

Degree Sequences of Rational Maps Need Not Be Piecewise Linear Recurrent

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Abstract

Benedetto, Ingram, Jones, Manes, Silverman and Tucker asked whether, for every dominant rational map f of $\mathbb{P}^N$, there are finitely many linear recursions with constant coefficients $L^{(1)}, \dots, L^{(r)}$ and an index function $\iota: \mathbb{N} \rightarrow \{1, \dots, r\}$ such that $\deg(f^n) = L_n^{(\iota(n))}$ for all $n \geq 1$ (Bull. Amer. Math. Soc. 56 (2019); Question 14.6 of arXiv:1806.04980). We read the question, as its notation suggests, as asking for finitely many fixed sequences, each satisfying a linear recursion with constant coefficients, with coefficients and initial values in any field of characteristic zero, and an arbitrary index function; the different reading in which the recursions are applied to the degree sequence itself, with coefficients that change with n , is not addressed. Under our reading the answer is no. We show that if a sequence of positive integers a_n with $a_n^{1/n} \rightarrow \delta$ agrees on the fibres of an index function with finitely many such sequences, then $\delta = |\theta|$ for an algebraic number θ . The proof specializes the data to algebraic numbers and then uses the Skolem–Mahler–Lech theorem and Evertse’s lower bound for sums of S -units, a consequence of the subspace theorem, in the form used by Bell, Diller and Jonsson. Bell, Diller and Jonsson (Acta Math. 2020) proved that an explicit dominant rational map of $\mathbb{P}^2$, defined over $\mathbb{Z}$, has a transcendental dynamical degree; hence its degree sequence has no such description. For $N \geq 3$ the same follows from the birational examples of Bell, Diller, Jonsson and Krieger. The deduction is short, and its novelty is modest. We also show that degree sequences of monomial maps are of the form in the question, which explains why the example of Hasselblatt and Propp mentioned in the source does not answer it. This is an unrefereed note.

1 Introduction

1.1 The question

Let K be a field. Following [BIJ⁺19, §14], a rational map $f: \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ over K is given by homogeneous polynomials $f_0, \dots, f_N \in K[x_0, \dots, x_N]$ of the same degree without a common factor of positive degree; this common degree is $\deg f$, and f is *dominant* if $f_0, \dots, f_N$ satisfy no non-trivial polynomial relation. The iterate f^n is again such a map, obtained by composing and removing common factors. If f is dominant, then so is f^n , and $\deg(f^n) \geq 1$, since a map of degree 0 is constant. Because $\deg(f^{m+n}) \leq \deg(f^m) \deg(f^n)$, the *dynamical degree*

$$\delta_f := \lim_{n \rightarrow \infty} \deg(f^n)^{1/n}$$

exists [BIJ⁺19, Definition 14.2 and the paragraph after it].

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The integers $\deg(f^n)$ need not satisfy a linear recursion with constant coefficients: for every $N \geq 3$ there are rational maps of $\mathbb{P}^N$ whose degree sequences satisfy none [BIJ⁺19, Theorem 14.4], a result of Hasselblatt–Propp and Bedford–Kim [HP07, BK08]. An example is the monomial map $[YW, ZW, X^2, XW]$ of $\mathbb{P}^3$ with the degree sequence $2, 3, 4, 6, 9, 12, 17, 25, \dots$; the source remarks that this construction “does not provide a negative answer to the following question” [BIJ⁺19, Example 14.5]. That question, [BIJ⁺19, Question 14.6], asks whether for a dominant rational map $f: \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ one can find finitely many linear recursions with constant coefficients $L_n^{(1)}, \dots, L_n^{(r)}$ and an index function $\iota: \mathbb{N} \rightarrow \{1, \dots, r\}$ such that $\deg(f^n) = L_n^{(\iota(n))}$ for every $n \geq 1$. The source states no expected answer.

In both arXiv versions of [BIJ⁺19] the question is Question 14.6. In version 2 it is followed by Remark 14.7 (degree sequences of birational maps of surfaces [DF01] and of polynomial maps of the affine plane [FJ11] satisfy a linear recursion) and by the integrality conjecture of Bellon and Viallet, Conjecture 14.8; version 1 has no such remark and numbers the conjecture 14.7. The number in the published version could not be checked, because the publisher’s web page could not be accessed; it is most likely Question 13.6, since later papers cite items of the same section of the published version under numbers 13.x. The question is record AMR-126-0025 of the UnsolvedMath collection [ula26], listed as open.

The reading. Let F be a field. A sequence $(L_n)_{n \geq 1}$ in F is a *linear recurrence sequence* (LRS) if there are $k \geq 0$ and $c_1, \dots, c_k \in F$ with

$$L_{n+k} = c_1 L_{n+k-1} + \dots + c_k L_n \quad \text{for all } n \geq 1 \quad (1)$$

($k = 0$ means $L_n = 0$ for all n). If F has characteristic zero, we call a sequence $(a_n)_{n \geq 1}$ of integers *piecewise linear recurrent over F* if there are $r \geq 1$, LRS $L^{(1)}, \dots, L^{(r)}$ in F and a map $\iota: \mathbb{N} \rightarrow \{1, \dots, r\}$ with $a_n = L_n^{(\iota(n))}$ for all $n \geq 1$, the equality being taken in $F \supseteq \mathbb{Z}$. We read [BIJ⁺19, Question 14.6] as asking whether the degree sequence of every dominant rational map is piecewise linear recurrent over some field of characteristic zero; the source does not say where the constants lie, and $F = \mathbb{C}$ contains every choice of integer, rational, algebraic, real or complex constants. The index function is arbitrary, as in the source. This is what the notation of the source says ($L_n^{(i)}$ is the n -th term of a sequence), and it is the reading under which the remark on the example of Hasselblatt and Propp is correct (Proposition 1.4). Over a field of characteristic $p > 0$ the question would be trivial, since $\deg(f^n) \bmod p$ takes at most p values.

We do not address a different reading, in which the recursions are applied to the degree sequence itself: $\deg(f^n) = \sum_{j=1}^k c_j^{(\iota(n))} \deg(f^{n-j})$ for finitely many coefficient vectors $c^{(1)}, \dots, c^{(r)}$ chosen by an index function. For that reading a transcendental growth rate is no obstruction: if $\rho \in (0, 1)$ is algebraic and irrational, the integers $a_n = 2^{n - \lfloor n\rho \rfloor} 3^{\lfloor n\rho \rfloor}$ satisfy $a_n \in \{2a_{n-1}, 3a_{n-1}\}$ for all $n \geq 1$, and $\lim a_n^{1/n} = 2(3/2)^\rho$ is transcendental by the theorem of Gelfond and Schneider [Sch35].

1.2 Results

Theorem 1.1. *Let $(a_n)_{n \geq 1}$ be a sequence of positive integers such that $\delta := \lim_{n \rightarrow \infty} a_n^{1/n}$ exists in $\mathbb{R}$. If (a_n) is piecewise linear recurrent over a field of characteristic zero, then $\delta = |\theta|$ for an algebraic number $\theta \in \overline{\mathbb{Q}} \subset \mathbb{C}$. In particular, δ is algebraic.*

The theorem applies to every degree sequence $a_n = \deg(f^n)$. To apply it, we use the maps of Bell, Diller and Jonsson [BDJ20]. Let $g: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the quadratic involution

$$g[x_0 : x_1 : x_2] = [x_0(x_1 + x_2 - x_0) : x_1(x_2 + x_0 - x_1) : x_2(x_0 + x_1 - x_2)], \quad (2)$$

and for a Gaussian integer $\zeta = a + bi$ let h_ζ be the monomial map given in the affine coordinates $(y_1, y_2) = (x_1/x_0, x_2/x_0)$ by $h_\zeta(y_1, y_2) = (y_1^a y_2^b, y_1^{-b} y_2^a)$. Bell, Diller and Jonsson proved that if

$\zeta^n \notin \mathbb{R}$ for all $n \geq 1$, then $f = g \circ h_\zeta$ has a transcendental dynamical degree, over every field of characteristic different from 2 [BDJ20, Main Theorem and Introduction]. For $\zeta = 1 + 2i$ one finds $h_\zeta = [x_0^3 x_1^2 : x_1^3 x_2^2 : x_0^4 x_2]$ and $f = [f_0 : f_1 : f_2]$ with

$$\begin{aligned} f_0 &= x_0^7 x_1^2 x_2 - x_0^6 x_1^4 + x_0^3 x_1^5 x_2^2, \\ f_1 &= x_0^4 x_1^3 x_2^3 + x_0^3 x_1^5 x_2^2 - x_1^6 x_2^4, \\ f_2 &= -x_0^8 x_2^2 + x_0^7 x_1^2 x_2 + x_0^4 x_1^3 x_2^3. \end{aligned} \tag{3}$$

Theorem 1.2. *Let K be a field of characteristic different from 2, and let $f: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the rational map over K given by (3). Then f is dominant, $\deg f = 10$, and the degree sequence $(\deg(f^n))_{n \geq 1}$ is not piecewise linear recurrent over any field of characteristic zero. Hence the answer to [BIJ⁺19, Question 14.6], in the reading of Section 1.1, is negative.*

The dynamical degree of this map is $\delta_f = 6.8575574092 \dots$ [BDJ20, Introduction], and $\deg(f^n) = 10, 66, 454, 3114$ for $n = 1, \dots, 4$ (Section 7).

Corollary 1.3. *For every $N \geq 2$ there is a dominant rational map of $\mathbb{P}^N$ over $\mathbb{Q}$, birational if $N \geq 3$, whose degree sequence is not piecewise linear recurrent over any field of characteristic zero. For $N = 1$ the degree sequence of every dominant rational map is $(d^n)_{n \geq 1}$ with $d = \deg f$, a single LRS.*

For a matrix $A \in M_N(\mathbb{Z})$ with $\det A \neq 0$, let h_A be the monomial map of $\mathbb{P}^N$ given in the affine coordinates $y_i = x_i/x_0$ by $h_A(y)_j = \prod_{i=1}^N y_i^{A_{ji}}$ for $j = 1, \dots, N$; thus $h_\zeta = h_A$ with $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

Proposition 1.4. *Let $A \in M_N(\mathbb{Z})$ with $\det A \neq 0$. There are $r \leq (N+1)^{N+1}$ LRS $L^{(1)}, \dots, L^{(r)}$ with values in $\mathbb{Z}$, all satisfying the recursion given by the characteristic polynomial of A , such that $\deg(h_A^n) = \max_{1 \leq i \leq r} L_n^{(i)}$ for all $n \geq 1$. In particular, the degree sequence of h_A is piecewise linear recurrent over $\mathbb{Q}$.*

1.3 What is known and what is new

The question is due to Benedetto, Ingram, Jones, Manes, Silverman and Tucker [BIJ⁺19]. Earlier results exclude a *single* linear recursion for particular maps: monomial maps of Hasselblatt–Propp and Bedford–Kim [HP07, BK08] and, for degrees in higher codimension, of Lin–Wulcan [LW14]; and the maps $g \circ h_\zeta$ of Bell, Diller and Jonsson, for which Nguyen proved that the generating series $\sum_n \deg(f^n) z^n$ has a natural boundary [Ngu26, Corollary B], so that it is not a rational function. These results do not decide [BIJ⁺19, Question 14.6]: by Proposition 1.4 the degree sequences of monomial maps are piecewise linear recurrent, and the generating series of the degree sequence of h_{1+2i} already has a natural boundary [Ngu26, Theorem D].

The essential input of this note is the theorem of Bell, Diller and Jonsson that the map of Theorem 1.2 has a transcendental dynamical degree [BDJ20], which disproved the integrality conjecture, and its extension to birational maps of $\mathbb{P}^N$, $N \geq 3$, by Bell, Diller, Jonsson and Krieger [BDJK24]. The second input is the growth of non-degenerate linear recurrence sequences of algebraic numbers: such a sequence u_t whose characteristic roots of largest modulus have modulus M satisfies $|u_t| \geq M^t e^{-\varepsilon t}$ for all large t (Lemma 3.4). Lower bounds of this kind rest on the subspace theorem; we derive the one we need from Evertse’s inequality for sums of S -units [Eve84] (see also [vdPS82]), in the form stated by Bell, Diller and Jonsson [BDJ20, Theorem 3.5]. The case of two complex conjugate roots and constant coefficients is [BDJK24, Corollary 4.2], derived there from the same inequality. What this note adds is Theorem 1.1, whose proof combines these tools with a specialization argument, the resulting negative answer (Theorem 1.2 and Corollary 1.3), and Proposition 1.4. The deduction is short, specialists may regard it as known, and we claim no priority. We did not find it in the sources we searched; see the paragraph “Scope and priority” at the end.

2 Linear recurrence sequences

Throughout, $\mathbb{N} = \{1, 2, 3, \dots\}$, $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and $|\cdot|$ is the absolute value of $\mathbb{C}$. LRS are defined by (1).

Lemma 2.1 (specialization). *Let F be a field of characteristic zero, let $(L_n)_{n \geq 1}$ be an LRS in F , and let $S \subseteq \mathbb{N}$ be a set such that $L_n \in \mathbb{Z}$ for all $n \in S$. Then there is an LRS $(L'_n)_{n \geq 1}$ whose values and recursion coefficients lie in $\overline{\mathbb{Q}}$, such that $L'_n = L_n$ for all $n \in S$.*

Proof. Let $c_1, \dots, c_k$ be the coefficients in (1), and let $A := \mathbb{Q}[c_1, \dots, c_k, L_1, \dots, L_k] \subseteq F$ be the $\mathbb{Q}$ -subalgebra generated by the coefficients and the initial values. By (1) and induction on n , every L_n lies in A . The ring A is a non-zero finitely generated $\mathbb{Q}$ -algebra; let $\mathfrak{m}$ be a maximal ideal of A . The field $E := A/\mathfrak{m}$ is a finitely generated $\mathbb{Q}$ -algebra, so it is a finite extension of $\mathbb{Q}$ by Zariski's lemma [AM69, Proposition 7.9]. Choose a field embedding $\sigma: E \rightarrow \overline{\mathbb{Q}}$, and let $\varphi: A \rightarrow \overline{\mathbb{Q}}$ be the composition of $A \rightarrow E$ with σ . It is a ring homomorphism with $\varphi(1) = 1$, hence the identity on $\mathbb{Z}$. Applying φ to (1) shows that $L'_n := \varphi(L_n)$ is an LRS with the coefficients $\varphi(c_j) \in \overline{\mathbb{Q}}$. For $n \in S$ we have $L_n = a \cdot 1$ with $a \in \mathbb{Z}$, hence $L'_n = \varphi(a \cdot 1) = a = L_n$. $\square$

No Noetherian argument is needed: the infinitely many relations $L_n = a \cdot 1$, $n \in S$, are equalities in the ring A , and φ preserves all of them at once. This is the only place in the proof of Theorem 1.1 where the characteristic of F matters.

Lemma 2.2 (structure). *Let $(u_n)_{n \geq 1}$ be an LRS whose values and recursion coefficients lie in $\overline{\mathbb{Q}}$. Then either $u_n = 0$ for all large n , or there are $n_0 \geq 1$, $m \geq 1$, pairwise distinct $\theta_1, \dots, \theta_m \in \overline{\mathbb{Q}} \setminus \{0\}$ and non-zero polynomials $g_1, \dots, g_m \in \overline{\mathbb{Q}}[X]$ such that*

$$u_n = \sum_{i=1}^m g_i(n) \theta_i^n \quad \text{for all } n \geq n_0. \quad (4)$$

Proof. Let $c_1, \dots, c_k \in \overline{\mathbb{Q}}$ be the coefficients in (1). If all c_j vanish (this includes $k = 0$), then $u_n = 0$ for $n \geq k + 1$. Otherwise let $k' := \max\{j : c_j \neq 0\}$, $n_0 := 1 + k - k'$ and $w_h := u_{n_0+h}$ for $h \geq 0$. Writing $n + k = n_0 + h$ in (1), we get $w_h = c_1 w_{h-1} + \dots + c_{k'} w_{h-k'}$ for all $h \geq k'$, and $c_{k'} \neq 0$. Let $X^{k'} - c_1 X^{k'-1} - \dots - c_{k'} = \prod_{i=1}^{m'} (X - \theta_i)^{e_i}$ with pairwise distinct θ_i ; they are non-zero because $c_{k'} \neq 0$, and algebraic because the c_j are. By the structure theorem for linear recurrence sequences [Eve23, Theorem 8.17] (stated there for the recursion of minimal length, whose characteristic polynomial divides ours) there are polynomials $\tilde{g}_i \in \mathbb{C}[X]$ of degree less than e_i with $w_h = \sum_i \tilde{g}_i(h) \theta_i^h$ for all $h \geq 0$. Conversely, each sequence $h \mapsto h^j \theta_i^h$ with $j < e_i$ satisfies the recursion of (w_h) , since $(E - \theta_i)^{e_i}$ annihilates it, E being the shift operator. The solutions of the recursion form a vector space of dimension $k' = \sum_i e_i$, parametrized by $(w_0, \dots, w_{k'-1})$; hence the linear map from the coefficient vectors of the $\tilde{g}_i$ to $(w_0, \dots, w_{k'-1})$ is surjective between spaces of dimension k' , thus bijective. Its matrix has the entries $h^j \theta_i^h \in \overline{\mathbb{Q}}$, and $w_0, \dots, w_{k'-1} \in \overline{\mathbb{Q}}$; so the coefficients of the $\tilde{g}_i$ lie in $\overline{\mathbb{Q}}$. If all $\tilde{g}_i$ vanish, then $u_n = 0$ for $n \geq n_0$. Otherwise drop the indices with $\tilde{g}_i = 0$, renumber, and put $g_i(X) := \theta_i^{-n_0} \tilde{g}_i(X - n_0)$; then $u_n = w_{n-n_0} = \sum_i g_i(n) \theta_i^n$ for $n \geq n_0$. $\square$

Lemma 2.3 (residue classes). *Let u be as in (4). There is an integer $q \geq 1$ with the following property. For each $s \in \{0, \dots, q-1\}$, either $u_{qt+s} = 0$ for all $t \geq n_0$, or there are a non-empty finite set Λ_s , pairwise distinct $\beta_l \in \overline{\mathbb{Q}} \setminus \{0\}$ ($l \in \Lambda_s$) no quotient of two of which is a root of unity, and non-zero polynomials $h_{l,s} \in \overline{\mathbb{Q}}[X]$, such that*

$$u_{qt+s} = \sum_{l \in \Lambda_s} h_{l,s}(t) \beta_l^t \quad \text{for all } t \geq n_0; \quad (5)$$

moreover, each β_l equals θ_i^q for some i .

Proof. Only finitely many quotients θ_i/θ_j with $i \neq j$ are roots of unity; let q be the least common multiple of their orders ($q = 1$ if there are none). Then $\theta_i^q = \theta_j^q$ whenever θ_i/θ_j is a root of unity. Let $\beta_1, \dots, \beta_p$ be the distinct values among $\theta_1^q, \dots, \theta_m^q$. For $t \geq n_0$ we have $qt + s \geq n_0$, and (4) gives

$$u_{qt+s} = \sum_{l=1}^p h_{l,s}(t) \beta_l^t, \quad h_{l,s}(t) := \sum_{i: \theta_i^q = \beta_l} \theta_i^s g_i(qt + s) \in \overline{\mathbb{Q}}[t].$$

If $\beta_l/\beta_{l'}$ were a root of unity for some $l \neq l'$, choose i, i' with $\theta_i^q = \beta_l$ and $\theta_{i'}^q = \beta_{l'}$; then $(\theta_i/\theta_{i'})^q$, hence $\theta_i/\theta_{i'}$, is a root of unity, so $\beta_l = \theta_i^q = \theta_{i'}^q = \beta_{l'}$, a contradiction. Let Λ_s be the set of l with $h_{l,s} \neq 0$. If it is empty, $u_{qt+s} = 0$ for $t \geq n_0$; otherwise (5) holds. $\square$

3 Growth of non-degenerate power sums

Let K be a number field of degree d and $M(K)$ its set of places. As in [BDJ20, §3.2], a finite place v , given by a prime ideal $\mathfrak{p}$ of the ring of integers $\mathcal{O}_K$, has $|x|_v = N(\mathfrak{p})^{-\text{ord}_{\mathfrak{p}}(x)}$ for $x \neq 0$; a real place given by $\tau: K \rightarrow \mathbb{R}$ has $|x|_v = |\tau(x)|$; and a complex place given by a pair $\tau, \bar{\tau}: K \rightarrow \mathbb{C}$ has $|x|_v = |\tau(x)|^2$. Then $\prod_{v \in M(K)} |x|_v = 1$ for $x \in K \setminus \{0\}$. For a finite set $S \subset M(K)$ containing all infinite places, $\mathcal{O}_{K,S} := \{x \in K : |x|_v \leq 1 \text{ for all } v \notin S\}$ is the ring of S -integers, its units are the S -units, and for $\mathbf{x} = (x_1, \dots, x_p) \in \mathcal{O}_{K,S}^p$ we put $H_S(\mathbf{x}) := \prod_{v \in S} \max_l |x_l|_v$.

Theorem 3.1 (Evertse; [BDJ20, Theorem 3.5]). *Let $S \subset M(K)$ be a finite set of places containing all infinite places, let $p \geq 2$ and $\varepsilon > 0$. There is a constant $c = c(K, S, p, \varepsilon) > 0$ such that for every $\mathbf{x} = (x_1, \dots, x_p) \in \mathcal{O}_{K,S}^p$ with $\sum_{l \in I} x_l \neq 0$ for every non-empty $I \subseteq \{1, \dots, p\}$, and for every $v_0 \in S$,*

$$|x_1 + \dots + x_p|_{v_0} \geq c \frac{\max_l |x_l|_{v_0}}{H_S(\mathbf{x})^\varepsilon \prod_{v \in S} \prod_{l=1}^p |x_l|_v}.$$

Remark 3.2 (provenance). We quote Theorem 3.1 from [BDJ20, Theorem 3.5], where it is attributed to Evertse [Eve84] and to [EG15, Proposition 6.2.1]. The same statement, with products over an arbitrary non-empty subset $T \subseteq S$ in place of v_0 on both sides, is [BDJK24, Theorem 4.1], quoted there from [EG15, Proposition 6.2.1]; Theorem 3.1 is its case $T = \{v_0\}$. We did not read [EG15]. In [Eve84, Theorem 2] the inequality is stated, with the same normalization of the absolute values, for vectors $\mathbf{x} \in \mathcal{O}_K^p$ and every non-empty $T \subseteq S$, in the form

$$\prod_{v \in T} |x_1 + \dots + x_p|_v \geq C \frac{\prod_{v \in T} \max_l |x_l|_v}{\|\mathbf{x}\|^\varepsilon \prod_{v \in S} \prod_{l=1}^p |x_l|_v},$$

where $\|\mathbf{x}\|$ is the maximum of $|\sigma(x_l)|$ over the coordinates x_l and the embeddings $\sigma: K \rightarrow \mathbb{C}$, and $C = C(\varepsilon, S, K, p)$. Theorem 3.1 follows from this form. Both sides of the inequality of Theorem 3.1 are unchanged when $\mathbf{x}$ is multiplied by an S -unit s of K : the left side and $\max_l |x_l|_{v_0}$ are multiplied by $|s|_{v_0}$, while $H_S(\mathbf{x})$ and $\prod_{v \in S} \prod_l |x_l|_v$ do not change, because $\prod_{v \in S} |s|_v = 1$ by the product formula. By Dirichlet's S -unit theorem the vectors $(\log |s|_v)_{v \in S}$ form a lattice of full rank in the hyperplane $\sum_{v \in S} u_v = 0$ of $\mathbb{R}^S$; let ρ be its covering radius for the maximum norm, let r be the number of infinite places and S_f the set of finite places in S . Given $\mathbf{x} \in \mathcal{O}_{K,S}^p$ with no vanishing subsum, put $m_v := \log \max_l |x_l|_v$ for $v \in S$, so that $\sum_{v \in S} m_v = \log H_S(\mathbf{x})$, and choose s with $|\log |s|_v - p_v| \leq \rho$ for all $v \in S$, where $p_v := -m_v - \rho$ for $v \in S_f$ and $p_v := (\log H_S(\mathbf{x}) + |S_f| \rho)/r - m_v$ for infinite v ; the point $(p_v)_{v \in S}$ lies in the hyperplane. Then $\max_l |sx_l|_v \leq 1$ for every finite place v , so $s\mathbf{x} \in \mathcal{O}_K^p$, and $\max_l |sx_l|_v \leq C_0 H_S(\mathbf{x})^{1/r}$ for every infinite v , with $C_0 := e^{\rho(1+|S_f|/r)}$; hence $\|s\mathbf{x}\| \leq C_0 H_S(\mathbf{x})^{1/r}$, because $H_S(s\mathbf{x}) = H_S(\mathbf{x}) \geq 1$ for a non-zero integral vector. Applying the inequality of [Eve84] to $s\mathbf{x}$ with $T = \{v_0\}$ and dividing by $|s|_{v_0}$ gives Theorem 3.1 with ε/r in place of ε and $c = C C_0^{-\varepsilon}$.

Theorem 3.3 (Skolem–Mahler–Lech). *Let $\beta_1, \dots, \beta_p \in \mathbb{C} \setminus \{0\}$ be such that no quotient $\beta_l/\beta_{l'}$ with $l \neq l'$ is a root of unity, and let $h_1, \dots, h_p \in \mathbb{C}[X]$ be polynomials, not all zero. Then $\sum_{l=1}^p h_l(t)\beta_l^t = 0$ for only finitely many integers $t \geq 1$.*

This is the theorem of Skolem, Mahler and Lech [Lec53] in the form of [Eve23, Theorem 8.18].

Lemma 3.4 (growth). *Let $p \geq 1$, let $\beta_1, \dots, \beta_p \in \overline{\mathbb{Q}} \setminus \{0\}$ be pairwise distinct, no quotient of two of them being a root of unity, let $h_1, \dots, h_p \in \overline{\mathbb{Q}}[X]$ be non-zero, and let $v_t := \sum_{l=1}^p h_l(t)\beta_l^t$ for integers $t \geq 0$. Put $M := \max_l |\beta_l|$ and $D := \max_l \deg h_l$. Then:*

- (a) *there is $C > 0$ with $|v_t| \leq C(t+1)^D M^t$ for all $t \geq 0$;*
- (b) *for every $\varepsilon' > 0$ we have $|v_t| \geq M^t e^{-\varepsilon' t}$ for all large t .*

In particular, $|v_t|^{1/t} \rightarrow M$ as $t \rightarrow \infty$.

Proof. Part (a) follows from the triangle inequality. For (b), let l^* be an index with $|\beta_{l^*}| = M$; since $h_{l^*} \neq 0$, there are $c_1 > 0$ and t_3 with $|h_{l^*}(t)| \geq c_1$ for $t \geq t_3$. If $p = 1$, then $|v_t| \geq c_1 M^t$ for $t \geq t_3$, and (b) follows. Let $p \geq 2$. Choose a number field $K \subset \mathbb{C}$ containing all β_l and all coefficients $c_{l,j}$ of the h_l , let $d := [K : \mathbb{Q}]$, let v_0 be the place given by the inclusion $K \subset \mathbb{C}$, and let $e_0 := 1$ if v_0 is real and $e_0 := 2$ otherwise, so that $|x|_{v_0} = |x|^{e_0}$. Let S consist of the infinite places and of the finite places at which some β_l has non-zero order or some $c_{l,j}$ has negative order. Then every β_l is an S -unit, every $c_{l,j}$ is an S -integer, and $x_l(t) := h_l(t)\beta_l^t \in \mathcal{O}_{K,S}$ for all integers $t \geq 0$.

By Theorem 3.3, applied to each of the finitely many non-empty $I \subseteq \{1, \dots, p\}$, there is t_2 such that $\sum_{l \in I} x_l(t) \neq 0$ for all such I and all $t \geq t_2$. For a finite $v \in S$ we have $|h_l(t)|_v \leq \max_j |c_{l,j}|_v$, by the ultrametric inequality and $|t|_v \leq 1$. For an infinite place v , given by τ , we have $|h_l(t)|_v \leq (\sum_j |\tau(c_{l,j})|(t+1)^D)^{e_v}$ with $e_v \in \{1, 2\}$ and $\sum_{v \in S} e_v = d$. Hence there is $C_1 \geq 1$, independent of t , with $\prod_{v \in S} \max_l |h_l(t)|_v \leq C_1(t+1)^{Dd}$ for all $t \geq 0$. Since β_l is an S -unit, the product formula gives $\prod_{v \in S} |\beta_l|_v = \prod_{v \in M(K)} |\beta_l|_v = 1$. Therefore, with $T := \prod_{v \in S} \max_l |\beta_l|_v \geq \prod_{v \in S} |\beta_1|_v = 1$,

$$\prod_{v \in S} \prod_{l=1}^p |x_l(t)|_v = \prod_{l=1}^p \prod_{v \in S} |h_l(t)|_v \leq C_1^p (t+1)^{pDd}, \quad H_S(\mathbf{x}(t)) \leq C_1(t+1)^{Dd} T^t,$$

and $\max_l |x_l(t)|_{v_0} \geq |x_{l^*}(t)|_{v_0} \geq c_1^{e_0} M^{e_0 t}$ for $t \geq t_3$. Fix $\varepsilon \in (0, 1]$. For $t \geq \max(t_2, t_3)$, Theorem 3.1 gives

$$|v_t|^{e_0} = |x_1(t) + \dots + x_p(t)|_{v_0} \geq c c_1^{e_0} M^{e_0 t} C_1^{-\varepsilon-p} (t+1)^{-(\varepsilon+p)Dd} T^{-\varepsilon t} \geq c_2 M^{e_0 t} (t+1)^{-2pDd} T^{-\varepsilon t},$$

with $c_2 := c c_1^{e_0} C_1^{-1-p}$. Taking e_0 -th roots and using $t+1 \geq 1$, $T \geq 1$, we get

$$|v_t| \geq c_2^{1/e_0} M^t (t+1)^{-2pDd} T^{-\varepsilon t}.$$

Given $\varepsilon' > 0$, choose $\varepsilon \in (0, 1]$ with $\varepsilon \log T \leq \varepsilon'/2$; then $c_2^{1/e_0} (t+1)^{-2pDd} \geq e^{-\varepsilon' t/2}$ for all large t , and (b) follows. The last assertion follows from (a) and (b). $\square$

The proof uses that the β_l and the coefficients of the h_l are algebraic, since Theorem 3.1 concerns a number field; this is why the data are first specialized by Lemma 2.1. For two complex conjugate roots and constant polynomials h_l , (b) is [BDJK24, Corollary 4.2].

4 Proof of Theorem 1.1

Let $a_n = L_n^{(\iota(n))}$ for all $n \geq 1$, where $L^{(1)}, \dots, L^{(r)}$ are LRS in a field F of characteristic zero.

Step 1. The sets $\iota^{-1}(i)$, $1 \leq i \leq r$, cover $\mathbb{N}$, so one of them, say $S := \iota^{-1}(i)$, is infinite. Put $L := L^{(i)}$; then $L_n = a_n \in \mathbb{Z}$ for $n \in S$. Nothing else about ι is used.

Step 2. By Lemma 2.1 there is an LRS u with values and recursion coefficients in $\overline{\mathbb{Q}}$ such that $u_n = a_n$ for all $n \in S$.

Step 3. If $u_n = 0$ for all large n , then $a_n = 0$ for the infinitely many large $n \in S$, which is impossible. So by Lemma 2.2 u has the form (4), and we take q as in Lemma 2.3. Choose $s \in \{0, \dots, q-1\}$ such that the set $T_s := \{t \geq n_0 : qt + s \in S\}$ is infinite. The sequence $v_t := u_{qt+s}$, $t \geq n_0$, is not identically zero, since $v_t = a_{qt+s} \geq 1$ for $t \in T_s$. By Lemma 2.3, v_t has the form (5) with $\Lambda_s \neq \emptyset$.

Step 4. By Lemma 3.4, $|v_t|^{1/t} \rightarrow M_s := \max_{l \in \Lambda_s} |\beta_l|$ as $t \rightarrow \infty$, and $M_s = |\theta_i|^q$ for some i by Lemma 2.3. On the other hand, for $t \in T_s$,

$$|v_t|^{1/t} = a_{qt+s}^{1/t} = (a_{qt+s}^{1/(qt+s)})^{(qt+s)/t} \longrightarrow \delta^q \quad (t \rightarrow \infty, t \in T_s),$$

because $a_n^{1/n} \rightarrow \delta$, $\delta \geq 1$ and $(qt+s)/t \rightarrow q$. Comparing the two limits along the infinite set T_s gives $\delta^q = |\theta_i|^q$, hence $\delta = |\theta_i|$. Finally $\theta_i \in \overline{\mathbb{Q}}$ and its complex conjugate $\bar{\theta}_i \in \overline{\mathbb{Q}}$, so $\delta^2 = \theta_i \bar{\theta}_i \in \overline{\mathbb{Q}}$, and δ is algebraic. $\square$

Remark 4.1. (1) The proof gives more: δ is the modulus of a characteristic root of the specialization of one of the recursions. It does not show that δ is an algebraic integer, since the set S is arbitrary. (2) Recursions with a constant term, $L_{n+k} = c_0 + \sum_j c_j L_{n+k-j}$, are covered: such a sequence satisfies the homogeneous recursion with characteristic polynomial $(X-1)(X^k - c_1 X^{k-1} - \dots - c_k)$. So are recursions that hold only for $n \geq n'$: they are recursions of order $k+n'-1$ whose last $n'-1$ coefficients vanish, which the definition (1) and Lemma 2.2 allow.

5 The map of Bell, Diller and Jonsson

Lemma 5.1. *Let K be a field of characteristic different from 2 and $\zeta = 1 + 2i$.*

- (a) *In homogeneous coordinates $h_\zeta = [x_0^3 x_1^2 : x_1^3 x_2^2 : x_0^4 x_2]$, and these monomials have no common factor.*
- (b) *$f = g \circ h_\zeta$ is given by the forms (3), which have degree 10 and no common factor; so $\deg f = 10$.*
- (c) *f is dominant.*

Proof. (a) We have $h_\zeta(y_1, y_2) = (y_1 y_2^2, y_1^{-2} y_2) = (x_1 x_2^2 / x_0^3, x_0 x_2 / x_1^2)$; multiplying the coordinates $(1, x_1 x_2^2 / x_0^3, x_0 x_2 / x_1^2)$ by $x_0^3 x_1^2$ gives the three monomials, and no variable divides all of them.

(b) Put $(u_0, u_1, u_2) := (x_0^3 x_1^2, x_1^3 x_2^2, x_0^4 x_2)$ and $P_0 := u_1 + u_2 - u_0$, $P_1 := u_2 + u_0 - u_1$, $P_2 := u_0 + u_1 - u_2$. By (2), $g \circ h_\zeta = [u_0 P_0 : u_1 P_1 : u_2 P_2]$, and expanding gives (3). Let φ be an irreducible common factor of f_0, f_1, f_2 . It is not a variable: $x_0 \nmid f_1$ because of the term $-x_1^6 x_2^4$, $x_1 \nmid f_2$ because of $-x_0^8 x_2^2$, and $x_2 \nmid f_0$ because of $-x_0^6 x_1^4$. Hence φ divides no monomial, so $\varphi \mid P_j$ for $j = 0, 1, 2$, and then $\varphi \mid P_0 + P_1 = 2u_2$. Since $2 \neq 0$ in K , φ divides the monomial u_2 , a contradiction.

(c) Expanding gives $g \circ g = [mx_0 : mx_1 : mx_2]$ with $m = (x_1 + x_2 - x_0)(x_2 + x_0 - x_1)(x_0 + x_1 - x_2) \neq 0$, so g is a birational involution [BDJ20, §2.1]. The monomial map h_ζ is dominant, since its exponent matrix $\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ has determinant $5 \neq 0$ [BDJ20, §1.3]. A composition of dominant rational maps is dominant. $\square$

Lemma 5.2. *$(1 + 2i)^n \notin \mathbb{R}$ for all $n \geq 1$.*

Proof. If $(1+2i)^n$ were real, it would equal its complex conjugate $(1-2i)^n$, and $\rho := (1+2i)/(1-2i) = (-3+4i)/5$ would be a root of unity, hence an algebraic integer. But the minimal polynomial of ρ over $\mathbb{Q}$ is $X^2 + \frac{6}{5}X + 1$, which does not have integer coefficients. $\square$

Two conventions in [BDJ20]. The introduction of [BDJ20] uses $h(y_1, y_2) = (y_1^a y_2^b, y_1^{-b} y_2^a)$, which is our h_ζ , whereas [BDJ20, §1.3] defines h_ζ by the matrix $\begin{pmatrix} \operatorname{Re} \zeta & -\operatorname{Im} \zeta \\ \operatorname{Im} \zeta & \operatorname{Re} \zeta \end{pmatrix}$, that is, as $(y_1^a y_2^{-b}, y_1^b y_2^a)$. The coordinate swap $\sigma[x_0 : x_1 : x_2] = [x_0 : x_2 : x_1]$ conjugates the second map to the first and commutes with g . Hence the two versions of $g \circ h_\zeta$ are conjugate by a linear automorphism and have the same degree sequence, and ζ satisfies the hypothesis $\zeta^n \notin \mathbb{R}$ ($n \geq 1$) if and only if $\bar{\zeta}$ does.

Proof of Theorem 1.2. By Lemma 5.1, f is dominant and $\deg f = 10$. By Lemma 5.2, $\zeta = 1 + 2i$ satisfies $\zeta^n \notin \mathbb{R}$ for all $n \geq 1$, so $\lambda(f) = \lim_n (\deg f^n)^{1/n}$ is transcendental [BDJ20, Main Theorem and Introduction]; the proof is in [BDJ20, §§1–3], where $\lambda(f)$ is shown to be the unique positive solution of $\sum_{j \geq 1} \deg(h_\zeta^j) \lambda^{-j} = 1$ [BDJ20, Proposition 2.8]. In [BDJ20, §1.2] the degree of a rational map of $\mathbb{P}^2$ is the common degree of coprime defining forms, as in Section 1.1, so $\lambda(f) = \delta_f$. If $(\deg(f^n))_{n \geq 1}$ were piecewise linear recurrent over a field of characteristic zero, Theorem 1.1 would make δ_f algebraic, a contradiction. $\square$

Proof of Corollary 1.3. For $N = 2$ take the map of Theorem 1.2 over $K = \mathbb{Q}$. For $N \geq 3$, Bell, Diller, Jonsson and Krieger proved that there are birational maps $f: \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ with transcendental dynamical degree, defined by polynomials with coefficients in $\{-1, 0, 1\}$ [BDJK24, Theorem 1.1 and the paragraph after it], hence over $\mathbb{Q}$. There, $\deg(f^n)$ is the degree of the preimage of a general hyperplane, which is the common degree of coprime defining forms of f^n : if $f^n = [F_0 : \dots : F_N]$ with coprime forms of degree e , the preimage of a general hyperplane $\sum_i a_i x_i = 0$ is the hypersurface $\sum_i a_i F_i = 0$ of degree e . Theorem 1.1 gives the claim. For $N = 1$, a dominant rational map is given by coprime binary forms of degree $d \geq 1$ without a common zero over an algebraic closure; compositions of such maps have again no common zero, so $\deg(f^n) = d^n$ [BIJ⁺19, Remark 14.3]. $\square$

6 Monomial maps

Proof of Proposition 1.4. For $B \in M_N(\mathbb{Z})$ we have $h_B \circ h_{B'} = h_{BB'}$ on the torus $\{x_0 x_1 \dots x_N \neq 0\}$, so $h_A^n = h_{A^n}$ as rational maps. In homogeneous coordinates, the coordinates of h_B are the Laurent monomials $x^{e_0}, \dots, x^{e_N}$ with $e_0 := 0$ and

$$e_j(B) := \left(-\sum_{i=1}^N B_{ji}, B_{j1}, \dots, B_{jN} \right) \in \mathbb{Z}^{N+1} \quad (1 \leq j \leq N),$$

each of total degree 0. Multiply all of them by x^μ with $\mu_i := \max_{0 \leq j \leq N} (-e_{j,i}) \geq 0$ for $0 \leq i \leq N$. The resulting monomials have non-negative exponents and the common degree $\sum_i \mu_i$, and for each i one of them is not divisible by x_i . So they have no common factor, and

$$\deg h_B = \sum_{i=0}^N \max_{0 \leq j \leq N} (-e_{j,i}(B)). \quad (6)$$

By the Cayley–Hamilton theorem, the entries of A^n , and hence the integers $-e_{j,i}(A^n)$, are LRS in n satisfying the recursion given by the characteristic polynomial of A ; so is the zero sequence $-e_{0,i}$. A sum of maxima is the maximum of the sums over all choices: for each map $\gamma: \{0, \dots, N\} \rightarrow \{0, \dots, N\}$ put $L_n^{(\gamma)} := \sum_{i=0}^N -e_{\gamma(i),i}(A^n)$, an LRS with the same recursion. By (6), $\deg(h_A^n) = \max_\gamma L_n^{(\gamma)}$, and choosing for $\iota(n)$ a maximizing γ gives $\deg(h_A^n) = L_n^{(\iota(n))}$, with at most $(N+1)^{N+1}$ sequences. $\square$

For the maps h_ζ , (6) agrees with the formula $\deg h_\zeta = \max_{\gamma \in \{-2, \pm 2i, 1 \pm 2i\}} \operatorname{Re}(\gamma\zeta)$ of [BDJ20, §1.3].

The example of Hasselblatt and Propp. The map $[YW, ZW, X^2, XW]$ of $\mathbb{P}^3$ [BIJ⁺19, Example 14.5] is the monomial map h_A with $A = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$ in the coordinates $(y_1, y_2, y_3) = (Y/X, Z/X, W/X)$: indeed $y_1 \mapsto y_2/y_1$, $y_2 \mapsto 1/(y_1 y_3)$ and $y_3 \mapsto 1/y_1$. Formula (6) gives the degrees 2, 3, 4, 6, 9, 12, 17, 25, 33, 45 for $n = 1, \dots, 10$, in agreement with the source and with direct iteration (Section 7). By Proposition 1.4 this degree sequence, which satisfies no single linear recursion [BIJ⁺19, Example 14.5], is piecewise linear recurrent; this is why the example does not answer the question. By the proof of Theorem 1.1 (Remark 4.1(1)), its growth rate is the modulus of a root of the characteristic polynomial $X^3 + X^2 + X - 1$ of A ; since it is at least 1, it is $1.3562\dots$, the modulus of the two complex roots (the real root is $0.5436\dots$).

7 Computations and reproducibility

No theorem, lemma or proposition of this note depends on a computation. The program `check_note.py` of the reproducibility package (Python 3, standard library, about 20 seconds) checks the finite and exact parts of the proofs and some numbers quoted from the literature: (A) the arithmetic of Lemma 5.2; (B) the forms in Lemma 5.1 and (3), the identities $f_j = u_j P_j$ and $P_0 + P_1 = 2u_2$ (with its permutations), the terms excluding a common variable, $g \circ g = [mx_0 : mx_1 : mx_2]$, and the commutation of g with the coordinate swap; (C) formula (6) against direct iteration (the map of Hasselblatt and Propp for $n \leq 40$, and 1,200 cases for random matrices of size 2 and 3), the representation of Proposition 1.4 for the map of Hasselblatt and Propp for $n \leq 60$, and the agreement with [BDJ20, §1.3]; (D) an enclosure, certified by exact arithmetic and a bound for the tail, of the root of $\sum_j d_j \lambda^{-j} = 1$ [BDJ20, Proposition 2.8]: $\delta_f = 6.857557409279008598\dots$ for $\zeta = 1 + 2i$ and $13.449607681785413449\dots$ for $\zeta = -3 + 4i$, as printed in [BDJ20] to ten decimals; (E) $\deg(f^n) = 10, 66, 454, 3114$ for $n = 1, \dots, 4$, by restricting the iterates to three random lines modulo $2^{61} - 1$ and removing the common factor after each step (a special line or prime could only lower the result), equal to the values of the recursion of [BDJ20, Proposition 2.7]; (F) the example at the end of Section 1.1. All 38 checks pass; (D) and (E) check conventions and are not part of a proof.

Verification. The result was first obtained with a written proof and a program checking the conventions (the degrees of the map of Hasselblatt and Propp, the forms (3), $\deg f^2 = 66$ by exact polynomial arithmetic, and δ_f). Two independent verification runs, both AI-assisted, followed; each re-derived the proofs and wrote its own programs, and neither read the other's report. The first run compared the quoted statements with the texts of [BIJ⁺19, BDJ20, BDJK24, Eve23] and computed $\deg(f^n)$ for $n \leq 4$ on random lines modulo $2^{61} - 1$. The second run computed the degrees of the iterates of $g \circ h_\zeta$ up to the sixth for $\zeta = 1 + 2i$ and the fifth for $\zeta = -3 + 4i$ modulo primes near 2^{20} , in agreement with [BDJ20, Proposition 2.7], compared (6) with direct iteration in 2,400 random cases, re-ran the first program with identical output, repeated the literature search and checked the bibliography against Crossref. Neither run found an error in the mathematics. The corrections they required were made: the numbering of the source in its two arXiv versions; an explicit statement of the reading in the abstract and the introduction; the provenance of Theorem 3.1, with the reduction in Remark 3.2; the remark on the two conventions in [BDJ20]; two statements about the literature (the published number of the question, which we could not verify, and a later citation of the source, which concerns a different question); the title of [EG15] and the data of [BDJ20, BDJK24]. The reference to [BDJK24, Corollary 4.2] follows a suggestion of the first run. The program `check_note.py`, the reference to [Ngu26, Theorem D] and the last sentence of Section 6 were added when this note was written.

A third AI-assisted verification run examined the note in its written form on 18 October 2026, as if nothing had been checked before. It downloaded the sources again and compared every quoted statement with the TeX of both arXiv versions of [BIJ⁺19], of [BDJ20, BDJK24, Ngu26] and with [Eve23], and it rendered the scan of [Eve84], so that the displayed inequality of its Theorem 2 could be read. It re-derived every lemma, theorem and proposition line by line, including the three items just mentioned, and found no error in the mathematics. Its own programs, written from the definitions with a computer algebra system and included in the reproducibility package, checked the forms (3), their coprimality over $\mathbb{Q}$ and modulo several primes, $g \circ g$, the coordinate swap, the dominance of f , $\deg f^2 = 66$ by an exact trivariate gcd, formula (6) against direct composition in dimensions 2, 3 and 4, the representation of Proposition 1.4, an independent exact certification of the enclosure of δ_f , the recursion of [BDJ20, Proposition 2.7], and $\deg(f^n)$ for $n \leq 6$ ($\zeta = 1 + 2i$) and $n \leq 4$ ($\zeta = -3 + 4i$) on new random lines modulo $2^{31} - 1$, all in agreement with the note and with that recursion; it re-ran `check_note.py` from the archive with identical output, repeated the literature search, checked the bibliography against Crossref and read all pages. The corrections it required were made: the description of [Eve84, Theorem 2] in Remark 3.2, with the passage to Theorem 3.1; the mention of constant coefficients for [BDJK24, Corollary 4.2]; and the records of the package.

Scope and priority. The note answers [BIJ⁺19, Question 14.6] negatively in the reading of Section 1.1 (finitely many fixed sequences satisfying linear recursions with constant coefficients in a field of characteristic zero, and an arbitrary index function); the reading in which the recursions are applied to the degree sequence itself, with coefficients changing with n , is not addressed. The question is due to Benedetto, Ingram, Jones, Manes, Silverman and Tucker [BIJ⁺19]. The transcendence of the dynamical degree is due to Bell, Diller and Jonsson [BDJ20] and, for $N \geq 3$, to Bell, Diller, Jonsson and Krieger [BDJK24]; Lemma 3.4 rests on the subspace theorem through the inequality of Evertse [Eve84] (see also [vdPS82]), in the form used in [BDJ20]. Earlier results exclude a single linear recursion for particular maps [HP07, BK08, LW14], [Ngu26, Corollary B]. What the note adds is the short deduction that piecewise linear recurrence forces an algebraic growth rate (Theorem 1.1), which with [BDJ20] answers the question; its novelty is modest, specialists may regard it as known, and no priority is claimed. No printed answer to the question was found in the sources searched: both arXiv versions of [BIJ⁺19]; [BDJ20, BDJK24, Ngu26, Hu26, Sil26]; the works listed by Semantic Scholar as citing [BIJ⁺19] (80 records) or [BDJ20] (31 records), by title and abstract and several in full text; abstract searches in arXiv and searches in zbMATH Open on dynamical degrees, degree sequences and degree growth with recurrences or transcendence; five web searches; and the slides of a talk of Bell on transcendental dynamical degrees of birational maps (BIRS, 2024), which contain nothing on the question; all on 17 and 18 October 2026 (the lists of citing works in OpenAlex were not available on either day). A search that finds nothing is not a proof of novelty.

What was read: of [BIJ⁺19], Section 14 in the TeX sources of both arXiv versions; of [BDJ20], the TeX source of arXiv version 2 (introduction, Sections 1 and 2, Section 3.2 and the conclusions of the proof); of [BDJK24], the introduction with Theorem 1.1, Section 2.1, Theorem 4.1 and Corollary 4.2 of arXiv version 2; the introduction of [Ngu26], with Theorems A and D and Corollary B; the passages of [Hu26, Sil26] on [BIJ⁺19] and on transcendental dynamical degrees; Section 8.4 of [Eve23]; and Section 1 and Theorem 2 of [Eve84], in the Numdam scan. We did not read [EG15, vdPS82, HP07, BK08, LW14, DF01, FJ11]; what we say about them is taken from [BIJ⁺19, BDJ20, BDJK24, Ngu26, Eve84]. The results cited from [AM69, Lec53, Sch35] are standard, and so is Dirichlet's S -unit theorem, used in Remark 3.2. The proofs of the cited theorems were not checked.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has

not been peer reviewed.

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