

Hausdorff Dimensions of Computable Closed Subgroups of Finitely Generated Profinite Groups

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Abstract

In a 2025 list of open problems in computability theory, A. Nies notes that every left- Σ_2 real $r \in [0, 1]$ is the Hausdorff dimension of a computable subgroup of a computable profinite group, and asks which values occur when the group is topologically finitely generated. Reading “computable subgroup” as computable closed subgroup, and taking the dimension, as Mayordomo and Nies do, with respect to the given computable inverse system, we show that finite generation imposes no restriction: the values are exactly the left- Σ_2 reals in $[0, 1]$. All of them occur for computable closed subgroups of the 2-generated pro- p lamplighter group $\mathbb{Z}_p \ltimes \mathbb{F}_p[[t]]$, simultaneously for its presentation by the wreath products $C_p \wr C_{p^k}$ and for its p -power, iterated p -power, Frattini and dimension subgroup series; for its lower p -series the values are the left- Σ_2 reals in $[0, \frac{1}{2}]$, the numbers $\frac{1}{2} + \frac{m}{2p^n}$ with $0 < m < p^n$, and 1. If “computable subgroup” meant “closed subgroup generated by finitely many computable elements”, the question would become an effective version of the problem of finitely generated Hausdorff spectra, which we do not address. The question and the statement without finite generation are from Nies’s section of the list; the statement is given there without proof and follows from an example of Mayordomo and Nies (essentially due to Barnea and Shalev) and the theorem of Downey, Jockusch and Schupp that the lower densities of computable sets are exactly the left- Σ_2 reals. The Hausdorff spectra of this group are classical (Klopsch; Klopsch and Thillaisundaram; de las Heras and Klopsch). The note is a straightforward effective version of these results, built on a lemma that makes one computable set give the prescribed dimension along several filtrations at once; its novelty is modest. This is an unrefereed note.

1 Introduction

1.1 The question

Let $G = \varprojlim (L_n, p_n)$ be an infinite profinite group, given by finite groups L_n with $L_0 = 1$ and surjective homomorphisms $p_n: L_{n+1} \rightarrow L_n$, and let $U_n \subseteq L_n$ be the image of a closed subgroup $U \leq G$. The Hausdorff dimension of U in the natural ultrametric of the inverse system is

$$\dim_H(U) = \liminf_{n \rightarrow \infty} \frac{\log |U_n|}{\log |L_n|} \quad (1)$$

by a theorem of Barnea and Shalev [BS97, Theorem 2.4], reproved geometrically by Mayordomo and Nies [MN26, Theorem 4.1]. The group is *computable* if the L_n and the p_n are uniformly computable (Smith [Smi81], La Roche [LR81]; see [Nie25, slide 11]). In the Cantor group $C_2^\omega = \varprojlim C_2^n$ the subgroup of the sequences vanishing outside $R \subseteq \mathbb{N}$ has dimension equal to the lower density of R [MN26,

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Example 4.2] (an example that Mayordomo and Nies attribute essentially to [BS97, Lemma 4.1]), and the lower densities of computable sets are exactly the left- Σ_2 reals in $[0, 1]$, the reals r for which $\{q \in \mathbb{Q} : q < r\}$ is Σ_2^0 [DJS13, Theorem 5.1]. Accordingly, the section of A. Nies in the problem list of the 2025 Tianyuan workshop [BBC⁺25] states that for each left- Σ_2 real $r \in [0, 1]$ there are a computable profinite group and a computable subgroup of Hausdorff dimension r , and asks:

“Which values can occur for r when G is also topologically finitely generated?”

The question is printed as Problem 16 of [BBC⁺25]; problems share their counter with theorems there, and it is the thirteenth problem of the list. The list gives no definitions, no proof and no reference for the preceding statement. The question is record AMR-150-0035 of the UnsolvedMath dataset [ula26], which cites the 2026 list [BGH⁺26]; in the $\text{\LaTeX}$ source of that list the text occurs only in a copy of the 2025 list placed after `\end{document}`, so it is not part of the compiled 2026 document.

We use the conventions of Mayordomo and Nies [MN26, Nie25], recalled in Section 2: computable inverse systems, computable closed subgroups (computable subtrees of the tree of the system) and the dimension (1) for the given system. For a topologically finitely generated pro- p group Nies singles out the natural inverse system $(G/G^{p^n})_n$ [Nie25, slide 19], the p -power series; we treat it and the other standard series as well.

1.2 Results

Let p be a prime, $B := \mathbb{F}_p[[t]]$ and $W := \mathbb{Z}_p \ltimes B$, where $1 \in \mathbb{Z}_p$ acts as multiplication by $1 + t$. This is the pro- p lamplighter group $C_p \hat{\wr} \mathbb{Z}_p$, the inverse limit of the wreath products $W_k = C_p \wr C_{p^k}$; it is topologically 2-generated and computable (Section 4). For $A \subseteq \mathbb{N}$ let $U_A := \{\sum_{j \in A} c_j t^j : c_j \in \mathbb{F}_p\}$, the closed subgroup of B of the power series supported in A .

Theorem 1.1. (a) *Let G be a computable profinite group and $U \leq G$ a computable closed subgroup. Then $\dim_H(U)$ is a left- Σ_2 real in $[0, 1]$. The same holds for Π_1^0 closed subgroups, and for the dimension with respect to any computable filtration of G (Definition 2.1).*

(b) *Let $r \in [0, 1]$ be left- Σ_2 . There is a computable set $A \subseteq \mathbb{N}$ such that U_A is a computable closed subgroup of W and*

$$\dim_W^{\mathcal{S}}(U_A) = r \quad \text{for } \mathcal{S} \in \{\mathcal{K}, \mathcal{P}, \mathcal{P}^*, \mathcal{F}, \mathcal{D}\}, \quad \dim_W^{\mathcal{L}}(U_A) = \frac{r}{2},$$

where $\mathcal{K}$ is the filtration by the kernels of the maps $W \rightarrow W_k$, and $\mathcal{P}, \mathcal{P}^*, \mathcal{F}, \mathcal{D}, \mathcal{L}$ are the p -power, iterated p -power, Frattini, dimension subgroup and lower p -series of W . These six filtrations are computable.

Corollary 1.2. *Read “computable subgroup” as “computable closed subgroup”. The dimensions (1) of the computable subgroups of topologically finitely generated computable profinite groups, with respect to the given computable inverse system, are exactly the left- Σ_2 reals in $[0, 1]$; all of them occur in W , for every prime p . The same holds with respect to $\mathcal{P}, \mathcal{P}^*, \mathcal{F}$ or $\mathcal{D}$, for the finitely generated pro- p groups in which this series is computable, for instance those given by this series as their inverse system. If “computable subgroup” meant “closed subgroup generated by finitely many computable elements”, the question would become an effective version of the problem of finitely generated Hausdorff spectra, which is not addressed here (Remark 5.5).*

Theorem 1.3. *For every prime p , the dimensions $\dim_W^{\mathcal{L}}(U)$ of the computable closed subgroups $U \leq W$, and also those of the Π_1^0 closed subgroups, are exactly the numbers in*

$$\{r \in [0, \frac{1}{2}] : r \text{ left-}\Sigma_2\} \cup \{\frac{1}{2} + \frac{m}{2p^n} : n \geq 1, 1 \leq m \leq p^n - 1\} \cup \{1\}.$$

Without effectivity this set becomes $[0, \frac{1}{2}] \cup \{\frac{1}{2} + \frac{m}{2p^n}\} \cup \{1\}$ [KT19, Corollary 2.11] (stated for odd p , valid for $p = 2$ by the appendix of [KT19]; Section 6 proves it for all p). We do not determine the values for $\mathcal{L}$ over all finitely generated groups.

1.3 Relation to known results

The question and the statement without finite generation are from Nies's section of [BBC⁺25]; the statement follows from [MN26, Example 4.2] and [DJS13, Theorem 5.1] (Remark 5.2). The Hausdorff spectra of W are classical: $[0, 1]$ for $\mathcal{P}$, $\mathcal{D}$, $\mathcal{F}$ by Klopsch [Klo99, Ch. VIII, §7] (as recalled in [dlHK22]), and the set above for $\mathcal{L}$ by Klopsch and Thillaisundaram [KT19, Corollary 2.11]; de las Heras and Klopsch [dlHK22], and de las Heras and Thillaisundaram for $p = 2$ [dlHT22b], found 2-generated pro- p groups with full normal spectra. The U_A are the coordinate subgroups of [KT19, Lemma 2.3], and Lemma 3.1 is part of [DJS13, Theorem 5.1]. By (13) the dimension of U_A is a lower limit of densities of A along the index sequence of the filtration. These sequences differ for the filtrations of Theorem 1.1(b), and a computable set of lower density r need not work for all of them (Remark 5.3). The new ingredient is the steering lemma (Lemma 3.2), a variant of [DJS13, Lemma 5.3] (which prescribes the lower and the upper density) in which the density is held at each target value on an interval of growing multiplicative length, so that the lower limit is the same along every index sequence of bounded ratio; Step 2 of the proof of Theorem 1.3 follows [KT19, Theorem 2.10]. The note is a straightforward effective version of known constructions, its novelty is modest, and no printed answer to the question was found in the sources searched (see “Scope and priority”).

2 Conventions and the upper bound

All profinite groups are infinite; a subgroup generated by a set means the closed subgroup generated by it. A *filtration* of a profinite group G is a descending chain $\mathcal{S} = (S_i)_{i \geq 0}$ of open normal subgroups with $S_0 = G$ and $\bigcap_i S_i = 1$. For a closed subgroup $U \leq G$ put

$$\dim_G^{\mathcal{S}}(U) := \liminf_{i \rightarrow \infty} \frac{\log|US_i : S_i|}{\log|G : S_i|} \in [0, 1] \quad (2)$$

(omitting the i with $S_i = G$); this is the Hausdorff dimension of U for the metric $d(g, h) = \inf\{|G : S_i|^{-1} : g^{-1}h \in S_i\}$ [BS97, Theorem 2.4], and finitely many terms of $\mathcal{S}$ do not affect it. For $G = \varprojlim (L_n, p_n)$ the kernels of the projections $G \rightarrow L_n$ form a filtration, and (2) becomes (1); conversely a filtration gives the inverse system $(G/S_i)_i$. All inverse systems below have $|L_{n+1}| > |L_n|$, as required in [MN26].

A profinite group $G = \varprojlim (L_n, p_n)$ ($L_0 = 1$, p_n surjective) is *computable* if the L_n (given by multiplication tables on initial segments of $\mathbb{N}$) and the p_n are uniformly computable. A closed subgroup $U \leq G$ is *computable* if the relation $\sigma \in U_n$ ($n \in \mathbb{N}$, $\sigma \in L_n$) is decidable, and a Π_1^0 *closed subgroup* if it is Π_1^0 ; by König's lemma the latter holds if U is the set of paths through a co-c.e. subtree of the tree of the system.

Definition 2.1. A filtration (S_i) of a computable profinite group $G = \varprojlim (L_n, p_n)$ is *computable* if there are computable functions $i \mapsto n(i)$ and $i \mapsto \bar{S}_i$ such that $\ker(G \rightarrow L_{n(i)}) \leq S_i$ and $\bar{S}_i \subseteq L_{n(i)}$ is the image of S_i .

Then $G/S_i \cong L_{n(i)}/\bar{S}_i$ and $|US_i : S_i| = |U_{n(i)}\bar{S}_i : \bar{S}_i|$. The filtration of the kernels of $G \rightarrow L_n$ is computable.

A real r is *left- Σ_2* if $\{q \in \mathbb{Q} : q < r\}$ is Σ_2^0 [DJS13, Definition 1.3]. Rationals are left- Σ_2 , and $r \in [0, \frac{1}{2}]$ is left- Σ_2 if and only if $2r$ is. For $A \subseteq \mathbb{N}$ and $m \geq 1$ let $\delta_A(m) := |A \cap [0, m)|/m$; the lower density of A is $\liminf_m \delta_A(m)$.

For a finitely generated pro- p group G the standard filtration series are [dlHK22, §2]

$$\begin{aligned}\mathcal{L}: P_1 &= G, \quad P_i = P_{i-1}^p[P_{i-1}, G] \quad (i \geq 2); & \mathcal{P}: \pi_i &= G^{p^i} \quad (i \geq 0); \\ \mathcal{D}: D_1 &= G, \quad D_i = D_{\lceil i/p \rceil}^p \prod_{1 \leq j < i} [D_j, D_{i-j}] \quad (i \geq 2); & \mathcal{P}^*: \pi_0^* &= G, \quad \pi_i^* = (\pi_{i-1}^*)^p \quad (i \geq 1); \\ \mathcal{F}: \Phi_0 &= G, \quad \Phi_i = \Phi_{i-1}^p[\Phi_{i-1}, \Phi_{i-1}] \quad (i \geq 1),\end{aligned}$$

the lower p -series, the dimension subgroup series, the p -power series, the iterated p -power series and the Frattini series; here $H^q = \langle h^q : h \in H \rangle$. The p -power series is the natural inverse system of [Nie25, slide 19].

Lemma 2.2 (upper bound). *Let G be a computable profinite group, $\mathcal{S} = (S_i)$ a computable filtration of G , and $U \leq G$ a computable or Π_1^0 closed subgroup. Then $\dim_G^{\mathcal{S}}(U)$ is a left- Σ_2 real in $[0, 1]$; in particular so is $\dim_H(U)$.*

Proof. Let $u_i := |US_i : S_i|$ and $g_i := |G : S_i| = |L_{n(i)}|/|\bar{S}_i|$, which is computable; $1 \leq u_i \leq g_i$, and $g_i \geq 2$ for large i . For $q \in \mathbb{Q}$,

$$q < \dim_G^{\mathcal{S}}(U) \iff \exists \varepsilon \in \mathbb{Q}_{>0} \exists N \forall i \geq N : \log u_i \geq (q + \varepsilon) \log g_i$$

(if the lower limit exceeds $q + 2\varepsilon$, the quotients exceed $q + \varepsilon$ for large i ; the converse is clear). For $q + \varepsilon = c/d$ with integers c and $d \geq 1$ the condition says $u_i \geq s_i$, where s_i is the least integer with $s_i^d \geq g_i^c$. If U is computable, $u_i = |U_{n(i)}\bar{S}_i : \bar{S}_i|$ is computable; if U is a Π_1^0 closed subgroup, “ $u_i \geq s$ ” says that some s elements of $L_{n(i)}$ in distinct cosets of $\bar{S}_i$ lie in $U_{n(i)}$, a Π_1^0 condition uniformly in i and s . In both cases the right-hand side is Σ_2^0 . $\square$

This is Theorem 1.1(a).

3 Left- Σ_2 reals and computable sets

Lemma 3.1 ([DJS13, Theorem 5.1]). *A real $r \in [0, 1]$ is left- Σ_2 if and only if $r = \liminf_k q_k$ for a computable sequence $(q_k)_{k \geq 0}$ of rationals in $[0, 1]$.*

This is the equivalence of (ii) and (iii) in [DJS13, Theorem 5.1]; the q_k may be taken in $[0, 1]$, since replacing q_k by $\min(\max(q_k, 0), 1)$ does not change the lower limit when $r \in [0, 1]$. (If $r = \liminf_k q_k$, then $q < r$ if and only if $\exists \varepsilon \in \mathbb{Q}_{>0} \exists N \forall k \geq N : q_k \geq q + \varepsilon$, which is Σ_2^0 ; the converse is proved in [DJS13] with the hat trick.)

Lemma 3.2 (steering). *Let $(q_k)_{k \geq 0}$ be a computable sequence of rationals in $[0, 1]$ and $r := \liminf_k q_k$. There is a computable set $A \subseteq \mathbb{N}$ such that $\liminf_{i \rightarrow \infty} \delta_A(m_i) = r$ for every strictly increasing sequence $(m_i)_{i \geq 0}$ of positive integers with $\sup_i m_{i+1}/m_i < \infty$. In particular the lower asymptotic density of A is r .*

Proof. Put $q'_k := \min(\max(q_k, \frac{1}{k+2}), 1 - \frac{1}{k+2})$. Then $q'_k \in (0, 1)$, $|q'_k - q_k| \leq \frac{1}{k+2}$, and $\liminf_k q'_k = r$. We define a computable function $\rho: \mathbb{N}_{\geq 1} \rightarrow \mathbb{Q} \cap [0, 1]$ such that

$$\rho(n+1) \in I_n := \left[\frac{n\rho(n)}{n+1}, \frac{n\rho(n)+1}{n+1} \right] \quad (n \geq 1), \quad (3)$$

that is, $0 \leq (n+1)\rho(n+1) - n\rho(n) \leq 1$. Note that $\rho(n) \in I_n \subseteq [0, 1]$. The construction runs in phases $k = 0, 1, 2, \dots$; phase k occupies the integers n with $N_k \leq n \leq N_{k+1}$, where $N_0 := 1$ and $\rho(1) := q'_0$.

Approach. For $n = N_k, N_k + 1, \dots$ in turn: if $q'_k \in I_n$, put $\rho(n+1) := q'_k$, $M_k := n+1$, and stop; otherwise let $\rho(n+1)$ be the endpoint of I_n nearer to q'_k .

Hold. Put $\rho(n) := q'_k$ for $M_k < n \leq N_{k+1} := (k+2)M_k$.

The approach terminates. If $q'_k < \rho(N_k)$, then as long as $q'_k < \min I_n$ the rule keeps $n\rho(n) = c := N_k\rho(N_k) > 0$ constant, so that $\rho(n) = c/n > q'_k$ and $\min I_n = c/(n+1) \rightarrow 0$; at the first n with $\min I_n \leq q'_k$ we have $q'_k < \rho(n) \leq \max I_n$, so $q'_k \in I_n$. If $q'_k > \rho(N_k)$, the rule keeps $n(1 - \rho(n)) = N_k(1 - \rho(N_k)) > 0$ constant and the argument is symmetric; if $q'_k = \rho(N_k)$ it stops at once. Thus ρ , (M_k) and (N_k) are computable, (3) holds, and $N_{k+1} \geq 2M_k > 2N_k$. Moreover $\rho(N_k) = q'_{k-1}$ for $k \geq 1$, and in phase k the values $\rho(n)$ move monotonically from q'_{k-1} to q'_k and then stay there:

$$\min(q'_{k-1}, q'_k) \leq \rho(n) \leq \max(q'_{k-1}, q'_k) \quad (N_k \leq n \leq N_{k+1}, k \geq 1). \quad (4)$$

Let $g(0) := 0$, $g(n) := n\rho(n)$ for $n \geq 1$, and $A := \{n \in \mathbb{N} : \lfloor g(n+1) \rfloor > \lfloor g(n) \rfloor\}$. By (3) and $g(1) \in [0, 1]$ we have $0 \leq g(n+1) - g(n) \leq 1$, so $\lfloor g(n+1) \rfloor - \lfloor g(n) \rfloor \in \{0, 1\}$ and $|A \cap [0, m]| = \lfloor g(m) \rfloor$ for $m \geq 1$. Hence

$$\rho(m) - \frac{1}{m} < \delta_A(m) = \lfloor m\rho(m) \rfloor / m \leq \rho(m) \quad (m \geq 1), \quad (5)$$

and A is computable.

Let (m_i) be strictly increasing with $m_{i+1} \leq Cm_i$ for all i . For each $k \geq C$ with $M_k > m_0$ let i be least with $m_i \geq M_k$. Then $i \geq 1$ and $M_k \leq m_i \leq Cm_{i-1} < CM_k \leq (k+2)M_k = N_{k+1}$, so m_i lies in $[M_k, N_{k+1}]$, where $\rho = q'_k$, and $\delta_A(m_i) \leq q'_k$ by (5). These indices i tend to infinity with k , so $\liminf_i \delta_A(m_i) \leq \liminf_k q'_k = r$. Conversely, m_i lies in a phase $k(i)$ with $k(i) \rightarrow \infty$, and by (5) and (4), $\delta_A(m_i) > \min(q'_{k(i)-1}, q'_{k(i)}) - 1/m_i$, whose lower limit is $\geq r$. $\square$

4 The pro- p lamplighter group

Fix a prime p . The additive group of $B = \mathbb{F}_p[[t]]$ is a compact elementary abelian pro- p group (coefficientwise topology); put $B_j := t^j B$, so that $|B : B_j| = p^j$. For $\lambda \in \mathbb{Z}_p$ the binomial series $(1+t)^\lambda := \sum_{i \geq 0} \binom{\lambda}{i} t^i$ (coefficients read modulo p) defines a continuous homomorphism $\lambda \mapsto (1+t)^\lambda$ from $\mathbb{Z}_p$ to the unit group of B , with $(1+t)^{p^a \mu} = ((1+t)^\mu)^{p^a}$, $(1+t)^{p^n} = 1 + t^{p^n}$, and with coefficient of t equal to $\lambda \bmod p$. Hence, if $\lambda = p^a \lambda'$ with $\lambda' \in \mathbb{Z}_p^\times$, then

$$(1+t)^{-\lambda} - 1 = ((1+t)^{-\lambda'} - 1)^{p^a} \in t^{p^a} B^\times, \quad (6)$$

because $(1+t)^{-\lambda'} - 1 = -\lambda' t + O(t^2)$. We write the elements of $W = \mathbb{Z}_p \ltimes B$ as $x^\lambda v$ ($\lambda \in \mathbb{Z}_p$, $v \in B$), with $x^\lambda v x^{-\lambda} = (1+t)^\lambda v$; the group B is written additively. Then

$$(x^\lambda v)(x^\mu w) = x^{\lambda+\mu}((1+t)^{-\mu} v + w), \quad (x^\lambda v)^{-1} = x^{-\lambda}(-(1+t)^\lambda v). \quad (7)$$

The compact group W is topologically generated by $x = x^1 0$ and $y = x^0 1$: the closed subgroup generated by them contains $x^{\mathbb{Z}_p}$ and the conjugates $x^n y x^{-n} = (1+t)^n$, $n \geq 0$, whose $\mathbb{F}_p$ -span is $\mathbb{F}_p[t]$, which is dense in B .

Lemma 4.1. *Let $\lambda, \mu \in \mathbb{Z}_p$ and $v, w \in B$.*

(a) *For $n \geq 1$, $(x^\lambda v)^n = x^{n\lambda}(\sum_{j=0}^{n-1} (1+t)^{-j\lambda})v$. For $e \geq 1$ and $u := (1+t)^{-\lambda} - 1$,*

$$(x^\lambda v)^{p^e} = x^{p^e \lambda} u^{p^e - 1} v. \quad (8)$$

(b) *With $[g, h] := g^{-1} h^{-1} g h$,*

$$[x^\lambda v, x^\mu w] = ((1+t)^{-\mu} - 1)v - ((1+t)^{-\lambda} - 1)w \in B. \quad (9)$$

- (c) For $a, b \geq 0$ the set $S(a, b) := \{x^\lambda w : \lambda \in p^a \mathbb{Z}_p, w \in B_b\}$ is a closed subgroup. If $p^a \geq b$, it is normal of index p^{a+b} , and $x^\lambda w \mapsto (\lambda \bmod p^a, w \bmod t^b)$ is an isomorphism of $W/S(a, b)$ onto $(\mathbb{Z}/p^a) \ltimes \mathbb{F}_p[t]/(t^b)$, where 1 acts as multiplication by $1+t$.

Proof. (a) Induction on n with (7). For $n = p^e$ and $\lambda \neq 0$, $z := (1+t)^{-\lambda}$ satisfies $u \sum_{j < p^e} z^j = z^{p^e} - 1 = u^{p^e}$ in characteristic p , and B is an integral domain; for $\lambda = 0$ both sides of (8) are trivial. (b) For $g = x^\lambda v$ and $h = x^\mu w$, (7) gives $gh = x^{\lambda+\mu}((1+t)^{-\mu}v + w)$ and $hg = x^{\lambda+\mu}((1+t)^{-\lambda}w + v)$, and $[g, h] = (hg)^{-1}gh$. (c) $S(a, b)$ is a closed subgroup by (7), as B_b is an ideal. Let $p^a \geq b$. For $s \in S(a, b)$ and $g \in W$, $g^{-1}sg = s[s, g]$ with $[s, g] \in B_b + B_{p^a} = B_b$ by (9) and (6). If $\lambda - \mu \in p^a \mathbb{Z}_p$, then $(1+t)^{\lambda-\mu} \equiv 1 \pmod{B_b}$, and $(x^\lambda v)^{-1}x^\mu w = x^{\mu-\lambda}(w - (1+t)^{\lambda-\mu}v)$ lies in $S(a, b)$ if and only if $\lambda \equiv \mu \pmod{p^a}$ and $v \equiv w \pmod{B_b}$. So the map is well defined and bijective on cosets, and it is a homomorphism by (7). $\square$

By (8) and (6), for $\lambda \in p^a \mathbb{Z}_p$, $v \in B_\beta$ and $e \geq 1$,

$$(x^\lambda v)^{p^e} \in S(a+e, p^a(p^e-1)+\beta), \quad (10)$$

and for $\lambda = p^a$ the map $v \mapsto u^{p^e-1}v$ is multiplication by $t^{p^a(p^e-1)}$ times a unit, so it maps B_β onto $B_{p^a(p^e-1)+\beta}$. By (9) and (6), for $\lambda \in p^\alpha \mathbb{Z}_p$, $v \in B_\beta$, $\mu \in p^{\alpha'} \mathbb{Z}_p$, $w \in B_{\beta'}$,

$$[x^\lambda v, x^\mu w] \in B_{\min(p^\alpha+\beta, p^{\alpha'}+\beta')}, \quad (11)$$

and the elements $[v, x^{p^{\alpha'}}] = -t^{p^{\alpha'}}(1+t^{p^{\alpha'}})^{-1}v$, $v \in B_\beta$, exhaust $B_{p^{\alpha'}+\beta}$.

The wreath presentation. Put $K_k := S(k, p^k)$ and $W_k := W/K_k \cong (\mathbb{Z}/p^k) \ltimes \mathbb{F}_p[t]/(t^{p^k})$ for $k \geq 0$. Since $\mathbb{F}_p[s]/(s^{p^k}-1) = \mathbb{F}_p[s]/((s-1)^{p^k}) \cong \mathbb{F}_p[t]/(t^{p^k})$ via $s \mapsto 1+t$, and $C_p \wr C_{p^k} = \mathbb{F}_p[C_{p^k}] \ltimes C_{p^k}$ with the generator acting as multiplication by s , we have $W_k \cong C_p \wr C_{p^k}$ and $|W_k| = p^{k+p^k}$. The inverse system $L_0 := 1, L_{k+1} := W_k$ ($k \geq 0$), with the canonical surjections, is computable by (7), has $|L_{n+1}| > |L_n|$, and has limit W , since $\bigcap_k K_k = 1$. This is the presentation of W that we use; its filtration $\mathcal{K} = (W, K_0, K_1, \dots)$ is the filtration of Theorem 1.1(b). (The trivial group L_0 is prepended so that the system has the form required in Section 2; it does not affect dimensions.) So W is a computable, topologically 2-generated pro- p group.

Proposition 4.2. Let $f_i := (p^i - 1)/(p - 1)$ and $e_i := \lceil \log_p i \rceil$. The standard series of W are

$$P_i(W) = S(i-1, i-1), \quad \pi_i(W) = \pi_i^*(W) = S(i, p^i-1), \quad \Phi_i(W) = S(i, f_i), \quad D_i(W) = S(e_i, i-1)$$

for all $i \geq 1$ (for π_i, π_i^* and Φ_i also for $i = 0$). In each case $p^a \geq b$ for $S(a, b)$.

Proof. A closed subgroup contains $S(a, b)$ as soon as it contains x^{p^a} and B_b , and it is contained in $S(a, b)$ as soon as its generators are. The inequalities $p^a \geq b$ are clear ($p^{i-1} \geq i-1$, $p^i \geq f_i$, $p^{e_i} \geq i$).

$\mathcal{L}$. $P_1 = W = S(0, 0)$. Let $P_i = S(i-1, i-1)$. By (10) the p -th powers of elements of P_i lie in $S(i, (p-1)p^{i-1} + i-1) \subseteq S(i, i)$, and $x^{p^i} = (x^{p^{i-1}})^p$ is one of them. By (11) with $(\alpha, \beta, \alpha', \beta') = (i-1, i-1, 0, 0)$, $[P_i, W] \subseteq B_{\min(i, p^{i-1})} = B_i$, and the elements $[v, x]$, $v \in B_{i-1}$, exhaust B_i . Hence $P_{i+1} = S(i, i)$.

$\mathcal{P}, \mathcal{P}^*$. By (10) with $a = v_p(\lambda)$, every p^i -th power lies in $S(i, p^a(p^i-1)) \subseteq S(i, p^i-1)$, and x^{p^i} and $(xv)^{p^i} = x^{p^i}t^{p^{i-1}}\epsilon v$ ($\epsilon \in B^\times$, $v \in B$) generate $S(i, p^i-1)$. If $\pi_i^* = S(i, p^i-1)$, the p -th powers of its elements lie in $S(i+1, p^i(p-1) + p^i-1) = S(i+1, p^{i+1}-1)$, and $x^{p^{i+1}}$ and the elements $(x^{p^i}v)^p$, $v \in B_{p^i-1}$, generate it.

$\mathcal{F}$. $\Phi_0 = W$. If $\Phi_i = S(i, f_i)$, then the p -th powers of its elements lie in $S(i+1, p^i(p-1) + f_i) \subseteq S(i+1, f_{i+1})$ (as $f_{i+1} = f_i + p^i$), and $x^{p^{i+1}}$ occurs; by (11) with $\alpha = \alpha' = i$ and $\beta = \beta' = f_i$, $[\Phi_i, \Phi_i] \subseteq B_{p^i + f_i} = B_{f_{i+1}}$, and the elements $[v, x^{p^i}]$, $v \in B_{f_i}$, exhaust $B_{f_{i+1}}$. So $\Phi_{i+1} = S(i+1, f_{i+1})$.

$\mathcal{D}$. $D_1 = W = S(0, 0)$. Let $i \geq 2$, assume $D_j = S(e_j, j-1)$ for $j < i$, and put $e := e_i$ and $i' := \lceil i/p \rceil$. From $p^{e-1} < i \leq p^e$ we get $p^{e-2} < i' \leq p^{e-1}$ (and $i' = 1$ if $e = 1$), so $e_{i'} = e - 1$. The p -th powers of elements of $D_{i'} = S(e-1, i'-1)$ lie in $S(e, (p-1)p^{e-1} + i' - 1) \subseteq S(e, i-1)$, because $(p-1)p^{e-1} + i' \geq (p-1)p^{e-1} + i/p \geq i$ as $p^e \geq i$; and $x^{p^e} = (x^{p^{e-1}})^p$ occurs. For $1 \leq j < i$, (11) gives $[D_j, D_{i-j}] \subseteq B_{\min(p^{e_i-j} + j-1, p^{e_j+i-j-1})} \subseteq B_{i-1}$, since $p^{e_m} \geq m$; and the elements $[v, x]$, $v \in B_{i-2}$, of $[D_{i-1}, D_1]$ exhaust B_{i-1} . So $D_i = S(e, i-1)$. $\square$

In the finite quotients W_k these formulas agree with [KT19, Proposition 2.6]. Each of the six filtrations $\mathcal{S} = (S(a_i, b_i))_i$ above (with $\mathcal{K}$) is computable in the sense of Definition 2.1: $S(a, b) \supseteq K_k$ when $a \leq k$ and $b \leq p^k$, so one can take $n(i) = k+1$ with $k = \max(a_i, e(b_i))$, where $e(c)$ denotes the least $e \geq 0$ with $p^e \geq c$ (so $e_i = e(i)$), and the image of $S(a_i, b_i)$ in W_k is given by Lemma 4.1(c).

Lemma 4.3. *Let $A \subseteq \mathbb{N}$.*

- (a) U_A is a closed elementary abelian subgroup of B , and $|U_A : U_A \cap B_m| = p^{|A \cap [0, m]|}$ for $m \geq 0$.
- (b) If $p^a \geq b$, then $U_A \cap S(a, b) = U_A \cap B_b$ and

$$\log_p |U_A S(a, b) : S(a, b)| = |A \cap [0, b]|. \quad (12)$$

Consequently, for a filtration $\mathcal{S} = (S(a_i, b_i))_i$ with $p^{a_i} \geq b_i$,

$$\dim_W^{\mathcal{S}}(U_A) = \liminf_{i \rightarrow \infty} \frac{|A \cap [0, b_i]|}{a_i + b_i}. \quad (13)$$

- (c) Let $p^a \geq b$. A coset $x^\lambda w S(a, b)$ meets U_A if and only if $\lambda \in p^a \mathbb{Z}_p$ and the truncation of w modulo t^b is supported in A . So if A is computable, then U_A is a computable closed subgroup of W (take $(a, b) = (k, p^k)$), and its images in the quotients $W/S(a_i, b_i)$ of a computable sequence (a_i, b_i) with $p^{a_i} \geq b_i$ are uniformly decidable.

Proof. (a) U_A is the intersection of the closed subgroups $\{w \in B : \text{the coefficient of } t^j \text{ in } w \text{ is } 0\}$, $j \notin A$, and truncation modulo t^m maps $U_A/(U_A \cap B_m)$ isomorphically onto $\mathbb{F}_p^{A \cap [0, m]}$. (b) An element $x^\lambda w$ of $S(a, b)$ lies in B if and only if $\lambda = 0$, so $U_A \cap S(a, b) = U_A \cap B_b$, and $|U_A S(a, b) : S(a, b)| = |U_A : U_A \cap B_b|$; now use (a), and (2) with $\log_p |W : S(a_i, b_i)| = a_i + b_i$. (c) For $u \in U_A$, $(x^\lambda w)^{-1}u = x^{-\lambda}(u - (1+t)^\lambda w)$ lies in $S(a, b)$ if and only if $\lambda \in p^a \mathbb{Z}_p$ and $u \equiv w \pmod{B_b}$, because $(1+t)^\lambda \equiv 1 \pmod{B_b}$ for $\lambda \in p^a \mathbb{Z}_p$. $\square$

5 Proof of the main theorem

Theorem 5.1. *Let A be the set of Lemma 3.2 for a computable sequence (q_k) of rationals in $[0, 1]$ with $\liminf_k q_k = r$. Let $(a_i, b_i)_{i \geq 1}$ be a computable sequence of pairs of integers $a_i \geq 0$, $b_i \geq 1$ with $p^{a_i} \geq b_i$, such that (a_i) is non-decreasing and unbounded, (b_i) is strictly increasing, $\sup_i b_{i+1}/b_i < \infty$, and $a_i/b_i \rightarrow 0$. Then $\mathcal{S} := (W, S(a_1, b_1), S(a_2, b_2), \dots)$ is a computable filtration of W , U_A is computable, and $\dim_W^{\mathcal{S}}(U_A) = r$. If instead $a_i = b_i$ for all i , with (b_i) strictly increasing and $\sup_i b_{i+1}/b_i < \infty$, then $\dim_W^{\mathcal{S}}(U_A) = r/2$.*

Proof. The subgroups $S(a_i, b_i)$ are open and normal (Lemma 4.1(c)), descending, with trivial intersection, and the filtration is computable as explained after Proposition 4.2; U_A is computable by Lemma 4.3(c). By (13) the quotient in (2) equals $\delta_A(b_i) \cdot b_i/(a_i + b_i)$. The second factor tends to 1 and the first lies in $[0, 1]$, so the lower limit equals $\liminf_i \delta_A(b_i) = r$ by Lemma 3.2. If $a_i = b_i$, the quotient is $\delta_A(b_i)/2$. $\square$

Proof of Theorem 1.1. Part (a) is Lemma 2.2. For (b), Lemma 3.1 gives a computable sequence (q_k) of rationals in $[0, 1]$ with $\liminf_k q_k = r$, and Lemma 3.2 a computable set A . By the construction of $\mathcal{K}$ and by Proposition 4.2, up to finitely many initial terms the six filtrations are of the form of Theorem 5.1, with $(a_i, b_i) = (k, p^k)$ for $\mathcal{K}$, $(i, p^i - 1)$ for $\mathcal{P} = \mathcal{P}^*$, $(i, (p^i - 1)/(p - 1))$ for $\mathcal{F}$ ($i \geq 1$), $(\lfloor \log_p i \rfloor, i - 1)$ for $\mathcal{D}$ and $(i - 1, i - 1)$ for $\mathcal{L}$ ($i \geq 2$). The ratios b_{i+1}/b_i are at most $p + 1$, and $a_i/b_i \rightarrow 0$ except for $\mathcal{L}$, where $a_i = b_i$. Theorem 5.1 gives the values r and $r/2$. $\square$

Proof of Corollary 1.2. By Lemma 2.2, every value is a left- Σ_2 real in $[0, 1]$, for the given computable inverse system and for every computable filtration, in particular for a computable p -power series. Conversely, W is a topologically finitely generated computable profinite group with the presentation (L_n) , whose filtration is $\mathcal{K}$; its series $\mathcal{P}, \mathcal{P}^*, \mathcal{F}, \mathcal{D}$ are computable; and by Theorem 1.1(b) every left- Σ_2 real in $[0, 1]$ is the dimension of a computable closed subgroup with respect to each of them. $\square$

Remark 5.2 (the statement without finite generation). In $C_2^\omega = \varprojlim C_2^n$ (projections forgetting the last coordinate) the subgroup U_R of the sequences vanishing outside a computable set R is a computable closed subgroup of dimension $\liminf_n \delta_R(n)$ [MN26, Example 4.2] (essentially [BS97, Lemma 4.1], as noted in [MN26]). By Lemmas 3.1 and 3.2 (or [DJS13, Theorem 5.1]) every left- Σ_2 real in $[0, 1]$ arises, and by Lemma 2.2 only these. This is the statement of [BBC⁺25], which the list gives without proof. The group C_2^ω is not finitely generated.

Remark 5.3 (the steering is needed). Let $p = 3$, $A := \mathbb{N} \setminus \bigcup_{m \geq 0} [4 \cdot 9^m, 8 \cdot 9^m)$ and $c_m := (9^m - 1)/2$. Then $|A \cap [0, n]|$ is $n - c_m$ on $[9^m, 4 \cdot 9^m]$, $4 \cdot 9^m - c_m$ on $[4 \cdot 9^m, 8 \cdot 9^m]$ and $n - c_{m+1}$ on $[8 \cdot 9^m, 9^{m+1}]$. Hence the lower density of A is $\lim_m \delta_A(8 \cdot 9^m) = \frac{7}{16}$, while $\delta_A(9^m - 1) = \frac{1}{2}$ and $\delta_A(3 \cdot 9^m - 1) \rightarrow \frac{5}{6}$; by (13), $\dim_W^{\mathcal{P}}(U_A) = \frac{7}{16}$ but $\dim_W^{\mathcal{L}}(U_A) = \frac{1}{2}$.

Remark 5.4 (the index filtration). Let I_j be the intersection of the open normal subgroups N of W of index at most p^j ; write $v(z)$ for the t -adic valuation of $z \in B$, and let $e(\cdot)$ be as after Proposition 4.2. For such N , $N \cap B$ is a nonzero closed ideal B_b of $\mathbb{F}_p[[t]]$ (it is invariant under multiplication by $1+t$), and if $p^a \mathbb{Z}_p$ is the image of N in W/B , then $N = \langle g \rangle B_b$ with $g = x^{p^a} z$. By (9) and (6), $[g, y] = 1 - (1+t)^{-p^a}$ and $[g, x] = ((1+t)^{-1} - 1)z$, of valuations p^a and $v(z) + 1$, lie in B_b ; so $p^a \geq b$, $z \in B_{b-1}$ and $|W : N| = p^{a+b}$. Let $a + b \leq j$. Then $x^{p^j} \in N$: if $a < j$, then $g^{p^{j-a}} = x^{p^j} w$ with $w \in B_{p^a} \subseteq B_b$ by (8) and (6); if $a = j$, then $b = 0$. Moreover $B_{\beta_j} \subseteq B_b$, where $\beta_j := \max\{c : c + e(c) \leq j\}$, because $b + e(b) \leq a + b \leq j$. As $S(j, 0)$ and $S(e(\beta_j), \beta_j)$ are among these N , $I_j = S(j, \beta_j)$. Since $\beta_{j+1} - \beta_j \in \{0, 1\}$ and $j - \beta_j = O(\log j)$, (13) gives $\dim_W^{\mathcal{I}}(U_A) = \frac{r}{2}$ for $\mathcal{I} = (I_j)$ and the set A of Theorem 1.1(b): for W this canonical filtration behaves like $\mathcal{L}$.

Remark 5.5 (other readings). (1) A closed subgroup generated by finitely many computable elements of a computable profinite group is a computable closed subgroup (its image in L_n is generated by the images of the generators), so its dimension is left- Σ_2 by Lemma 2.2. Which left- Σ_2 reals occur for such subgroups of finitely generated groups is an effective version of the problem of finitely generated Hausdorff spectra, not addressed here. Classically, the finitely generated spectra of W [KT19, Theorem 2.10] and of the groups of [dHT22a] are countable, while the closure of the first Grigorchuk group has finitely generated spectrum $[0, 1]$ for the 2-central lower series and the dimension subgroup series, answering a question of de las Heras and Thillaisundaram [FAGUA26]; more pro-2 groups with complete finitely generated spectrum are given in [FAM26]. (2) The question does not ask for normal subgroups; the normal spectra of W are $\{0, 1\}$ for $\mathcal{P}, \mathcal{D}, \mathcal{F}$ and $\{0, \frac{1}{2}, 1\}$ for $\mathcal{L}$ [KT19, Corollary 2.8].

6 The lower p -series

Proof of Theorem 1.3. By Proposition 4.2, $P_i := P_i(W) = S(i-1, i-1)$ and $\log_p |W : P_i| = 2(i-1)$.

Step 1: closed subgroups of B . Let $M \leq B$ be closed. For $j \geq 0$ the index $|M \cap B_j : M \cap B_{j+1}|$ is 1 or p ; let $J(M)$ be the set of j for which it is p , that is, the set of valuations of the nonzero elements of M . Then $|M : M \cap B_i| = p^{|J(M) \cap [0, i]|}$, and since $M \cap P_i = M \cap B_{i-1}$,

$$\dim_W^{\mathcal{L}}(M) = \liminf_{i \rightarrow \infty} \frac{|J(M) \cap [0, i-1]|}{2(i-1)} = \frac{1}{2} \liminf_{m \rightarrow \infty} \delta_{J(M)}(m) \in [0, \frac{1}{2}].$$

For $M = U_A$ we have $J(M) = A$.

Step 2: closed subgroups not contained in B . We follow the proof of [KT19, Theorem 2.10]. Let $K \leq W$ be closed with $K \not\subseteq B$. Its image in $W/B \cong \mathbb{Z}_p$ is a nonzero closed subgroup, hence equal to $p^n \mathbb{Z}_p$ for some $n \geq 0$. Choose $h = x^{p^n} z \in K$ ($z \in B$) and put $M := K \cap B$. Every $g \in K$ has image $p^n \mu$ for a unique $\mu \in \mathbb{Z}_p$, and $gh^{-\mu} \in M$; so $K = \{h^\mu w : \mu \in \mathbb{Z}_p, w \in M\}$. Conjugation by $h^{\pm 1}$ acts on B as multiplication by $(1+t)^{\pm p^n} = (1+s)^{\pm 1}$, where $s := t^{p^n}$; so the closed subgroup M is invariant under multiplication by the polynomials in $1+s$, hence by $\mathbb{F}_p[[s]]$. Consequently $J(M) + p^n \subseteq J(M)$. Let $J_0 \subseteq \mathbb{Z}/p^n$ be the set of residues of the elements of $J(M)$ and $m := |J_0|$. For each residue in J_0 , $J(M)$ contains all large integers in that class and no integer of a class outside J_0 ; hence $|J(M) \cap [0, N]| = Nm/p^n + O(1)$.

Let $i-1 > n$ and $c := i-1-n$. By (8) and (6), $h^{p^c} = x^{p^{i-1}} w$ with $w \in B_{p^n(p^c-1)} \subseteq B_{i-1}$, because $p^n(p^c-1) \geq 2^{n+c-1} \geq n+c$. So $h^{p^c} \in P_i$, and since P_i is a closed subgroup, $h^{p^c \nu} \in P_i$ for all $\nu \in \mathbb{Z}_p$. The image of K in $W/BP_i \cong \mathbb{Z}/p^{i-1}$ has order p^c , and $K \cap BP_i = \{h^{p^c \nu} w : \nu \in \mathbb{Z}_p, w \in M\}$. As the elements $h^{p^c \nu}$ lie in $K \cap P_i$, the group $(K \cap BP_i)/(K \cap P_i)$ is the image of M , isomorphic to $M/(M \cap B_{i-1})$. Therefore

$$\log_p |KP_i : P_i| = \log_p |K : K \cap P_i| = (i-1-n) + |J(M) \cap [0, i-1]| = (i-1) \left(1 + \frac{m}{p^n}\right) + O(1),$$

and $\dim_W^{\mathcal{L}}(K) = \frac{1}{2} + \frac{m}{2p^n}$, a genuine limit, with $0 \leq m \leq p^n$. The case $m = 0$ gives $\frac{1}{2}$, and $m = p^n$ gives 1.

Step 3: realisation. (a) If $r \in [0, \frac{1}{2}]$ is left- Σ_2 , so is $2r$, and Theorem 1.1(b) gives a computable closed subgroup U_A with $\dim_W^{\mathcal{L}}(U_A) = r$. (b) Let $n \geq 1$, $1 \leq m \leq p^n - 1$, $A(n, m) := \{j \in \mathbb{N} : j \bmod p^n \in \{1, \dots, m\}\}$ and

$$K_{n,m} := \{x^\lambda w : \lambda \in p^n \mathbb{Z}_p, w \in U_{A(n,m)}\}.$$

Multiplication by a power series in t^{p^n} maps $U_{A(n,m)}$ into itself, because $A(n, m) + p^n \mathbb{N} \subseteq A(n, m)$; since $(1+t)^\mu$ is such a power series for $\mu \in p^n \mathbb{Z}_p$, $K_{n,m}$ is a closed subgroup by (7). It is computable: as in Lemma 4.3(c), an element $x^\lambda w$ of W maps into the image of $K_{n,m}$ in W_k if and only if $\lambda \in p^{\min(n,k)} \mathbb{Z}_p$ and the truncation of w modulo t^{p^k} is supported in $A(n, m)$. Here $K_{n,m} \cap B = U_{A(n,m)}$, $J = A(n, m)$ and $J_0 = \{1, \dots, m\}$, so $\dim_W^{\mathcal{L}}(K_{n,m}) = \frac{1}{2} + \frac{m}{2p^n}$ by Step 2. (c) $\dim_W^{\mathcal{L}}(W) = 1$.

Step 4. By Lemma 2.2, applied to the computable filtration $\mathcal{L}$, the dimension of a computable or Π_1^0 closed subgroup is left- Σ_2 ; by Steps 1 and 2 it lies in $[0, \frac{1}{2}]$ or is of the form $\frac{1}{2} + \frac{m}{2p^n}$ with $0 < m < p^n$, or equals 1. Conversely every value of the set in Theorem 1.3 occurs by Step 3, for computable, hence Π_1^0 , closed subgroups. (With arbitrary sets A in Step 3(a), Steps 1–3 give the full spectrum of [KT19, Corollary 2.11], for all p .) $\square$

7 Computations and reproducibility

No theorem depends on a computation. The program `check_note.py` (standard library, about 30 seconds) re-checks the finite parts: the group law and the formulas (6)–(9) in W_k for nine pairs (p, k)

with $p \leq 7$; Lemma 4.1(c) and the five standard series, computed from their definitions, for twelve pairs (p, k) with $p \leq 7$ (up to $|W_k| = 7^{51}$; 551 terms agree with Proposition 4.2; brute force in the three smallest groups; wrong formulas are detected); (12) in 5,900 comparisons; Lemma 4.3(c) on 120,114 cosets; Step 2 of the proof of Theorem 1.3 in 765 comparisons on 360 random subgroups, and the $K_{n,m}$; the integer inequalities of the proofs; Lemma 3.2 on six rational sequences, in exact arithmetic; and Remarks 5.3 and 5.4 in small cases. The package also contains the three programs of the first version (about 20 seconds) and the programs of the three verification runs, with their outputs.

Verification. The results were first obtained with written proofs and three programs. Two independent verification runs, both AI-assisted, followed; each re-derived the proofs and wrote its own programs, and neither read the other’s report. The first checked the statement against the sources of both lists and [MN26, Nie25], and recomputed the series of W_k , the count of Theorem 1.3 (501 instances) and the steering construction. The second used a different model of W_k (the series for fourteen pairs (p, k) , 1,608 terms; (12) in 11,860 comparisons; the count of Theorem 1.3 in 1,594 comparisons on 796 random subgroups; the steering construction in closed form up to about 10^{70}), showed with a naive set that the steering is needed, computed the index filtration for small j , re-ran the first programs with identical outputs, repeated the literature search and checked the bibliography against Crossref. Neither run found a mathematical error. Their corrections were made: the problem number, the group L_0 , Step 2 of the proof of Theorem 1.3, the reading caveat, the numbering of [KT19] and the case $p = 2$, a remark superseded by [FAGUA26], running times, journal versions, the hypotheses of Theorem 5.1, and the naming of the results made effective. When the note was written, the direct proof of Lemma 3.1 checked by both runs was replaced by the reference to [DJS13], and computable filtrations, Remark 5.3, the proof in Remark 5.4 and `check_note.py` were added; the first two runs did not examine these additions.

A third run, also AI-assisted, examined the note as written, and these additions first. It read again the L^AT_EX sources of [BBC⁺25] (arXiv version 1), [BGH⁺26] (version 2), [MN26] (version 3), [KT19] (version 2), [dlHK22] (version 2) and [DJS13] (version 1, with its printed numbering), and the slides [Nie25]; it re-derived every proof line by line and wrote its own programs (in the package, `independent_run_2/`): the product rule (7) against the wreath product with the shift action; (6)–(11) and Lemma 4.1(c) on random elements; the five standard series of W_k from their definitions for fourteen pairs (p, k) with $p \leq 13$ (635 terms; $\mathcal{D}$ also by Lazard’s formula; brute force on sets in the three smallest groups); (12) in 4,496 comparisons and Lemma 4.3(c) by brute force; the count of Theorem 1.3 in 832 comparisons on 420 random subgroups, and the $K_{n,m}$; all normal subgroups of index at most p^j of $W/S(j, j)$ for $p = 2, 3, 5, 7$ and $j \leq 6, 4, 3, 2$ respectively, which confirm Remark 5.4; the construction of Lemma 3.2 step by step up to $2 \cdot 10^5$ and in closed form up to about 10^{56} , along the index sequences of all the filtrations; and the counts and dimensions of Remark 5.3. It re-ran the package from the archive: all recorded outputs were reproduced up to timing lines, and the programs of the second run whose random seeds are not recorded gave the same results with other seeds. It found no error in the theorems. It required that the statement without finite generation be credited as in the sources, that [DJS13] be cited by its Theorem 5.1 (two statements are numbered 1.4 in its arXiv version), that Lemma 3.2 be called a variant rather than a strengthening of [DJS13, Lemma 5.3], and corrections of wording, of the counts in Section 7 and of the package, which were made; it also repeated the literature search.

Scope and priority. The note answers [BBC⁺25, Problem 16] in the reading of Corollary 1.2; it does not treat subgroups generated by finitely many computable elements (Remark 5.5), the lower p -series over all finitely generated groups, or groups whose standard series are not computable. The

question and the statement without finite generation are from Nies’s section of [BBC⁺25]. The example behind that statement is due to Mayordomo and Nies [MN26, Example 4.2] (essentially to Barnea and Shalev [BS97, Lemma 4.1]), the Hausdorff spectra of W are due to Klopsch [Klo99] and to Klopsch and Thillaisundaram [KT19], the coordinate subgroups to [KT19, Lemma 2.3], and Lemma 3.1 and the characterisation of lower densities to Downey, Jockusch and Schupp [DJS13]. What the note adds (the steering lemma, the simultaneous effective realisation in one finitely generated group, and Theorem 1.3) is a straightforward effective version of these results; its novelty is modest, specialists may regard it as known, and no priority is claimed. No printed answer to the question was found in the sources searched: the L^AT_EX sources of [BBC⁺25, BGH⁺26]; [MN26] (arXiv version 3; the journal version through its Crossref record); Nies’s Logic Blogs 2023–2024 (arXiv:2503.12673) and 2025–2026; Nies’s slides of 2024–2026, among them [Nie25]; abstracts of Nies’s arXiv papers of 2025–2026; the literature on Hausdorff spectra, among them [dlHK22, dlHT22b, dlHT22a, dlHKT25, FAGUA26, FAM26], and on computable profinite and topological groups, among them [Smi81, LR81, KMN25]; searches in arXiv, zbMATH Open, Crossref, OpenAlex and Semantic Scholar, and four web searches, on 17 and 18 October 2026 (the third verification run repeated the searches in arXiv, Crossref, OpenAlex and zbMATH Open on 18 October). A search that finds nothing is not a proof of novelty. What was read: the sources of [BBC⁺25, BGH⁺26, MN26, KT19, dlHK22, DJS13] and of the Logic Blog 2023–2024 in the parts used, the slides [Nie25], and the abstracts of [dlHT22b, dlHT22a, dlHKT25, FAGUA26, FAM26]; what we say about [BS97, Klo99, Smi81, LR81] is taken from [MN26, KT19, dlHK22, Nie25]. The proofs of the cited theorems were not checked.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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