in \pm\frac{1}{4} + \mathbb{Z}$ gives $16c$ odd. Examples: $\delta = -1$, $c = -6$ (x -coordinates $3, 0, -3$); $\delta = 3$, $c = -11$; $\delta = 7$, $c = -74$ ($-11, -2, 7$). Theorem 1.1(c), (d) give integral 4-cycles for $\delta = -1$ ($c = -2$; $-2, 2, 0, 0$) and $\delta = 1$ ($c = -\nu^2$, $\nu \in \mathbb{Z} \setminus \{0\}$). So integrality does not rescue [dH24, Section 11, Conjecture 4] as printed. Conversely, for a fixed integer $\delta \neq 1$ only finitely many integers c carry a rational 3-cycle: by Corollary 1.3(a) such a c is $c_\delta(p/q)$ with p/q in lowest terms, and $c_\delta(p/q) = -N(p, q)/(4p^2q^2(p - q)^2)$ with $N(p, q) = q^6N_\delta(p/q) \in \mathbb{Z}[p, q]$, whose coefficients of p^6 and of q^6 and whose sum of coefficients are all $(\delta - 1)^2$ (as $N_\delta(0) = N_\delta(1) = (\delta - 1)^2$); so $c \in \mathbb{Z}$ forces each of p , q and $p - q$ to divide $(\delta - 1)^2 \neq 0$. For $\delta = 3$ the only such c is -11 . Whether infinitely many integers c carry a rational 4-cycle for a fixed $\delta \notin \{\pm 1\}$ is not decided here.

Remark 3.3 (the interpolation construction of the source). Let $y_0, y_1, y_2 \in \mathbb{Q}$ be pairwise distinct and $y_3 = y_2$, the substitution used in [dH24, Section 11]. The linear equations $a_0 + a_1 y_n + a_2 y_n^2 - \delta y_{n-1} = y_{n+1}$ (n modulo 4), which say that $g(x, y) = (y, a_2 y^2 + a_1 y + a_0 - \delta x)$ has the cycle $(y_0, y_1) \mapsto (y_1, y_2) \mapsto (y_2, y_3) \mapsto (y_3, y_0)$, have determinant $-(y_0 - y_1)(y_0 - y_2)(y_1 - y_2)^2 \neq 0$, and their solution has

$$\delta = \frac{y_0 - y_2}{y_1 - y_2}, \quad a_2 = -\frac{(y_0 - y_2)^2 + (y_1 - y_2)^2}{(y_0 - y_2)(y_1 - y_2)^2} \neq 0.$$

By Lemma 2.1(b), g is conjugate over $\mathbb{Q}$ to some $f_{c,\delta}$, and a direct computation (done by the program of Section 6) shows that c and the cycle become exactly those of Theorem 1.1(c). As $(y_0 - y_2)/(y_1 - y_2)$ takes every value in $\mathbb{Q} \setminus \{0, 1\}$, the construction of the source gives a rational 4-cycle for every $\delta \notin \{0, 1\}$. For cycles of length 3 the equations (n modulo 3) are, for any given δ , a Vandermonde system for (a_0, a_1, a_2) , and for pairwise distinct y_0, y_1, y_2

$$a_2 = -\frac{(\delta - 1)(y_0^2 + y_1^2 + y_2^2 - y_0 y_1 - y_1 y_2 - y_2 y_0)}{(y_0 - y_1)(y_0 - y_2)(y_1 - y_2)},$$

which is nonzero exactly when $\delta \neq 1$. So the same linear algebra gives rational 3-cycles for every $\delta \neq 1$ (their normal forms are the cycles of Corollary 1.3(a), with $\tau = (y_0 - y_2)/(y_1 - y_2)$), and for $\delta = 1$ it shows that the x -coordinates of a rational 3-cycle are not pairwise distinct.

4 Rational 3-cycles

Lemma 4.1. *Let $a, b, d \in \mathbb{Q}$ be not all equal. The equations (E1)–(E3) have a solution $(c, \delta) \in \mathbb{Q}^2$ if and only if*

$$D(a, b, d) = \det \begin{pmatrix} 1 & -a & d - b^2 \\ 1 & -b & a - d^2 \\ 1 & -d & b - a^2 \end{pmatrix} = 0,$$

and the solution is then unique. Moreover $D(u - \frac{1}{2}, v - \frac{1}{2}, w - \frac{1}{2}) = -F(u, v, w)$.

Proof. (E1)–(E3) are the linear equations $c - \delta a = d - b^2$, $c - \delta b = a - d^2$, $c - \delta d = b - a^2$ for (c, δ) . As a, b, d are not all equal, the coefficient matrix, with rows $(1, -a)$, $(1, -b)$, $(1, -d)$, has rank 2; so a solution exists if and only if the augmented matrix has rank 2, that is, $D(a, b, d) = 0$, and it is then unique and rational. Subtracting the first row from the others gives $D(a, b, d) = (a - b)(b - a^2 - d + b^2) - (a - d)(a - d^2 - d + b^2)$; substituting $a = u - \frac{1}{2}$, $b = v - \frac{1}{2}$, $d = w - \frac{1}{2}$ and expanding gives $-F(u, v, w)$. $\square$

Lemma 4.2. *For all m, h, k ,*

$$F(m + h, m + k, m) = 2m(h^2 - hk + k^2) + h^3 - h^2k + k^3.$$

Proof. In $F(m + h, m + k, m) = (m + h)^3 + (m + k)^3 + m^3 - (m + h)^2(m + k) - (m + k)^2m - m^2(m + h)$ the coefficient of m^3 is $3 - 3 = 0$, that of m^2 is $3h + 3k - (2h + k) - 2k - h = 0$, that of m is $3h^2 + 3k^2 - (h^2 + 2hk) - k^2 = 2(h^2 - hk + k^2)$, and the constant term is $h^3 + k^3 - h^2k$. $\square$

Proof of Theorem 1.2. (a) If (a, b) has period 3 with orbit $(a, b) \mapsto (b, d) \mapsto (d, a)$, then (E1)–(E3) hold and a, b, d are not all equal, since otherwise (a, b) would be fixed; by Lemma 4.1, $F(u, v, w) = 0$. Conversely, if $F(u, v, w) = 0$ and u, v, w are not all equal, Lemma 4.1 gives a unique (c, δ) satisfying (E1)–(E3), and (a, b) has period 3 by Lemma 2.1(a). Subtracting (E2) from (E1) gives $\delta(b - a) = (d - a) + (d^2 - b^2)$; since $b - a = v - u$, $d - a = w - u$ and $d^2 - b^2 = (w - v)(w + v - 1)$, this is

$$\delta(v - u) = (w - u) - (w - v) + (w^2 - v^2) = (v - u) + (w^2 - v^2),$$

which gives δ if $u \neq v$. If $u = v$, then $F(u, u, w) = (u - w)^2(u + w) = 0$ and $w \neq u$ give $w = -u \neq 0$; the same cycle based at (b, d) has the triple $(v, w, u) = (u, -u, u)$, to which the formula applies and gives $\delta = 1$. Finally $c = d - b^2 + \delta a$ is (E1).

(b) F and its partial derivatives $F_u = 3u^2 - 2uv - w^2$, $F_v = 3v^2 - 2vw - u^2$, $F_w = 3w^2 - 2wu - v^2$ vanish at $(1, 1, 1)$, so $(1 : 1 : 1)$ is a singular point of C . By Lemma 4.2 with $m = 1$ and (h, k) replaced by $(\lambda h, \lambda k)$,

$$F(1 + \lambda h, 1 + \lambda k, 1) = 2\lambda^2(h^2 - hk + k^2) + \lambda^3(h^3 - h^2k + k^3).$$

So $(1 : 1 : 1)$ is a double point with tangent cone $h^2 - hk + k^2 = 0$, in the affine coordinates $(h, k) = (u - 1, v - 1)$ of the chart $w = 1$. Its discriminant $-3k^2$ shows that the two tangents $h = \frac{1 \pm \sqrt{-3}}{2}k$ are distinct, defined over $\mathbb{Q}(\sqrt{-3})$ and conjugate: $(1 : 1 : 1)$ is a node, and since $h^2 - hk + k^2 = (h - \frac{k}{2})^2 + \frac{3}{4}k^2$, it is an isolated point of the real curve.

Every singular point of C is a common zero of F_u, F_v, F_w (and every such zero lies on C , because $uF_u + vF_v + wF_w = 3F$). On the line $w = 0$ we have $F_w = -v^2$ and $F_v = 3v^2 - u^2$, so $u = v = w = 0$, which is not a point of $\mathbb{P}^2$. In the chart $w = 1$, $F_u(u, v, 1) = -2uv + 3u^2 - 1$ is linear in v and equals -1 for $u = 0$; so at a common zero $u \neq 0$ and $v = (3u^2 - 1)/(2u)$. Substituting this, that is, taking the resultants with respect to v ,

$$4u^2F_v(u, \frac{3u^2-1}{2u}, 1) = 23u^4 - 12u^3 - 18u^2 + 4u + 3 = (u - 1)(23u^3 + 11u^2 - 7u - 3),$$

$$4u^2F_w(u, \frac{3u^2-1}{2u}, 1) = -9u^4 - 8u^3 + 18u^2 - 1 = -(u - 1)(9u^3 + 17u^2 - u - 1).$$

The two cubic factors have no common root: their resultant is $331776 = 2^{12} \cdot 3^4 \neq 0$ (equivalently, the Euclidean algorithm gives greatest common divisor 1). Hence $u = 1$, $v = 1$, and $(1 : 1 : 1)$ is the only singular point of C , over any field of characteristic 0.

If $F = \ell g$ over $\overline{\mathbb{Q}}$ with ℓ linear, every point of $\{\ell = 0\} \cap \{g = 0\}$ is singular on C . If the line $\{\ell = 0\}$ lies in the conic $\{g = 0\}$ or meets it in two points, C has more than one singular point; otherwise they meet in one point P with multiplicity 2, and P is either a point of multiplicity 3 of C (g singular at P) or has the double line $\ell^2 = 0$ as tangent cone (g smooth at P , tangent to ℓ). Neither is a node, so C is irreducible over $\overline{\mathbb{Q}}$, with one node, and its geometric genus is $\frac{(3-1)(3-2)}{2} - 1 = 0$; part (c) gives the parametrisation.

(c) Let $(u, v, w) \in \mathbb{Q}^3$ with u, v, w not all equal, and put $h = u - w$, $k = v - w$, so $(h, k) \neq (0, 0)$. By Lemma 4.2 with $m = w$, $F(u, v, w) = 2w(h^2 - hk + k^2) + h^3 - h^2k + k^3$, and $h^2 - hk + k^2 = (h - \frac{k}{2})^2 + \frac{3}{4}k^2 > 0$. So $F(u, v, w) = 0$ if and only if $w = w_0(h, k)$, which proves the bijection. If $h \neq k$, that is $u \neq v$, then by (a)

$$\delta - 1 = \frac{w^2 - v^2}{v - u} = \frac{-k(2w_0 + k)}{k - h},$$

where

$$2w_0 + k = \frac{-2(h^3 - h^2k + k^3) + 2k(h^2 - hk + k^2)}{2(h^2 - hk + k^2)} = -\frac{h(h - k)^2}{h^2 - hk + k^2};$$

so $\delta - 1 = -hk(h - k)/(h^2 - hk + k^2)$; if $h = k$, then $\delta = 1$ by (a), in accordance with the formula. $\square$

Proof of Corollary 1.3. (a) Fix $\delta \neq 1$. By Theorem 1.2, the triples (a, b, d) of x -coordinates of the rational points of period 3 of the maps $f_{c, \delta}$, $c \in \mathbb{Q}$, correspond to the $(h, k) \in \mathbb{Q}^2 \setminus \{(0, 0)\}$ with $-hk(h - k)/(h^2 - hk + k^2) = \delta - 1$. For these $hk(h - k) \neq 0$, so $\tau = h/k \in \mathbb{Q} \setminus \{0, 1\}$; conversely, since the left side is homogeneous of degree 1, each $\tau \in \mathbb{Q} \setminus \{0, 1\}$ comes from exactly one such pair, namely $k = (1 - \delta)(\tau^2 - \tau + 1)/(\tau(\tau - 1))$ and $h = \tau k$. Then $w_0 = (\delta - 1)(\tau^3 - \tau^2 + 1)/(2\tau(\tau - 1))$, and $(a, b, d) = (w_0 + h - \frac{1}{2}, w_0 + k - \frac{1}{2}, w_0 - \frac{1}{2})$ and $c = d - b^2 + \delta a$ are the expressions in the statement (a direct computation, also done by the program of Section 6). Basing the cycle at (b, d) instead of (a, b) replaces (a, b, d) by (b, d, a) , hence $(h, k) = (u - w, v - w)$ by $(v - u, w - u) = (k - h, -h)$ and τ

by $1 - \frac{1}{\tau}$; this map has order 3 and no fixed point in $\mathbb{Q}$, since $\tau^2 - \tau + 1 \neq 0$. So the three points of a cycle correspond to the three distinct values $\tau, 1 - \frac{1}{\tau}, \frac{1}{1-\tau}$.

The polynomial N_δ has degree 6 and leading coefficient $(\delta - 1)^2 \neq 0$, and $N_\delta(0) = N_\delta(1) = (\delta - 1)^2 \neq 0$; so N_δ is prime to $\tau^2(\tau - 1)^2$ and c_δ is a rational function of degree 6. It takes each value at most six times, so each map $f_{c,\delta}$ has at most two rational 3-cycles, and since $\mathbb{Q} \setminus \{0, 1\}$ is infinite, infinitely many values c occur.

(b) For $\delta = 1$, Theorem 1.2(c) gives $hk(h - k) = 0$. If $k = 0$, then $w_0 = -\frac{h}{2}$ and $(u, v, w) = (\frac{h}{2}, -\frac{h}{2}, -\frac{h}{2})$, which is the triple of Theorem 1.1(b) with $\mu = -\frac{h}{2}$; if $h = 0$ or $h = k$, the triple is a cyclic shift of such a triple. So the rational 3-cycles of the maps $f_{c,1}$ are exactly the cycles of Theorem 1.1(b), with $c = -1 - (\mu - \frac{1}{2})^2$, $\mu \neq 0$. As μ runs through $\mathbb{Q} \setminus \{0\}$, $s = \mu - \frac{1}{2}$ runs through $\mathbb{Q} \setminus \{-\frac{1}{2}\}$, and $c = -1 - s^2$ depends only on s^2 ; so the values of c are exactly the numbers $-1 - s^2$, $s \in \mathbb{Q}$. $\square$

For $\delta = 0$, Corollary 1.3(a) is a rational parametrisation of the rational 3-cycles of $z \mapsto z^2 + c$, with $c_0(\tau) = -(\tau^6 - 2\tau^5 + 4\tau^4 - 8\tau^3 + 9\tau^2 - 4\tau + 1)/(4\tau^2(\tau - 1)^2)$; such parametrisations are classical [Mor92, WR94] (we have not compared the formulas).

Proof of Corollary 1.4. Infinitely many c with a rational 3-cycle: Corollary 1.3(a) for $\delta \neq 1$, Theorem 1.1(b) for $\delta = 1$; a rational 4-cycle: Theorem 1.1(c). As 3 and 4 do not divide 2, the exceptional δ form all of $\mathbb{Q}$, the exceptional c for each δ and the exceptional pairs form infinite sets. For integral data, Remark 3.2 gives infinitely many integers δ with an integer c and an integral 3-cycle, which refutes (R1) and (R3), and Theorem 1.1(d) gives at $\delta = 1$ the integral 4-cycles of $f_{-\nu^2,1}$ for all $\nu \in \mathbb{Z} \setminus \{0\}$, which refutes (R2) there; the finiteness statement is proved in Remark 3.2. $\square$

5 Computations and a question

This section reports computations; they are not theorems, except for statements about explicitly listed maps. The program of Section 6 re-checks every cycle quoted here. The searches rest on the following bounds.

Lemma 5.1. *Let $c, \delta \in \mathbb{Q}$ and let (x_n) be the sequence of x -coordinates of a rational cycle of $f_{c,\delta}$.*

- (a) $|x_n| \leq R$ for all n , where R is the positive root of $R^2 - (1 + |\delta|)R - |c| = 0$.
- (b) Let p be a prime. If $|\delta|_p \leq 1$, then $|x_n|_p \leq \max(1, |c|_p^{1/2})$ for all n ; if $|\delta|_p > 1$, then $|x_n|_p \leq \max(|\delta|_p, |c|_p^{1/2})$ for all n .
- (c) If $|\delta|_p \leq 1$ and $v_p(c) < 0$, then $v_p(c)$ is even.

Proof. The maxima below are attained, as the sequence is periodic. (a) If $M = \max_n |x_n| = |x_{n+1}|$ exceeds R , then $|x_{n+2}| \geq M^2 - |c| - |\delta|M > M$. (b) Let $M = \max_n |x_n|_p = |x_{n+1}|_p$. If $|\delta|_p \leq 1$ and $M > \max(1, |c|_p^{1/2})$, then M^2 exceeds $|c|_p$ and $M \geq |\delta x_n|_p$, so $|x_{n+2}|_p = M^2 > M$. If $|\delta|_p > 1$ and $M > \max(|\delta|_p, |c|_p^{1/2})$, then $M > 1$ and $x_n = (x_{n+1}^2 + c - x_{n+2})/\delta$ has $|x_n|_p = M^2/|\delta|_p > M$. (c) Let $e = \min_n v_p(x_n)$. If $2e < v_p(c)$, then $e < 0$, and where $v_p(x_{n+1}) = e$ we get $v_p(x_{n+2}) = 2e < e$. If $2e > v_p(c)$, then $v_p(c) < \min(2e, e)$ and $v_p(x_{n+2}) = v_p(c) < e$. So $v_p(c) = 2e$. $\square$

The searches. (i) *Box search.* For the 31 values $\delta \in \{-8, \dots, 8\} \cup \{\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{1}{4}, \pm\frac{3}{4}\}$ and $L = D \cdot \text{den}(\delta)$, with $D \in \{1, \dots, 10, 12, 14, 15, 16, 18, 20, 24\}$ for integral δ and $D \in \{1, 2, 3, 4, 6, 8, 12\}$ otherwise, all $x_0, x_1, x_2 \in \frac{1}{L}\mathbb{Z}$ with exact common denominator L and $|Lx_i| \leq \min(8L, 96)$ were taken, with $c = x_2 - x_1^2 + \delta x_0$, and the orbit of (x_0, x_1) was followed exactly for

at most 40 steps. Since $c \in \frac{1}{L^2}\mathbb{Z}$ and $\text{den}(\delta)$ divides L , Lemma 5.1(b) shows that the orbit of a periodic point through such a triple stays in $\frac{1}{L}\mathbb{Z}$; so for every δ this finds every rational cycle of length at most 40 with three consecutive x -coordinates of exact common denominator L in the list and numerators at most $\min(8L, 96)$ in absolute value, but for non-integral δ only denominators divisible by $\text{den}(\delta)$ are examined. (ii) *Complete search for each map*. For the same 31 values of δ and every $c = N/M$ in lowest terms with $|c| \leq C_\delta$ and $M = m^2k$, $m \leq m_\delta$, k dividing the denominator of δ , all rational periodic points of $f_{c,\delta}$, of any period, were determined: by Lemma 5.1(a), (b) they lie in an explicit finite set $(\frac{1}{L}\mathbb{Z} \cap [-R, R])^2$, on which $f_{c,\delta}$ is a partial map whose cycles were found exactly. Between 6,001 and 116,601 values of c were treated per δ ; for example $(C_\delta, m_\delta) = (20, 24)$ for $\delta = -1$, $(8, 12)$ for $\delta = 1$, $(120, 12)$ for $\delta = -4$ and $(3, 16)$ for $\delta = -\frac{1}{4}$. For integral δ the maps whose c has a non-square denominator have no rational periodic point, by Lemma 5.1(c). (iii) *5-cycles with a repeated x -coordinate* (after a rotation, $x_0 = x_1$ or $x_0 = x_2$) were determined by resultants, with δ as a variable, for each of the 1110 nonzero values $\delta = n/m$ with $|n| \leq 30$ and $1 \leq m \leq 30$; at $\delta = 1$ the resultants vanish identically and the method gives nothing.

δ	periods found	remarks
-1	1, 2, 3, 4, 6, 8	Ingram's set; in (ii) period 8 only for $c = -9/16$
1	1, 2, 3, 4, 5, 6, 7, 9, 10	
-4	1, 2, 3, 4, 5	in (ii) period 5 only for $c = -469/16$
-1/4	1, 2, 3, 4, 5	period 5 for $c = -469/256$ (the dual)
0	1, 2, 3	the map $z^2 + c$
the other 26 values	subsets of $\{1, 2, 3, 4\}$	

Table 1: Periods of rational periodic points found by the searches (i) and (ii) together. Computations, not theorems; the ranges are in the text and, for each δ , in the reproducibility package.

Results (Table 1). For $\delta = -1$ both searches find exactly Ingram's periods; in (ii) period 8 occurs only for $c = -\frac{9}{16}$, where Ingram found it [Ing14], and period 6 for eleven values of c , such as $c = -\frac{7}{9}$ with x -coordinates $-\frac{4}{3}, \frac{4}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}$. For $\delta = 1$ both find the periods $1, \dots, 7, 9, 10$; for example $c = -\frac{9}{16}$ has a 7-cycle with x -coordinates $-\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{3}{4}, \frac{1}{4}, \frac{1}{4}$. The map $f_{-469/16, -4}$ has the 5-cycle with x -coordinates $-\frac{21}{4}, \frac{29}{4}, \frac{9}{4}, \frac{19}{4}, \frac{9}{4}$, and its rational periodic points are exactly this cycle and one 2-cycle; by Proposition 1.5 the same holds for $f_{-469/256, -1/4}$. Search (iii) finds no other rational 5-cycle with a repeated x -coordinate for the 1109 values $\delta \neq 1$ in its range. Apart from $\delta = \pm 1$, every 4-cycle found is that of Theorem 1.1(c), except for six: at $\delta = -4$ for $c \in \{-\frac{685}{16}, -\frac{365}{16}, -\frac{5749}{144}\}$, at $\delta = -\frac{1}{4}$ for $c \in \{-\frac{685}{256}, -\frac{365}{256}\}$, and at $\delta = \frac{5}{2}$ for $c = -\frac{2785}{256}$ (with x -coordinates $-\frac{69}{16}, -\frac{29}{16}, \frac{51}{16}, \frac{61}{16}$).

Question 5.2. *Is it true that for all but finitely many $\delta \in \mathbb{Q}$ every rational periodic point of every map $f_{c,\delta}$, $c \in \mathbb{Q}$, has period at most 4? Does period 5 occur for infinitely many $\delta \in \mathbb{Q}$?*

By Theorem 1.1 the bound 4 cannot be lowered; by the cycles quoted above an exceptional set would contain ± 1 , -4 and $-\frac{1}{4}$, and by Proposition 1.5 it would be invariant under $\delta \mapsto 1/\delta$. For $\delta = 0$ the first question contains Poonen's conjecture [Poo98]. We do not claim an answer.

6 Computations and reproducibility

The proofs in Sections 2 to 4 do not depend on computers. The reproducibility package, which accompanies the source files of this note, contains the exact program of this note, `check_note.py` (Python, standard library only, exact rational arithmetic). It checks every identity used in the proofs as an identity of polynomials or rational functions (Theorem 1.1, the remarks of Section 3, Lemmas 2.1, 4.1

and 4.2, the resultants and the greatest common divisor in Theorem 1.2(b), the parametrisation and the formulas of Theorem 1.2(c) and Corollary 1.3(a), and Proposition 1.5); it iterates the map along the families of Theorem 1.1 for 1958 values n/m ($|n| \leq 40$, $1 \leq m \leq 40$) of each parameter, checks every cycle quoted in this note, determines all rational periodic points of $f_{-469/16, -4}$, $f_{-469/256, -1/4}$, $f_{-9/16, -1}$ and $f_{-11, 3}$ by the method of Lemma 5.1, and tests Corollary 1.3(b) on the 295 values $c = N/M$ with $M \in \{1, 4, \dots, 36\}$ a square and $|c| \leq 3$. It performs 92 checks in about two seconds. The package also contains the programs of the searches (i)–(iii) with their recorded outputs: the box search (standard library; three processes of about three minutes each), the complete search (Python with NumPy; from 0.5 to 43 seconds per value of δ) and the resultant computation (Python with SymPy; a few seconds), and, in `independent_run_2/`, the five programs of the third verification run with their outputs.

Verification. The results were first obtained with a written proof and programs (the identities in SymPy, the box search (i), and a search by Gröbner bases which is not used here). Two independent verification runs, both AI-assisted, followed; each re-derived the proofs and wrote its own programs, and neither read the other’s report. The first run recomputed every identity from the printed formulas, found the family of Theorem 1.1(d) and the resultant argument of Theorem 1.2(b), ran a box search of its own, and compared the statement with the TeX source of [dH24] and the statements quoted from [Ing14]. The second run iterated the map along the families, recomputed the period-3 locus by a Gröbner basis and by a resultant, wrote the searches (ii) and (iii), re-ran the original programs with identical results, found that the construction at the beginning of [dH24, Section 11] already yields rational 4-cycles, repeated the literature search and checked the bibliography against Crossref. Neither run found an error in the theorems. The corrections they required were made: the reference for Morton’s theorem; the reading (R2), which the first version had called trivially true, is refuted as well; the corrected statement is now a question; the construction of the source is cited and the claim of novelty reduced; the numbering of Ingram’s statements; Remark 3.2; the family of Theorem 1.1(d); the proof that the node is the only singular point; the sources of the statements quoted from papers we did not read; and the authors of a preprint found in the search. The parametrisation by (h, k) , the explicit fibres of Corollary 1.3(a), the normal form in Remark 3.3, the list of exceptional 4-cycles in Section 5 and `check_note.py` were added when this note was written.

A third run, also AI-assisted, examined the note as written, and these additions first. It re-read Section 11 of [dH24] in the TeX source of arXiv:2312.03907v1 and in the journal’s HTML edition, the introduction of [Ing14] in the TeX source of arXiv:1111.3609v1, the zbMATH reviews of [MR22, Sil94] and the Crossref records of all references; it re-derived every proof line by line and wrote its own programs (in the package, `independent_run_2/`): the identities of Theorems 1.1 and 1.2, Corollary 1.3, and Proposition 1.5 and of the remarks of Section 3 in SymPy, by iterating the map; the singular locus of C by Gröbner bases and resultants, and the numbers of points of C over $\mathbb{F}_p$ for twelve primes, which confirm that C is absolutely irreducible with one node, non-split over $\mathbb{Q}$; a map with two rational 3-cycles for $\delta = -1$ ($c = -186874/2025$), so that the bound of Corollary 1.3(a) is attained; the complete periodic sets of twelve maps and Corollary 1.3(b) on 591 values of c ; a box search over the ranges of (i) which lists every cycle and also examines, for non-integral δ , denominators not divisible by $\text{den}(\delta)$; a complete search over the ranges of (ii) for all 31 values of δ by a different algorithm; and the computation (iii) in a different formulation. Every period of Table 1, the list of six 4-cycles, the two 5-cycles and every quoted cycle were reproduced, and the package was re-run from the archive with identical outputs. It found no error in the theorems. It required that the statement on integral data in Corollary 1.4 be restricted to what is proved (for a fixed integer $\delta \neq 1$ only finitely many integers c carry a rational 3-cycle, Remark 3.2), and some changes of wording and of the package, which were made; it also repeated the literature search.

Scope and priority. The note shows that [dH24, Section 11, Conjecture 4], as printed, is false for every $\delta \in \mathbb{Q}$ in each of the readings (R1)–(R3), and it describes all rational 3-cycles; which periods occur for a given δ is not determined, and a corrected statement is only asked (Question 5.2). The conjecture is Ingram’s [dH24, Section 11], and the rational periodic points for $\delta = -1$ are studied in his paper [Ing14]. The construction at the beginning of [dH24, Section 11] already gives rational 4-cycles for every $\delta \notin \{0, 1\}$ and, by the same linear algebra, 3-cycles for every $\delta \neq 1$ (Remark 3.3), so the counterexamples are essentially implicit in the source. The case $\delta = 0$ is the classical map $z^2 + c$, including its 3-cycle for $c = -\frac{29}{16}$ [Mor92, WR94, Mor98, MR22]. What the note adds is the explicit families of Theorem 1.1 and the complete description of the rational 3-cycles (Theorem 1.2 and Corollary 1.3); the remark on duality is elementary. The novelty is modest, specialists may regard it as known, and no priority is claimed. No printed answer to the question was found in the sources searched: both editions of [dH24]; [Ing14]; [KKPS25], on Hénon maps of higher degree; the works citing [dH24] in Semantic Scholar (10) and zbMATH Open (4) and those citing [Ing14] in zbMATH Open (6), by title and abstract; more than thirty arXiv abstract searches on arithmetic properties of Hénon maps; zbMATH Open, Crossref and four web searches, on 17 and 18 October 2026, repeated in part by the third verification run on 18 October 2026 (OpenAlex was not available). A search that finds nothing is not a proof of novelty. Silverman [Sil94] proved that periodic points of Hénon maps over number fields have bounded height and, according to the zbMATH review, gives numerical examples of Hénon maps with rational periodic points; we could not read the paper, and these examples may include rational cycles of length 3 or 4 for particular Jacobians.

What was read: of [dH24], Section 11 in the TeX source of arXiv:2312.03907v1 and in the journal’s HTML edition (amj.math.stonybrook.edu, article 23-70), which agree in the passages used; of [Ing14], the introduction of arXiv:1111.3609v1 (TeX source); [KKPS25] in its TeX source, by keyword; of [MR22, Sil94] only their zbMATH reviews. What we say about [Mor92, WR94] and [MR22, Theorems 1 and 4] is taken from the zbMATH review of [MR22], and about [Mor98, FPS97, Sto08] from the abstract of [Sto08]; Poonen’s conjecture [Poo98] is standard. The proofs of the cited theorems were not checked.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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