

On Tabachnikov's Question on the Density of Reflected Wave Fronts: Balls and Ellipses

Alper Ferudun*

October 18, 2026

Abstract

Tabachnikov asked in 2026 whether, for a domain with smooth boundary in Euclidean space and a generic point P of it, the time- t wave front (the positions at time t of the unit-speed photons that leave P in all directions and are reflected optically at the boundary) becomes dense as $t \rightarrow \infty$; he added that even the round disk is open. We use the definition of Kang and Knill: the front becomes dense if it meets every open ball for all sufficiently large t . We give a partial answer. In a ball of $\mathbb{R}^d$, $d \geq 2$, the front becomes dense for every source other than the centre, and for the centre it does not. For the disk this also follows from a theorem of Colin de Verdière and Vicente on the length of the front inside a subdomain (2020; Exp. Math. 2024), whose proof is a sketch and who do not state the density corollary; the ball reduces to the disk by rotational symmetry. We give a complete proof: on the invariant tori the motion is a linear flow whose frequencies satisfy a twist condition for every Fourier mode, and the Riemann–Lebesgue lemma gives weak convergence of the distribution of the position of a photon with a uniformly random direction to a limit of full support. In an ellipse with semi-major axis a the front becomes dense exactly when P is not a focus; from a focus all photons meet at a focus at the times $2a, 4a, \dots$. Here the twist condition comes from elliptic coordinates and the limits of the frequencies at the ends of the two regimes of motion. We also recall, with an explicit C^∞ table of Penrose type, that for non-convex smooth domains the answer is negative in general; this is known in substance. General convex and generic smooth domains, the setting of the conjectures of Kang and Knill, remain open. This is an unrefereed note.

1 Introduction

1.1 The question

Let $D \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with smooth boundary and let $P \in D$. A photon leaving P with velocity v , $|v| = 1$, moves at unit speed on straight lines in D and is reflected at ∂D by the law of optics, $v \mapsto v - 2\langle v, n \rangle n$ with n the unit normal; let $\gamma_v(t)$ be its position at time t . The *wave front* at time t is

$$W_t(P) = \{\gamma_v(t) : v \in S^{d-1}\}.$$

In the strictly convex tables of our theorems every trajectory starting at an interior point is defined for all times. In general, trajectories for which the reflection law does not define the motion are disregarded, as in [KK26, Section 3].

In his contribution to the collection of open problems [ABB⁺26], Tabachnikov considers such a domain and a generic point P of it, and asks whether $W_t(P)$ becomes dense as $t \rightarrow \infty$. He refers to

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
Unrefereed preprint. CC BY 4.0.

[KK26] for the discussion, notes “Even the case of a round circle is open”, and mentions the work on the length of fronts [dVV24, dV21]. Neither “generic” nor “dense” is defined there. We use the definition of Kang and Knill.

Definition 1.1 ([KK26, Definition 1]). The front $W_t(P)$ *becomes dense* in D if for every open ball U meeting D there is τ such that $W_t(P) \cap U \neq \emptyset$ for all $t > \tau$.

Equivalently, for every $\varepsilon > 0$ the ε -neighbourhood of $W_t(P)$ covers D for all large t [KK26, Section 3]. This asks more than density for a sequence of times, and much more than illumination (density of the union of the fronts over all t). Kang and Knill use “generic” informally; we state our results with the exceptional points named.

1.2 Results

Theorem 1.2 (balls). *Let $d \geq 2$, let B be the open unit ball of $\mathbb{R}^d$ and let $P \in B$.*

- (a) *If $P \neq 0$, then $W_t(P)$ becomes dense in B .*
- (b) *$W_t(0)$ is the sphere of radius $\text{dist}(t, 2\mathbb{Z})$ about 0 (the point 0 if $t \in 2\mathbb{Z}$), and it does not become dense.*

For the disk ($d = 2$), part (a) also follows from a theorem of Colin de Verdière and Vicente (Section 1.3); for $d \geq 3$ it is an immediate corollary of the case of the disk, by rotational symmetry about the line through 0 and P (Corollary 3.5).

Theorem 1.3 (ellipses). *Let $E = \{x^2/a^2 + y^2/b^2 < 1\}$ with $a > b > 0$, let $c = \sqrt{a^2 - b^2}$, and let $F_1 = (-c, 0)$, $F_2 = (c, 0)$ be the foci.*

- (a) *If $P \in E$ is not a focus, then $W_t(P)$ becomes dense in E .*
- (b) *For $n \geq 1$, $W_{2an}(F_2) = \{F_1\}$ if n is odd and $W_{2an}(F_2) = \{F_2\}$ if n is even; the same holds with F_1 and F_2 exchanged. So the front of a focus does not become dense.*

Part (b) is elementary (Proposition 4.5). In the disk and in the ellipse the proof of (a) gives more: for the directions of a suitable arc (in the disk, by symmetry, for all directions) the image of the uniform measure under $v \mapsto \gamma_v(t)$ converges weakly, as $t \rightarrow \infty$, to a measure whose support is the closure of the table (Theorem 3.4 and Section 4.4); for the ball in dimension $d \geq 3$ we only deduce density from the disk (Corollary 3.5). The exceptional points are those at which the caustic of the photons does not depend on the direction: the centre of the ball (all photons move on diameters) and the foci of the ellipse (all photons pass through the foci).

Remark 1.4 (non-convex tables). Without convexity the answer is negative in general: in the explicit C^∞ table of Penrose type of Section 5, the fronts of all points of a non-empty open set miss a fixed disk for all t . This is known in substance: as reported in [CG03, Section 1], Penrose’s region of 1958 [PP58] is not illuminable from some points, and Rauch [Rau78] gave a smooth region not illuminable from any finite set of points; for the piecewise smooth Penrose table the failure of density is stated in [KK26, Section 5], and in a note in [KK26, Section 7] credited to Tabachnikov. For convex tables every point of the table sees every other point, and the question stays open; it is the setting of [KK26, Conjectures 3–5]: generic convex tables (all P), all convex tables (some P), convex polygons (all P).

1.3 Relation to the literature

Colin de Verdière and Vicente [dVV24, Section 4] prove, for the unit disk, a source P with $a = |P|$ and a smooth domain K in the disk, that the length of $W_t(P) \cap K$ is asymptotic to $\frac{2t}{\pi} \int_K \Psi(|z|) dx dy$, where $\Psi(r) = \min(r, a) / \sqrt{1 - \min(r, a)^2}$ (Theorem 4.1 in the arXiv version). If $a > 0$, then $\Psi > 0$ on $(0, 1]$, so the length of the front inside every small disk K grows linearly; in particular the front meets K for all large t . So the disk case of Theorem 1.2(a) follows from their theorem. The corollary is not stated in [dVV24], and [KK26, Section 5] and [ABB⁺26] regard the disk as open. The proof of [dVV24, Theorem 4.1] is a sketch which refers to the proof of their Theorem 3.1, where the oscillatory integrals are said to tend to 0 by stationary phase; the non-degeneracy of the phase for every Fourier mode (Lemma 3.2 below) is not written out there. Our proof is self-contained. The second paper [dV21] treats surfaces without boundary with an integrable Hamiltonian (Theorem 5.1 there: linear growth of the length of the front) and, for geodesic flows on closed surfaces, the length inside a domain (Corollary 5.1 there); it does not treat billiards.

Chernov and Galperin [CG03] study a search light, a narrow beam of directions from one point, in circles and ellipses, and determine the illuminated set, the union over all times of the points reached by the beam [CG03, Theorems 2.1 and 2.3]. The twist of the billiard map drives their proofs as it drives ours, but illumination does not decide density of the front at each large time. The integrability of the elliptic billiard (confocal caustics, elliptic coordinates) is classical, see for instance [Tab05, Chapter 4]; what we use is derived in Section 4.1.

A neutral remark: [KK26, Section 5] quotes the length of the front in the unit disk as $\arcsin(|P|)t + o(t)$; the formula proved by Vicente, as stated in [dVV24, Theorem 3.1], is $2 \arcsin(|P|)t + o(t)$, and the computations of Section 6 agree with the latter.

Section 2 gives a criterion for density (a linear flow on a continuous family of invariant tori, with a twist condition for every Fourier mode); Sections 3 and 4 verify it for the disk, with the ball as a corollary, and for the ellipse. Section 5 contains the remark on non-convex tables, and Section 6 the computations, the verification and the scope of the note.

2 A density criterion

We write $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$, m for its Haar probability measure, and $|\cdot|$ also for Lebesgue measure on $\mathbb{R}$. Weak convergence of probability measures ν_t as the real parameter $t \rightarrow \infty$ means convergence along every sequence $t_n \rightarrow \infty$; we use the continuous mapping theorem and the portmanteau theorem in this form [Bil99, Section 2].

Lemma 2.1. *Let $J \subset \mathbb{R}$ be open, $\varphi \in C^1(J)$ real with $|\{\varphi' = 0\}| = 0$, and $A \in L^1(J)$. Then $\int_J A(\theta) e^{2\pi i t \varphi(\theta)} d\theta \rightarrow 0$ as the real parameter $t \rightarrow \infty$.*

Proof. The set $O = \{\varphi' \neq 0\}$ is open and of full measure in J . Given $\eta > 0$, choose finitely many disjoint compact intervals $K_1, \dots, K_M$ in O with $\int_{J \setminus \bigcup K_m} |A| < \eta$. On K_m the function φ is a C^1 diffeomorphism onto an interval, and the substitution $s = \varphi(\theta)$ turns $\int_{K_m} A e^{2\pi i t \varphi}$ into $\int \tilde{A}_m(s) e^{2\pi i t s} ds$ with $\tilde{A}_m = (A \circ \varphi^{-1}) |(\varphi^{-1})'|$, whose L^1 norm is $\int_{K_m} |A|$. By the Riemann–Lebesgue lemma these integrals tend to 0 as $t \rightarrow \infty$. Hence the lim sup of the modulus of the integral is at most η , for every $\eta > 0$. $\square$

Setting. Let $J \subset \mathbb{R}$ be a finite union of disjoint bounded open intervals, with the probability measure $\lambda_J = d\theta/|J|$ (the directions), let $I \subset \mathbb{R}$ be an interval (the caustic parameter), and let

$$\kappa : J \rightarrow I, \quad \omega : I \rightarrow \mathbb{R}^2, \quad X : I \times \mathbb{T}^2 \rightarrow \mathbb{R}^d, \quad \Theta^0 : J \rightarrow \mathbb{T}^2$$

be maps such that κ and X are continuous, Θ^0 is Borel, and $\omega \circ \kappa$ is C^1 . For $\theta \in J$ and $t \geq 0$ put

$$\Gamma_\theta(t) = X(\kappa(\theta), \Theta^0(\theta) + t\omega(\kappa(\theta))), \quad (1)$$

and let ν_t be the law of $\Gamma_\theta(t)$ when θ has the law λ_J .

Proposition 2.2. *In this setting assume that for every $k \in \mathbb{Z}^2 \setminus \{0\}$ the function $\varphi_k(\theta) = k \cdot \omega(\kappa(\theta))$ satisfies $|\{\theta \in J : \varphi'_k(\theta) = 0\}| = 0$. Then, as $t \rightarrow \infty$, ν_t converges weakly to*

$$\mu = \int_J X(\kappa(\theta), \cdot)_* \mathfrak{m} \, d\lambda_J(\theta).$$

Consequently, if $U \subset \mathbb{R}^d$ is open and $\mu(U) > 0$, there is τ such that for every $t > \tau$ some $\theta \in J$ has $\Gamma_\theta(t) \in U$.

Proof. Let $\Psi_t(\theta) = \Theta^0(\theta) + t\omega(\kappa(\theta))$ in $\mathbb{T}^2$, and let ρ_t be the law of $(\theta, \Psi_t(\theta))$ on the compact metric space $K = \bar{J} \times \mathbb{T}^2$. We show $\rho_t \rightarrow \lambda_J \otimes \mathfrak{m}$ weakly. The functions $f(\theta, \Psi) = b(\theta)e^{2\pi i k \cdot \Psi}$ with $b \in C(\bar{J})$ and $k \in \mathbb{Z}^2$ span an algebra which contains the constants, separates points and is closed under conjugation; by the Stone–Weierstrass theorem it is dense in $C(K)$, so it suffices to test such f . For $k = 0$, $\int f \, d\rho_t = \int b \, d\lambda_J$ for all t . For $k \neq 0$,

$$\int f \, d\rho_t = \frac{1}{|J|} \int_J b(\theta) e^{2\pi i k \cdot \Theta^0(\theta)} e^{2\pi i t \varphi_k(\theta)} \, d\theta \longrightarrow 0 = \int f \, d(\lambda_J \otimes \mathfrak{m})$$

by Lemma 2.1, the amplitude being bounded and Borel.

Let $G(\theta, \Psi) = X(\kappa(\theta), \Psi)$ on $J \times \mathbb{T}^2$, and $G = 0$ on $(\bar{J} \setminus J) \times \mathbb{T}^2$. Then G is Borel, $\nu_t = G_* \rho_t$, and G is continuous (as κ and X are) at every point of the open set $J \times \mathbb{T}^2$, whose complement is $\lambda_J \otimes \mathfrak{m}$ -null because $\bar{J} \setminus J$ is finite. By the continuous mapping theorem $\nu_t \rightarrow G_*(\lambda_J \otimes \mathfrak{m}) = \mu$. Finally, if U is open, the portmanteau theorem gives $\liminf_{t \rightarrow \infty} \nu_t(U) \geq \mu(U)$; if $\mu(U) > 0$ there is τ with $\nu_t(U) > 0$ for all $t > \tau$, and $\nu_t(U) > 0$ means that $\Gamma_\theta(t) \in U$ for a set of θ of positive measure. $\square$

The limit is taken along the real parameter t , so the conclusion holds for *all* $t > \tau$, as Definition 1.1 requires; this is where the Riemann–Lebesgue lemma for real $t \rightarrow \infty$ is used. We shall also use two continuity facts.

Lemma 2.3. *Let Y be a metric space.*

- (a) *Let $F : Y \times [\alpha, \beta] \rightarrow \mathbb{R}$ be continuous with $F(y, \cdot)$ strictly increasing for every y . Then $(y, s) \mapsto F(y, \cdot)^{-1}(s)$ is continuous on $\{(y, s) : F(y, \alpha) \leq s \leq F(y, \beta)\}$.*
- (b) *Let $\Lambda : Y \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be continuous, with every $\Lambda_y = \Lambda(y, \cdot)$ a homeomorphism. Then $(y, \Theta) \mapsto \Lambda_y^{-1}(\Theta)$ is continuous.*

Proof. In both cases let $(y_n, s_n) \rightarrow (y, s)$ and let x_n be the preimage of s_n under $F(y_n, \cdot)$, resp. Λ_{y_n} . The x_n lie in a compact set, and every limit point x^* satisfies $F(y, x^*) = s$, resp. $\Lambda_y(x^*) = s$, by continuity; so x^* is the unique preimage, and x_n converges to it. $\square$

3 The disk and the ball

3.1 The disk as a linear flow

Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk and, after a rotation, $P = p \in (0, 1)$. For $\beta \in (0, \pi)$ let $g_\beta(t)$ be the position at time t of the photon with initial direction $e^{i\beta}$; these photons are part of the front, so it suffices to show that $\{g_\beta(t) : 0 < \beta < \pi\}$ becomes dense. For $\rho \in (0, 1)$ put

$$\alpha = 2 \arccos \rho, \quad \ell = 2\sqrt{1 - \rho^2}, \quad h_\rho(f) = e^{-i f \alpha} (1 - f + f e^{i \alpha}), \quad Y(\rho, \Psi) = e^{2\pi i \Psi_1} h_\rho(\Psi_2),$$

$$\omega(\rho) = \left(\frac{\arccos \rho}{2\pi\sqrt{1-\rho^2}}, \frac{1}{2\sqrt{1-\rho^2}} \right). \quad (2)$$

Since $h_\rho(0) = h_\rho(1) = 1$, Y is a continuous function on $(0, 1) \times \mathbb{T}^2$.

Lemma 3.1. *For $\beta \in (0, \pi)$ let $\rho = p \sin \beta$. There is $\Psi^0(\beta) \in \mathbb{T}^2$, continuous in β , such that $g_\beta(t) = Y(\rho, \Psi^0(\beta) + t\omega(\rho))$ for all $t \geq 0$.*

Proof. The angular momentum $L = \text{Im}(\bar{z}\dot{z})$ is constant during free motion and is unchanged by a reflection at $|z| = 1$, since $\text{Im}(\bar{z}(\dot{z} - 2\text{Re}(\bar{z}\dot{z})z)) = \text{Im}(\bar{z}\dot{z})$ when $\bar{z}z = 1$. Initially $L = \text{Im}(pe^{i\beta}) = \rho$. The chord from $e^{i\psi}$ to $e^{i(\psi+\alpha)}$ has length $|e^{i\alpha} - 1| = \ell$ and, traversed at unit speed, angular momentum $\sin \alpha / \ell = \cos(\alpha/2) = \rho$; an inward unit vector at $e^{i\psi}$ is determined by its angular momentum, so the photon, once at $e^{i\psi}$, runs along this chord, is reflected at $e^{i(\psi+\alpha)}$, runs along the chord to $e^{i(\psi+2\alpha)}$, and so on. Going back from p in direction $-e^{i\beta}$, the photon meets the circle after the time $s_b = p \cos \beta + \sqrt{1 - p^2 \sin^2 \beta} > 0$, at the point $e^{i\psi_0} = p - s_b e^{i\beta}$; the forward time to the circle is $-p \cos \beta + \sqrt{1 - p^2 \sin^2 \beta} > 0$, and the two add up to ℓ , so $f_0 = s_b / \ell \in (0, 1)$. With $u = f_0 + t/\ell$, $n = \lfloor u \rfloor$ and $f = u - n$,

$$g_\beta(t) = (1 - f)e^{i(\psi_0 + n\alpha)} + fe^{i(\psi_0 + (n+1)\alpha)} = e^{i(\psi_0 + u\alpha)} h_\rho(f).$$

So $\Psi^0(\beta) = ((\psi_0 + f_0\alpha)/2\pi, f_0)$ works, since $t\omega(\rho) = (t\alpha/(2\pi\ell), t/\ell)$; it is continuous in β because s_b, f_0, α and $e^{i\psi_0}$ are. $\square$

Lemma 3.2 (twist). *For $k \in \mathbb{Z}^2 \setminus \{0\}$ the function*

$$g_k(\rho) = k \cdot \omega(\rho) = \frac{k_1 \arccos(\rho)/\pi + k_2}{2\sqrt{1-\rho^2}}$$

is real-analytic and not constant on $(-1, 1)$. Consequently, for $0 < p < 1$ the function $\varphi_k(\beta) = g_k(p \sin \beta)$ has only finitely many critical points in $(0, \pi)$.

Proof. Analyticity is clear, as $\sqrt{1-\rho^2} > 0$ on $(-1, 1)$. If $k_2 \neq 0$, the numerator tends to k_2 and the denominator to 0 as $\rho \rightarrow 1^-$, so $|g_k| \rightarrow \infty$. If $k_2 = 0$, then $k_1 \neq 0$, $g_k(0) = k_1/4$, and $g_k(\rho) \rightarrow k_1/(2\pi)$ as $\rho \rightarrow 1^-$, because $\arccos \rho = \arcsin \sqrt{1-\rho^2}$; and $1/4 \neq 1/(2\pi)$. So g_k is not constant, the zeros of g'_k are isolated in $(-1, 1)$ and finitely many lie in $[0, p]$. Now $\varphi'_k(\beta) = g'_k(p \sin \beta) p \cos \beta$ vanishes only at $\beta = \pi/2$ and where $p \sin \beta$ is one of these zeros, and each value of $\sin \beta$ is taken at most twice. $\square$

Lemma 3.3. *For $\rho \in (0, 1)$ the measure $Y(\rho, \cdot)_* \mathbf{m}$ has the density $(2\pi\sqrt{1-\rho^2}\sqrt{|z|^2-\rho^2})^{-1}$ on the open annulus $\{\rho < |z| < 1\}$ and is carried by its closure.*

Proof. Let Ψ have the law $\mathbf{m}$. Given Ψ_2 , the argument of $Y = e^{2\pi i \Psi_1} h_\rho(\Psi_2)$ is uniform; so $\arg Y$ is uniform and independent of $|Y| = |h_\rho(\Psi_2)|$. Since $\cos \alpha = 2\rho^2 - 1$,

$$|h_\rho(f)|^2 = (1 - f)^2 + f^2 + 2f(1 - f) \cos \alpha = \rho^2 + (1 - 2f)^2(1 - \rho^2).$$

With $s = |1 - 2\Psi_2|$, uniform on $(0, 1)$, we get $\Pr(|Y| \leq r) = \sqrt{(r^2 - \rho^2)/(1 - \rho^2)}$ for $\rho \leq r \leq 1$; differentiating and dividing by $2\pi r$ gives the density. $\square$

Theorem 3.4. *Let $0 < p < 1$. If β is uniform on $(0, \pi)$, the law of $g_\beta(t)$ converges weakly as $t \rightarrow \infty$ to $\mu_p = \frac{1}{\pi} \int_0^\pi Y(p \sin \beta, \cdot)_* \mathbf{m} d\beta$, and $\mu_p(U) > 0$ for every open set U meeting $\mathbb{D}$. In particular $W_t(p)$ becomes dense in $\mathbb{D}$.*

Proof. Lemma 3.1 puts $g_\beta(t)$ in the form (1) with $J = (0, \pi)$, $I = (0, 1)$, $\kappa(\beta) = p \sin \beta$, the analytic ω of (2), $X = Y$ and $\Theta^0 = \Psi^0$; by Lemma 3.2 the hypothesis of Proposition 2.2 holds. So the law converges weakly to μ_p . Let U be open with $U \cap \mathbb{D} \neq \emptyset$, and choose $z_0 \in U \cap \mathbb{D}$, $z_0 \neq 0$. For the β with $p \sin \beta < |z_0|$, a set of positive measure, z_0 lies in the annulus $\{p \sin \beta < |z| < 1\}$, so by Lemma 3.3 $Y(p \sin \beta, \cdot)_* \mathbf{m}(U) > 0$. Integrating, $\mu_p(U) > 0$, and Proposition 2.2 gives, for every open ball U meeting $\mathbb{D}$, a τ with $W_t(p) \cap U \neq \emptyset$ for all $t > \tau$. $\square$

3.2 Proof of Theorem 1.2

For $d = 2$, part (a) is Theorem 3.4. Part (b): from the centre every photon moves on a diameter and is at distance $\text{dist}(t, 2\mathbb{Z})$ from 0 at time t . For the times t with $\text{dist}(t, 2\mathbb{Z}) < 1/4$, which are unbounded, the front misses the ball of radius $1/4$ about $e_1/2$. For $d \geq 3$, part (a) is an immediate corollary of the case $d = 2$.

Corollary 3.5. *Let $d \geq 3$, $P \in B \setminus \{0\}$, $p = |P|$ and $e_1 = P/p$. Let $\zeta(x) = \langle x, e_1 \rangle + i|x - \langle x, e_1 \rangle e_1|$ and $\sigma(z) = \text{Re } z + i|\text{Im } z|$. Then $W_t(P) = \{x \in \mathbb{R}^d : \zeta(x) \in \sigma(W_t^{(2)}(p))\}$, where $W_t^{(2)}(p)$ is the front of p in the unit disk. Consequently $W_t(P)$ becomes dense in B .*

Proof. Every $v \in S^{d-1}$ can be written $v = \cos \beta e_1 + \sin \beta w$ with $\beta \in [0, \pi]$ and w a unit vector orthogonal to e_1 . The plane $\Pi_w = \text{span}(e_1, w)$ contains P and v , and it is invariant under the motion: a reflection at $x \in \partial B \cap \Pi_w$ is $v \mapsto v - 2\langle v, x \rangle x$. With $E_w(z) = (\text{Re } z)e_1 + (\text{Im } z)w$, the motion in $\Pi_w \cap B$ is the billiard in the unit disk, so $\gamma_v(t) = E_w(g_\beta(t))$, where g_β now denotes the planar photon from p with direction $e^{i\beta}$, $\beta \in [0, \pi]$. Since $\zeta(E_w(z)) = \sigma(z)$, the front lies in the stated set. Conversely, let $\zeta(x) = \sigma(g_\beta(t))$ with $\beta \in (-\pi, \pi]$. As $g_{-\beta}(t) = \overline{g_\beta(t)}$, we may take $\beta \in [0, \pi]$. Let $\hat{w}$ be the unit vector in the direction of $x - \langle x, e_1 \rangle e_1$ (any unit vector orthogonal to e_1 if this is 0); then $x = E_{\hat{w}}(g_\beta(t))$ if $\text{Im } g_\beta(t) \geq 0$ and $x = E_{-\hat{w}}(g_\beta(t))$ otherwise.

Let U be an open ball meeting B ; it contains a point $x_0 \in B \setminus \mathbb{R}e_1$. In the coordinates $x = s_1 e_1 + s_2 w$, $s_2 > 0$, the map ζ is $s_1 + i s_2$, so it is open on $\mathbb{R}^d \setminus \mathbb{R}e_1$, and $\zeta(U \cap B \setminus \mathbb{R}e_1)$ contains an open set V in the upper half of $\mathbb{D}$. By Theorem 3.4, for all large t some $g \in W_t^{(2)}(p)$ lies in V ; then $\sigma(g) = g = \zeta(x)$ for some $x \in U \cap B$, and $x \in W_t(P)$. $\square$

Remark 3.6. The limit μ_p is the limit of the normalised measure in the parameter β , not that of the normalised length measure of the front, which by [dVV24, Theorems 3.1 and 4.1] is $\Psi(|z|) dx dy / (\pi \arcsin p)$; both have a positive density on the disk minus the centre.

4 The ellipse

Throughout, $E = \{x^2/a^2 + y^2/b^2 < 1\}$ with $a > b > 0$, $c = \sqrt{a^2 - b^2}$, and $B = b/c$.

4.1 Elliptic coordinates

Put $z = x + iy = c \cosh w$, $w = \mu + i\nu$, so that $z = (c \cosh \mu \cos \nu, c \sinh \mu \sin \nu)$. This map is a conformal diffeomorphism of $\{\mu > 0\} \times \mathbb{R}/2\pi\mathbb{Z}$ onto $\mathbb{C} \setminus [F_1, F_2]$, and a local diffeomorphism wherever $\sinh w \neq 0$, with $|dz/dw|^2 = F := c^2(\sinh^2 \mu + \sin^2 \nu)$. The boundary ∂E is $\{\mu = \mu_0\}$, where $c \cosh \mu_0 = a$, $c \sinh \mu_0 = b$. In these coordinates the Euclidean metric is $F|dw|^2$, and unit-speed straight motion is the Hamiltonian flow of $H = (p_\mu^2 + p_\nu^2)/(2F)$ on $\{H = \frac{1}{2}\}$, with $p_\mu + ip_\nu = F(\dot{\mu} + i\dot{\nu})$; if the velocity is $e^{i\theta}$, then $p_\mu + ip_\nu = \overline{c \sinh w} e^{i\theta}$. On this level the Hamiltonian $\frac{1}{2}(p_\mu^2 + p_\nu^2) - \frac{1}{2}F = F(H - \frac{1}{2})$ has the same orbits, traversed in a time σ with $dt = F d\sigma$, and its equations separate:

$$\frac{d\mu}{d\sigma} = p_\mu, \quad \frac{dp_\mu}{d\sigma} = \frac{c^2}{2} \sinh 2\mu, \quad \frac{d\nu}{d\sigma} = p_\nu, \quad \frac{dp_\nu}{d\sigma} = \frac{c^2}{2} \sin 2\nu. \quad (3)$$

Hence $\kappa := p_\mu^2 - c^2 \sinh^2 \mu = c^2 \sin^2 \nu - p_\nu^2$ is constant during free motion. As the coordinates are conformal, ∂_μ is normal to ∂E , and the reflection is $p_\mu \mapsto -p_\mu$; so κ is a first integral of the billiard. This separation is classical, see [Tab05, Chapter 4]. For the line through $z = c \cosh w$ with direction $e^{i\theta}$,

$$\kappa = c^2 |\sinh w|^2 \cos^2(\theta - \theta^*) - c^2 \sinh^2 \mu, \quad \theta^* = \arg \sinh w. \quad (4)$$

For $z \in E$ this gives $-b^2 < \kappa \leq c^2$. As $p_\mu^2 \geq 0$ and $p_\nu^2 \geq 0$, there are two *regimes*: for $\kappa = -c^2\varepsilon^2$ with $0 < \varepsilon < B$ (rotational) the motion stays in $\{\sinh \mu \geq \varepsilon\}$, outside a confocal ellipse, and $p_\nu \neq 0$; for $\kappa = c^2\varepsilon^2$ with $0 < \varepsilon < 1$ (librational) it stays in $\{|\sin \nu| \geq \varepsilon\}$, between the branches of a confocal hyperbola, and $p_\mu \neq 0$. The value $\kappa = 0$ belongs to the lines through a focus and $\kappa = c^2$ to the minor axis.

4.2 Invariant tori and the linear flow

Rotational regime, $\kappa = -c^2\varepsilon^2$, $0 < \varepsilon < B$. Then $p_\mu^2 = c^2(\sinh^2 \mu - \varepsilon^2)$ and $p_\nu^2 = c^2(\sin^2 \nu + \varepsilon^2) > 0$; the sign of p_ν is constant. Let $U = \operatorname{arcosh}(B/\varepsilon)$ and define $u \in [-U, U]$ by $\sinh \mu = \varepsilon \cosh u$ and $\operatorname{sign} u = \operatorname{sign} p_\mu$, so that $p_\mu = c\varepsilon \sinh u$. A direct computation shows that (3) becomes

$$\frac{du}{d\sigma} = c\sqrt{1 + \varepsilon^2 \cosh^2 u}, \quad \frac{d\nu}{d\sigma} = \pm c\sqrt{\sin^2 \nu + \varepsilon^2}, \quad (5)$$

and the reflection at $\mu = \mu_0$ takes $u = U$ to $u = -U$. So for each sign of p_ν the states with this κ form a torus $(u, \nu) \in \mathbb{R}/2U\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ on which the motion in the time σ is a product of two circle flows.

Librational regime, $\kappa = c^2\varepsilon^2$, $0 < \varepsilon < 1$. Then $p_\mu^2 = c^2(\sinh^2 \mu + \varepsilon^2) > 0$ and $p_\nu^2 = c^2(\sin^2 \nu - \varepsilon^2)$. Using $c \cosh(-w) = c \cosh w$ we take $\nu \in [\nu_\varepsilon, \pi - \nu_\varepsilon]$, $\sin \nu_\varepsilon = \varepsilon$, and then $\mu \in [-\mu_0, \mu_0]$: on this band the map $w \mapsto c \cosh w$ is injective, a local diffeomorphism ($\sin \nu > 0$), and $\mu = \pm\mu_0$ are the two halves of ∂E . Let $q = \sqrt{1 - \varepsilon^2}$. Define $m \in \mathbb{R}/4\mu_0\mathbb{Z}$ by $\mu = m$ for $m \in [-\mu_0, \mu_0]$ ($p_\mu > 0$) and $\mu = 2\mu_0 - m$ for $m \in [\mu_0, 3\mu_0]$ ($p_\mu < 0$), and $\chi \in \mathbb{R}/2\pi\mathbb{Z}$ by $\cos \nu = q \sin \chi$ and $p_\nu = -cq \cos \chi$. Then (3) becomes

$$\frac{dm}{d\sigma} = c\sqrt{\sinh^2 \mu + \varepsilon^2}, \quad \frac{d\chi}{d\sigma} = c\sqrt{1 - q^2 \sin^2 \chi}, \quad (6)$$

so the states with this κ form one torus $(m, \chi) \in \mathbb{R}/4\mu_0\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ (the two bands of ν are identified by $w \sim -w$), again with a product flow.

In both regimes let P_μ, P_ν be the σ -periods of the two factors and T_μ, T_ν the integrals of $c^2 \sinh^2 \mu$ and $c^2 \sin^2 \nu$ over one period:

$$\text{rotational: } P_\mu = \frac{2}{c} \int_0^U \frac{du}{\sqrt{1 + \varepsilon^2 \cosh^2 u}}, \quad T_\mu = 2c \int_0^U \frac{\varepsilon^2 \cosh^2 u \, du}{\sqrt{1 + \varepsilon^2 \cosh^2 u}}, \quad (7)$$

$$P_\nu = \frac{1}{c} \int_0^{2\pi} \frac{d\nu}{\sqrt{\sin^2 \nu + \varepsilon^2}}, \quad T_\nu = c \int_0^{2\pi} \frac{\sin^2 \nu \, d\nu}{\sqrt{\sin^2 \nu + \varepsilon^2}};$$

$$\text{librational: } P_\mu = \frac{4}{c} \int_0^{\mu_0} \frac{d\mu}{\sqrt{\sinh^2 \mu + \varepsilon^2}}, \quad T_\mu = 4c \int_0^{\mu_0} \frac{\sinh^2 \mu \, d\mu}{\sqrt{\sinh^2 \mu + \varepsilon^2}}, \quad (8)$$

$$P_\nu = \frac{4}{c} \mathbf{K}(q), \quad T_\nu = 4c \mathbf{E}(q),$$

with the complete elliptic integrals $\mathbf{K}(q) = \int_0^{\pi/2} (1 - q^2 \sin^2 \chi)^{-1/2} d\chi$ and $\mathbf{E}(q) = \int_0^{\pi/2} (1 - q^2 \sin^2 \chi)^{1/2} d\chi$. All are real-analytic in ε on the open regime intervals. Let $\varphi_\mu, \varphi_\nu \in \mathbb{R}/\mathbb{Z}$ be the phases, the σ -times along the two factors divided by the periods, measured from fixed base points; then $d\varphi_\mu/d\sigma = 1/P_\mu$ and $d\varphi_\nu/d\sigma = 1/P_\nu$.

Lemma 4.1. *Fix κ in a regime and one of its tori. Let $\alpha_1(\varphi_\mu) = c^2 \sinh^2 \mu$ and $\alpha_2(\varphi_\nu) = c^2 \sin^2 \nu$ as functions of the phases, $A = T_\mu/P_\mu + T_\nu/P_\nu$, $q_1(\varphi_\mu) = \int_0^{\varphi_\mu} (P_\mu \alpha_1 - T_\mu)$, $q_2(\varphi_\nu) = \int_0^{\varphi_\nu} (P_\nu \alpha_2 - T_\nu)$, and*

$$\Lambda_\kappa(\varphi_\mu, \varphi_\nu) = \left(\varphi_\mu + \frac{q_1 + q_2}{AP_\mu}, \varphi_\nu + \frac{q_1 + q_2}{AP_\nu} \right).$$

Then Λ_κ is a C^1 diffeomorphism of $\mathbb{T}^2$ with Jacobian $F/A > 0$, and along the billiard motion $\Theta = \Lambda_\kappa(\varphi)$ satisfies $\Theta(t) = \Theta(0) + t\omega(\kappa)$, where

$$\omega(\kappa) = \left(\frac{1}{AP_\mu}, \frac{1}{AP_\nu} \right) = \left(\frac{1}{T_\mu + T_\nu P_\mu/P_\nu}, \frac{1}{T_\mu P_\nu/P_\mu + T_\nu} \right). \quad (9)$$

Proof. The integrands of q_1, q_2 are continuous and have mean 0 by the definition of T_μ, T_ν , so q_1, q_2 are C^1 functions on $\mathbb{R}/\mathbb{Z}$. Along the motion $\frac{d}{d\sigma}q_1 = \alpha_1 - T_\mu/P_\mu$ and $\frac{d}{d\sigma}q_2 = \alpha_2 - T_\nu/P_\nu$, while $\frac{dt}{d\sigma} = F = \alpha_1 + \alpha_2$; hence $t = A\sigma + q_1 + q_2 + \text{const}$, and $\Theta_\mu = \varphi_\mu + (q_1 + q_2)/(AP_\mu) = \text{const} + t/(AP_\mu)$, likewise for Θ_ν . The Jacobian matrix is $\begin{pmatrix} 1+a_1 & a_2 \\ b_1 & 1+b_2 \end{pmatrix}$ with $a_j = q'_j/(AP_\mu)$, $b_j = q'_j/(AP_\nu)$; since $a_1b_2 = a_2b_1$, its determinant is $1 + a_1 + b_2 = 1 + (\alpha_1 + \alpha_2 - A)/A = F/A$, which is positive because $\sinh \mu \geq \varepsilon$, resp. $\sin \nu \geq \varepsilon$, on the torus. So Λ_κ is a local diffeomorphism of $\mathbb{T}^2$ homotopic to the identity, hence a covering of degree one, hence a diffeomorphism. $\square$

Let $X_\kappa(\Theta) = c \cosh(\mu + i\nu)$, where (μ, ν) is the point of the torus with phases $\Lambda_\kappa^{-1}(\Theta)$. The image $X_\kappa(\mathbb{T}^2)$ is the closure of the open set

$$\mathcal{A}_\kappa = \{c \cosh(\mu + i\nu) : \sinh \mu > \varepsilon, \mu < \mu_0\} \quad \text{resp.} \quad \mathcal{A}_\kappa = \{c \cosh(\mu + i\nu) : |\mu| < \mu_0, \nu_\varepsilon < \nu < \pi - \nu_\varepsilon\}, \quad (10)$$

the region between ∂E and the confocal ellipse, resp. between the branches of the confocal hyperbola.

Lemma 4.2. *On each regime interval I and for each choice of torus, the map $(\kappa, \Theta) \mapsto X_\kappa(\Theta)$ is continuous on $I \times \mathbb{T}^2$.*

Proof. Consider the rotational μ -factor. With $u = U(\varepsilon)x$, $x \in [-1, 1]$, the phase is $\varphi_\mu = G(\varepsilon, x) = \frac{U(\varepsilon)}{P_\mu(\varepsilon)} \int_0^x \frac{dx'}{c\sqrt{1+\varepsilon^2 \cosh^2(U(\varepsilon)x')}} (base point $u = 0$); G is continuous on $(0, B) \times [-1, 1]$, strictly increasing in x , with $G(\varepsilon, \pm 1) = \pm \frac{1}{2}$. By Lemma 2.3(a), x is a continuous function of $(\varepsilon, \varphi_\mu) \in (0, B) \times [-\frac{1}{2}, \frac{1}{2}]$, and so is $\mu = \text{arsinh}(\varepsilon \cosh(U(\varepsilon)x))$, which takes the value μ_0 at both ends; so μ is continuous on $(0, B) \times \mathbb{R}/\mathbb{Z}$. The same argument applies to $\pm\nu$ (the sign of p_ν) in the rotational regime and to m and χ in the librational regime, on the fixed intervals $[0, 2\pi]$, $[-\mu_0, 3\mu_0]$ and $[0, 2\pi]$, since each of them moves with a velocity that is positive and continuous in ε and in the variable, by (5) and (6). Hence α_1, α_2 are continuous in (ε, φ) , the periods and the integrals T_μ, T_ν are continuous in ε , q_1, q_2 and Λ are continuous on $I \times \mathbb{T}^2$, and so is $(\kappa, \Theta) \mapsto \Lambda_\kappa^{-1}(\Theta)$ by Lemma 2.3(b). Composing with the continuous maps from the phases to (μ, ν) and with $c \cosh$ gives the claim. $\square$$

4.3 The twist condition

Lemma 4.3. *For every $k \in \mathbb{Z}^2 \setminus \{0\}$, $k \cdot \omega$ is real-analytic and not constant on the rotational interval $-b^2 < \kappa < 0$.*

Proof. Analyticity follows from (7) and (9). The end $\varepsilon \rightarrow B$ ($\kappa \rightarrow -b^2$): $U \rightarrow 0$, so $P_\mu, T_\mu \rightarrow 0$, and T_μ/P_μ , a weighted mean of $c^2\varepsilon^2 \cosh^2 u$ over $[0, U]$, tends to $c^2B^2 = b^2$, while P_ν, T_ν tend to their positive values at $\varepsilon = B$. So $\omega_\mu \rightarrow \infty$, and $\omega_\nu \rightarrow 1/(b^2P_\nu + T_\nu) = 1/\int_0^{2\pi} \sqrt{b^2 + c^2 \sin^2 \nu} d\nu = 1/|\partial E|$, the reciprocal of the perimeter, because $b^2 + c^2 \sin^2 \nu = a^2 \sin^2 \nu + b^2 \cos^2 \nu$.

The end $\varepsilon \rightarrow 0$ ($\kappa \rightarrow 0$); let $L = \log(1/\varepsilon)$. With $s = \varepsilon \cosh u$, $P_\mu = \frac{2}{c} \int_\varepsilon^B \frac{ds}{\sqrt{s^2 - \varepsilon^2} \sqrt{1+s^2}}$; since $0 \leq 1 - (1+s^2)^{-1/2} \leq Bs/2$ on $[0, B]$ and $\int_\varepsilon^B s ds / \sqrt{s^2 - \varepsilon^2} \leq B$, we get $P_\mu = \frac{2}{c} \text{arcosh}(B/\varepsilon) + O(1) = \frac{2}{c}L + O(1)$. Next, $P_\nu = \frac{4}{c} \int_0^{\pi/2} (\sin^2 \nu + \varepsilon^2)^{-1/2} d\nu$, and with $S = \sin \nu \geq 2\nu/\pi$,

$$0 \leq \frac{1}{\sqrt{S^2 + \varepsilon^2}} - \frac{1}{\sqrt{\nu^2 + \varepsilon^2}} \leq \frac{\nu^2 - S^2}{S\nu(S + \nu)} = \frac{\nu - S}{S\nu} \leq \frac{\pi\nu}{12},$$

while $\int_0^{\pi/2} (\nu^2 + \varepsilon^2)^{-1/2} d\nu = \operatorname{arsinh}(\pi/2\varepsilon) = L + O(1)$; so $P_\nu = \frac{4}{c}L + O(1)$ and $P_\nu/P_\mu \rightarrow 2$. Substituting $s = \sqrt{\varepsilon^2 + r^2}$ gives $T_\mu = 2c \int_0^{\sqrt{B^2 - \varepsilon^2}} s dr / \sqrt{1 + s^2}$ with a bounded integrand, so $T_\mu \rightarrow 2c(\sqrt{1 + B^2} - 1) = 2(a - c)$; and $T_\nu \rightarrow c \int_0^{2\pi} |\sin \nu| d\nu = 4c$ by dominated convergence. By (9), $\omega \rightarrow (1/(2(a - c) + 2c), 1/(4(a - c) + 4c)) = (1/(2a), 1/(4a))$.

If $k_1 \neq 0$, then $|k \cdot \omega| \rightarrow \infty$ at the first end and has a finite limit at the second. If $k_1 = 0$, the limits are $k_2/|\partial E|$ and $k_2/(4a)$, and they differ because $|\partial E| = \int_0^{2\pi} \sqrt{a^2 \sin^2 \nu + b^2 \cos^2 \nu} d\nu > \int_0^{2\pi} a |\sin \nu| d\nu = 4a$. $\square$

Lemma 4.4. *For every $k \in \mathbb{Z}^2 \setminus \{0\}$, $k \cdot \omega$ is real-analytic and not constant on the librational interval $0 < \kappa < c^2$.*

Proof. Analyticity follows from (8) and (9). Let $\varepsilon \rightarrow 0$ and $L = \log(1/\varepsilon)$.

(i) With $S = \sinh \mu \geq \mu$, the difference $(S^2 + \varepsilon^2)^{-1/2} - (\mu^2 + \varepsilon^2)^{-1/2}$ lies between 0 and $-(S - \mu)/(S\mu)$, so it is bounded by a constant times μ on $[0, \mu_0]$, and it tends to $1/\sinh \mu - 1/\mu$. By dominated convergence, with $\int_0^{\mu_0} (\mu^2 + \varepsilon^2)^{-1/2} d\mu = \log(2\mu_0/\varepsilon) + o(1)$ and $\int_0^{\mu_0} (1/\sinh \mu - 1/\mu) d\mu = \log \tanh(\mu_0/2) - \log(\mu_0/2)$,

$$P_\mu = \frac{4}{c}(\log(4/\varepsilon) + \log \tanh(\mu_0/2)) + o(1).$$

(ii) In the same way, writing $\mathbf{K}(q) = \int_0^{\pi/2} (\sin^2 \chi + \varepsilon^2 \cos^2 \chi)^{-1/2} d\chi$ and comparing with $\int_0^{\pi/2} (\chi^2 + \varepsilon^2)^{-1/2} d\chi = \log(\pi/\varepsilon) + o(1)$ (the difference of the integrands is bounded by $\varepsilon + (\chi - \sin \chi)/(\chi \sin \chi)$ and tends to $1/\sin \chi - 1/\chi$, whose integral is $\log(4/\pi)$), we get the classical $\mathbf{K}(q) = \log(4/\varepsilon) + o(1)$. Hence $P_\nu = \frac{4}{c}L + O(1)$ and $P_\mu - P_\nu \rightarrow D_0 := \frac{4}{c} \log \tanh(\mu_0/2) < 0$.

(iii) The integrand of T_μ differs from $\sinh \mu$ by at most $\min(\varepsilon, \varepsilon^2/(2 \sinh \mu))$, and $\sqrt{\cos^2 \chi + \varepsilon^2 \sin^2 \chi}$ differs from $\cos \chi$ by at most $\min(\varepsilon, \varepsilon^2/(2 \cos \chi))$; so $T_\mu = 4(a - c) + O(\varepsilon^2 L)$ and $T_\nu = 4c \mathbf{E}(q) = 4c + O(\varepsilon^2 L)$.

(iv) Hence $r = P_\mu/P_\nu = 1 + cD_0/(4L) + o(1/L)$, and (9) gives $\omega_\mu = 1/(T_\mu + T_\nu r) = 1/(4a + c^2 D_0/L + o(1/L))$ and $\omega_\nu = 1/(T_\mu/r + T_\nu) = 1/(4a - c(a - c)D_0/L + o(1/L))$, so

$$k \cdot \omega = \frac{k_1 + k_2}{4a} + \frac{cD_0}{16a^2 L} (k_2(a - c) - k_1 c) + o(1/L). \quad (11)$$

If $k_2(a - c) \neq k_1 c$, the coefficient of $1/L$ is not 0, and $k \cdot \omega$ is not constant.

(v) Otherwise $k = \lambda(a - c, c)$ with $\lambda \neq 0$, and $k \cdot \omega \rightarrow \lambda/4$ as $\varepsilon \rightarrow 0$. At the other end, $\varepsilon \rightarrow 1$ ($\kappa \rightarrow c^2$, $q \rightarrow 0$), with $G = \arctan(b/c) = \int_0^{\mu_0} d\mu / \cosh \mu$: $P_\nu \rightarrow 2\pi/c$, $T_\nu \rightarrow 2\pi c$, $P_\mu \rightarrow 4G/c$ and $T_\mu \rightarrow 4c \int_0^{\mu_0} (\cosh \mu - 1/\cosh \mu) d\mu = 4(b - cG)$, so $A \rightarrow cb/G$, $\omega \rightarrow (1/(4b), G/(2\pi b))$ and $k \cdot \omega \rightarrow \lambda((a - c)/(4b) + cG/(2\pi b))$. Equal limits would mean $b + c - a = (2c/\pi) \arctan(b/c)$. This equation is homogeneous; with $a = 1$, $c = \sin \phi$, $b = \cos \phi$, $0 < \phi < \pi/2$, so that $\arctan(b/c) = \pi/2 - \phi$, it reads $\cos \phi - 1 + (2\phi/\pi) \sin \phi = 0$, and after division by $2 \sin(\phi/2) \cos(\phi/2)$, $\tan(\phi/2) = (4/\pi)(\phi/2)$. This is impossible: $\tan$ is strictly convex on $[0, \pi/4]$ with $\tan 0 = 0$ and $\tan(\pi/4) = 1$, so $\tan x < 4x/\pi$ for $0 < x < \pi/4$. $\square$

4.4 Proof of Theorem 1.3(a)

Let $P \in E$ be not a focus, $P = c \cosh w_P$ with $w_P = \mu_P + i\nu_P$ and $\mu_P \geq 0$, and $s_P = \sinh w_P \neq 0$, $\theta^* = \arg s_P$. By (4) the photon with direction θ has $\kappa(\theta) = c^2 |s_P|^2 \cos^2(\theta - \theta^*) - c^2 \sinh^2 \mu_P$, a non-constant trigonometric polynomial.

Case 1: $\mu_P > 0$, i.e. $P \notin [F_1, F_2]$. The set $\{\kappa < 0\}$ is non-empty ($\kappa(\theta^* + \pi/2) = -c^2 \sinh^2 \mu_P$) and consists of open arcs on which κ takes values in $[-c^2 \sinh^2 \mu_P, 0) \subset (-b^2, 0)$ and tends to 0 at the

endpoints; there $p_\nu^2 = c^2 \sin^2 \nu_P - \kappa > 0$. Let J be one of these arcs; p_ν has constant sign on J , so all these photons move on tori of the same kind (rotational, one sign of p_ν).

Case 2: $\mu_P = 0$, i.e. $P = (c \cos \nu_P, 0)$ with $0 < \nu_P < \pi$. Then $\kappa(\theta) = c^2 \sin^2 \nu_P \sin^2 \theta$; let J be the component of $\{0 < \kappa < c^2\}$ which has 0 as an endpoint (this set excludes $\theta = \pm\pi/2$ only when P is the centre). All these photons are librational, and $\kappa \rightarrow 0$ as $\theta \rightarrow 0^+$.

In both cases, for $\theta \in J$ let $\Theta^0(\theta) = \Lambda_{\kappa(\theta)}(\varphi^0(\theta))$, where $\varphi^0(\theta)$ are the phases of the initial state. Then by Lemma 4.1 the photon is $\gamma_\theta(t) = X_{\kappa(\theta)}(\Theta^0(\theta) + t\omega(\kappa(\theta)))$, which is the form (1), with X continuous by Lemma 4.2. The map Θ^0 is continuous on J , hence Borel: the initial values of the uniformising variables are continuous in θ (in Case 1, $u_0 = \text{sign}(p_\mu) \text{arcosh}(\sinh \mu_P/\varepsilon)$, continuous also where $p_\mu = 0$, and $\nu_0 = \nu_P$; in Case 2, m_0 is μ_P or $2\mu_0 - \mu_P$ according to the constant sign of p_μ , and $(\sin \chi_0, \cos \chi_0) = (\cos \nu_P/q, -p_\nu/(cq))$), and the phases and Λ depend continuously on them and on κ (proof of Lemma 4.2). For $k \neq 0$ the derivative of $\varphi_k = k \cdot \omega \circ \kappa$ vanishes only where $\kappa' = 0$ (finitely many θ) or where $\kappa(\theta)$ is a zero of $(k \cdot \omega)'$; by Lemmas 4.3 and 4.4 these zeros form a discrete subset of the regime interval, and each level set of κ is finite, so the critical set of φ_k is countable. By Proposition 2.2 the law of $\gamma_\theta(t)$, θ uniform on J , converges weakly to $\mu_P = \frac{1}{|J|} \int_J (X_{\kappa(\theta)})_* m d\theta$.

Let U be open with $U \cap E \neq \emptyset$, and choose $z_0 \in U \cap E$ off the major axis. In Case 1 write $z_0 = c \cosh(\mu_1 + i\nu_1)$ with $0 < \mu_1 < \mu_0$; for the $\theta \in J$ with $\kappa(\theta) > -c^2 \sinh^2 \mu_1$, a non-empty open set, $z_0 \in \mathcal{A}_{\kappa(\theta)}$ by (10). In Case 2 write $z_0 = c \cosh(\mu_1 + i\nu_1)$ with $|\mu_1| < \mu_0$ and $0 < \nu_1 < \pi$; for the $\theta \in J$ near 0, $\kappa(\theta) < c^2 \sin^2 \nu_1$, so again $z_0 \in \mathcal{A}_{\kappa(\theta)}$. For each such θ the set $X_{\kappa(\theta)}^{-1}(U)$ is open and non-empty, so it has positive Haar measure; integrating, $\mu_P(U) > 0$. By Proposition 2.2, $W_t(P) \cap U \neq \emptyset$ for all large t . $\square$

4.5 The foci

Proposition 4.5. *For every $n \geq 1$, $W_{2an}(F_2) = \{F_1\}$ if n is odd and $\{F_2\}$ if n is even. Hence $W_t(F_2)$ does not become dense; the same holds for F_1 .*

Proof. At $X \in \partial E$ the normal bisects the angle between XF_1 and XF_2 , because the gradient of $|X - F_1| + |X - F_2|$ is the sum of the two unit vectors from the foci. So a photon leaving F_2 is reflected at its first boundary point X into the segment XF_1 , and passes F_1 (an interior point, where nothing happens) at time $|F_2X| + |XF_1| = 2a$. The photons along the major axis pass F_1 at time $2a$ as well: $(a - c) + (a + c)$ and $(a + c) + (a - c)$. Repeating the argument from F_1 gives the statement for all n . For instance the ball of radius $b/4$ about $(0, b/2)$ contains no focus, so it misses the front at all the times $2an$. $\square$

5 A remark on non-convex tables

This section makes Remark 1.4 explicit. The mechanism is that of Penrose's room [PP58], as described in [CG03, Section 1, Fig. 1]: an elliptic mirror reflects every ray that passes between the foci into a ray that passes between the foci. Nothing here is claimed as new.

Let E be the ellipse with $a = 2$, $b = \sqrt{3}$, so $c = 1$ and $F_{1,2} = (\mp 1, 0)$. Let $\psi(s) = e^{-2/s}$ for $s > 0$ and $\psi(s) = 0$ for $s \leq 0$; ψ is C^∞ , increasing, and convex on $(-\infty, 1]$. For $\eta > 0$ the set $Q_\eta = \{u, w \geq 0 : \psi(1 - u/\eta) + \psi(1 - w/\eta) \leq \psi(1)\}$ is convex, coincides with the quadrant $Q = \{u, w \geq 0\}$ outside $[0, \eta]^2$, and has a C^∞ boundary: inside the square it is a level curve of a function with non-zero gradient, which joins the two rays of ∂Q to infinite order because ψ is flat at 0.

Remove from E the two teeth $\{|x \mp 1| \leq \frac{1}{4}, y \leq 0\}$, after replacing each of their four upper corners by $Q_{1/8}$ (in coordinates u, w along the top and the side, pointing into the tooth); round the four corners where the sides of the teeth meet ∂E in the same way, in the local coordinates

$u = \pm(x - x_{\text{corner}})$, $w = 1 - x^2/4 - y^2/3$, inside disks of radius 0.1. The result is a bounded simply connected domain D with C^∞ boundary. Its upper part $D \cap \{y > 0\}$ is the upper half E^+ of E ; on the axis, $D \cap \{y = 0\} = I_L \cup I_0 \cup I_R$ with $I_0 = (-\frac{7}{8}, \frac{7}{8})$ and $I_R = -I_L = (\frac{9}{8}, 2)$, and the flat tops $[\frac{7}{8}, \frac{9}{8}] \times \{0\}$ and $[-\frac{9}{8}, -\frac{7}{8}] \times \{0\}$, which contain the foci, lie in ∂D . The lower part $D \cap \{y < 0\}$ has three components: R_1 between the teeth, whose closure meets the axis in $[-\frac{7}{8}, \frac{7}{8}]$, and Q_L, Q_R outside them, whose closures meet it in $[-2, -\frac{9}{8}]$ and $[\frac{9}{8}, 2]$. The disk V of radius 0.1 about $(1.6, -0.5)$ lies in Q_R , since $1.7^2/4 + 0.6^2/3 = 0.8425 < 1$.

Proposition 5.1. *For every $P \in R_1$ and every $t \geq 0$, $W_t(P) \cap V = \emptyset$. So the front of no point of the open set R_1 becomes dense in D .*

Proof. Focal property. Let a line ℓ meet ∂E at X and let ℓ' be its mirror image in the normal at X . If the inward directions of ℓ, ℓ', XF_1, XF_2 make the angles $\phi, -\phi, \phi_1, -\phi_1$ with the inner normal (which bisects the angle F_1XF_2 , see the proof of Proposition 4.5), then ℓ , and likewise ℓ' , strictly separates F_1 from F_2 if and only if $|\phi| < \phi_1$.

Let $s_0, s_1, \dots$ be the straight segments of a trajectory from P (grazing contacts are disregarded as in [KK26, Section 3]; continuing them straight would not change the argument). We show by induction: (i) s_j does not meet $Q_L \cup Q_R \cup I_L \cup I_R$; (ii) s_j does not lie on the axis, and if s_j meets E^+ , its line meets the axis in a point of $(-1, 1)$. Near the endpoints $(\pm\frac{7}{8}, 0)$ and $(\pm\frac{9}{8}, 0)$ of the flat tops the boundary is flat to infinite order, while a straight segment off the axis is not; so these endpoints cannot be reached from $\{y < 0\}$ without entering a tooth.

A segment that starts in R_1 stays in R_1 until it reaches the axis, which it can do only at a point of I_0 ; it then crosses into E^+ and does not return to the axis. This gives (i) and (ii) for s_0 , and for every s_{j+1} that starts at the end of s_j on $\partial D \cap \{y < 0\}$, since by (i) this point lies on the boundary of R_1 and s_{j+1} starts into R_1 . If s_j ends on the arc $\partial E \cap \{y > 0\}$, then s_j meets E^+ ; by (ii) its line separates the foci, and by the focal property so does the line of s_{j+1} , which therefore meets the axis in one point $x^* \in (-1, 1)$. Starting in E^+ , s_{j+1} either ends on the arc, or reaches the axis at x^* , where it crosses into R_1 if $|x^*| < \frac{7}{8}$ and ends on a flat top otherwise. If s_j ends at a point X_j of a flat top, endpoints included, it arrives from E^+ : below the open flat tops lie the teeth, the endpoints are not reached from below, and s_j is not on the axis. Its line meets the axis at X_j , so $|X_j| < 1$ by (ii); the reflection is the mirror image in the axis, and s_{j+1} runs in E^+ on a line through X_j . In all cases (i) and (ii) hold for s_{j+1} . Hence no segment meets $Q_R \supset V$. $\square$

6 Computations and reproducibility

No theorem, proposition, lemma or corollary of this note depends on a computation. The computations test the formulas against simulations of the billiard which do not use them, and they illustrate the theorems at finite times, which cannot prove density. The reproducibility package has five parts.

The program of this note, check_note.py (standard library only; it runs in about one second), checks: Lemma 3.1 and Corollary 3.5 against bounce-by-bounce simulations in the disk and the 3-ball (largest errors about 10^{-12}); the limits in Lemma 3.2; the identity of Lemma 3.3, exactly on a rational grid; that κ is unchanged by reflections; the frequencies (9) against the simulated rates of reflections and of sign changes of x in 8 trajectories of two ellipses, both regimes (relative agreement better than $3 \cdot 10^{-4}$); the limits in Lemmas 4.3 and 4.4 ($\omega_\nu = 1/|\partial E|$ to 6 digits at the end $\kappa \rightarrow -b^2$; the slow, logarithmic approach to $(1/(2a), 1/(4a))$; $P_\mu - P_\nu = D_0$ to 6 digits at $\varepsilon = 10^{-6}$; the values at $\kappa \rightarrow c^2$ to 7 digits); Proposition 4.5 exactly, in the ellipse $(a, b, c) = (5, 4, 3)$, for 1,428 rational directions; and the arithmetic of Section 5 exactly. All checks pass.

The programs with which the results were first obtained, with NumPy, SciPy, SymPy and mpmath, are in `original/`; *the programs of the two verification runs* are in `verification_run_A/` and

`verification_run_B/`, and the folder `independent_run_2/` holds *those of a further verification run on the final text* (see the next paragraph); all come with their recorded outputs. Among the recorded results, as illustration only: in the disk with $4 \cdot 10^5$ photons, the fraction of the cells of side 0.05 met by the front at $t = 10, 200, 5000$ is 0.17, 0.95, 1.000 for $p = 0.5$, 0.007, 0.36, 1.000 for $p = 0.1$, and 0.0009 at all times for $p = 0$; the radius of the largest ball missed by 10^6 sampled front points, maximised over 12 times in $[T, T + 20]$, drops to the sampling floor (≈ 0.009) by $T = 10^5$ for $p \geq 0.1$ and stays above 0.5 for the centre. The time $\tau(P, r)$ of Definition 1.1 can be large for P near the centre, where the caustic radii $p \sin \beta \leq p$ vary little: the ball of radius 0.03 about $(0.3, 0.2)$ was missed by $2 \cdot 10^5$ photons at 5 of 4,000 times in $[2000, 2020]$ for $p = 0.1$, and met at all of them for $p = 0.5$. In the ellipse $a = 2$, $b = \sqrt{3}$, the fraction of cells of side 0.1 met at $t = 20, 100, 400$ is 0.31, 0.82, 0.99 for $P = (0.3, 0.9)$; from a focus it stays below 0.005, and at $t = 2a, 4a, 6a$ all photons are within 10^{-12} of a focus. The length of the disk front divided by t near $t = 1600$ is 0.6074, 1.0473, 2.2427 for $p = 0.3, 0.5, 0.9$, against $2 \arcsin p = 0.6094, 1.0472, 2.2395$.

Verification. The results were first obtained with written proofs and the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each re-derived the proofs and wrote its own programs, and neither read the other's report. The first compared the question with [ABB⁺26] and [KK26] (LaTeX sources), read [dVV24, dV21], and re-derived all proofs, including (11), the limits at the ends of the regimes and the case analysis of Section 5; its programs reproduced Lemma 3.1 to $4 \cdot 10^{-12}$, the frequencies to 4–5 digits, all endpoint values and the behaviour at the foci, and found no photon reaching V in the table of Section 5. It required seven corrections; the mathematical ones were a proof of the continuity of $(\kappa, \Theta) \mapsto X_\kappa(\Theta)$ and of Θ^0 , which the first version only asserted (now Lemma 4.2 and Section 4.4), and two cases in the proof of Proposition 5.1 (no segment lies on the axis; the endpoints of the flat tops). The second run checked the statement and the literature, found [CG03] and in it the reports of [PP58, Rau78], re-derived the proofs, and checked with its own programs the disk model, the largest empty ball of the front up to $t = 10^6$ in the disk and $t = 10^5$ in the 3-ball, the frequencies in 36 trajectories of three ellipses to 10^{-5} , fronts in four ellipses, and the table of Section 5 with $2.8 \cdot 10^5$ photons and $4.9 \cdot 10^7$ reflections, none reaching V ; deliberately wrong inputs were rejected by its tests. It re-ran the programs of `original/` with identical outputs, confirmed the factor 2 in the length formula, and required four corrections of the text (the citation of [CG03], the reports of Penrose and Rauch, the ball as a corollary of the disk, and the numbering of results cited from [dV21, KK26]). All corrections required by the two runs were made.

A further independent verification run, also AI-assisted, examined the text of this note after it was written, without relying on the earlier checks. It compared the question, the definition and every cited statement with the LaTeX sources of [ABB⁺26, KK26, dVV24, dV21] and the publisher's file of [CG03], re-derived all proofs line by line, checked the bibliographic data on Crossref and repeated the literature search. Its own programs (`independent_run_2/`) compare Lemma 3.1 with a 40-digit simulation up to $t = 3000$ (agreement $3 \cdot 10^{-33}$) and Corollary 3.5 with simulations in balls of dimension 3 to 5; follow the Kolmogorov distance between the radial law of the disk front and that of μ_p up to $t = 10^5$; check Section 4.1 symbolically; test Lemma 4.1 directly, by computing the phases Θ from the states of 9 simulated trajectories in three ellipses, which satisfy $\Theta(t) - \Theta(0) - t\omega(\kappa) \in \mathbb{Z}^2$ to $3 \cdot 10^{-11}$ for $t \leq 1500$; compute the limits of Lemmas 4.3 and 4.4 and the coefficient of $1/L$ in (11) with up to 160 digits, for ε down to 10^{-60} ; confirm Proposition 4.5 with 50 digits; and simulate the table of Section 5 ($2.9 \cdot 10^6$ segments from sources in R_1 : none meets V , and the statements (i) and (ii) of the proof of Proposition 5.1 hold for every segment). It also ran the whole package again from the archive: the outputs of `check_note.py`, of the seven programs of `original/` and of three programs of the second run are byte-identical to the recorded ones, and the others differ only in the reported running times (one of them also in a warning of SciPy, which the re-run wrote to the error stream), except the

four simulations of the table of Section 5 by the second run, whose random seeds were not recorded; with the default seed they give other counts and the same results. This run found no mathematical error and required eight corrections, all made: the continuity of κ was added to the hypotheses of Proposition 2.2; the abstract now names the limit measure correctly (Remark 3.6); the paragraph after Theorem 1.3 now says for which directions the proofs give weak convergence; the description of [dV21] and a digit count in this section were corrected; the labels printed by `check_note.py` now follow the numbering of the note; and this run is recorded here, in the verification report and in the package.

Scope and priority. The note answers Tabachnikov’s question for balls (Theorem 1.2) and for ellipses (Theorem 1.3), with density in the sense of Definition 1.1. For general convex or generic smooth domains, the setting of [KK26, Conjectures 3–5], the question remains open, and so does its analogue for Riemannian manifolds mentioned in [ABB⁺26]. The question is Tabachnikov’s [ABB⁺26]; the definition and the conjectures are due to Kang and Knill [KK26]. The disk case of Theorem 1.2(a) follows from [dVV24, Theorem 4.1] of Colin de Verdière and Vicente, whose proof is a sketch and who do not state the corollary; the ball case reduces to the disk by symmetry. The method, a linear flow on invariant tori with a twist condition and oscillatory integrals, is that of [dVV24, dV21]; the twist of the circular and elliptic billiards is also behind the results of Chernov and Galperin [CG03] on illumination by a narrow beam, which do not concern the front at each time. The integrability of the elliptic billiard is classical [Tab05, Chapter 4]. The example of Section 5 is known in substance [PP58, Rau78, CG03, KK26]. What may be new is the ellipse case, Theorem 1.3(a), and a complete written proof in the case of the disk and the ball: apart from the consequence of [dVV24, Theorem 4.1] for the disk, no printed answer to the question was found in the sources searched (named below); a search that finds nothing is not a proof of novelty, and we make no priority claim.

What was read: the paragraph of [ABB⁺26], and [KK26] (arXiv version 4), in their LaTeX sources; [dVV24] in the LaTeX source of arXiv version 1 (the journal version was not available to us, so we do not know whether its Theorem 4.1 is in print); the relevant parts of [dV21] (Sections 1, 5 and 7); [CG03] in the publisher’s file, Sections 1 and 2. Of [Tab05] only the record of Chapter 4 was checked; [PP58, Rau78] were not seen. The searches, on 18 October 2026, used the arXiv (wave fronts with density, billiards, equidistribution, caustics and illumination; recent work of Knill, Colin de Verdière, Vicente and Tabachnikov), Crossref, zbMATH, OpenAlex and Semantic Scholar with the lists of works citing [KK26], [dVV24], [dV21] and [ABB⁺26] (apart from these papers themselves, there were none), six web searches, and a blog post of Knill of 20 December 2025 on polygons; the further verification run repeated the arXiv, Crossref, zbMATH and OpenAlex queries and made one of the web searches.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [ABB⁺26] Bernhard Albach, Jean-François Barraud, Misha Bialy, Johanna Bimmermann, Ana Chávez Cáliz, Mihai Damian, Lina Deschamps, Umberto Hryniewicz, Vincent Humilière, Boris Khesin, Levin Maier, Agustin Moreno, Alexandru Oancea, Olga Paris-Romaskevich, Alfonso Sorrentino, and Serge Tabachnikov. Open problems in billiards and quantitative symplectic geometry, 2026. Version 1; contribution of S. Tabachnikov, “Density of reflected wave fronts”. [arXiv:2602.12896](https://arxiv.org/abs/2602.12896).

- [Bil99] Patrick Billingsley. *Convergence of Probability Measures*. Wiley Series in Probability and Statistics. Wiley, New York, second edition, 1999. doi:10.1002/9780470316962.
- [CG03] N. I. Chernov and G. Galperin. Search light in billiard tables. *Regular and Chaotic Dynamics*, 8(2):225–241, 2003. doi:10.1070/RD2003v008n02ABEH000239.
- [dV21] Yves Colin de Verdière. Large time asymptotics of the wave fronts length II: surfaces with integrable Hamiltonians, 2021. arXiv:2101.01919.
- [dVV24] Yves Colin de Verdière and David Vicente. Large-time series expansion of the wave front length in the Euclidean disk. *Experimental Mathematics*, 33(2):254–260, 2024. Published online 9 May 2022; theorem numbers refer to arXiv version 1, “Large-time asymptotics of the wave fronts length I: the Euclidean disk”. arXiv:2011.14886, doi:10.1080/10586458.2022.2063208.
- [KK26] Emily Kang and Oliver Knill. Density of wave fronts. *The Mathematical Intelligencer*, 2026. Published online 10 June 2026; numbers refer to arXiv version 4. arXiv:2501.14611, doi:10.1007/s00283-026-10528-z.
- [PP58] L. Penrose and R. Penrose. Puzzles for Christmas. *New Scientist*, pages 1580–1581, 1597, 1958. Not seen by us; cited after [CG03].
- [Rau78] Jeffrey Rauch. Illumination of bounded domains. *The American Mathematical Monthly*, 85(5):359–361, 1978. Not seen by us; cited after [CG03]. doi:10.1080/00029890.1978.11994593.
- [Tab05] Serge Tabachnikov. *Geometry and Billiards*, volume 30 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 2005. doi:10.1090/stml/030.