

A Partial Answer to Propp's Problem 7 on Benzel Tilings: An Infinite Family of Type-103 Tilings and the Open Cases with $|a - b| \leq 7$ or $\max(a, b) \leq 60$

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Abstract

Propp asked whether every (a, b) -benzel with nonnegative Conway–Lagarias invariant has a tiling by right-pointing stones and bones of all three orientations, a tiling of type 103 (Problem 7 of his list of twenty problems). By published theorems of Defant, Li, Propp and Young, of Defant, Foster, Li, Propp and Young, and of Kim and Propp, the question is open only for the benzels $(d + 3J, 2d + 3J)$ with $d \geq 3$ and $1 \leq J \leq \frac{(d-2)(d+1)}{2}$ and for their mirror images. We give a partial answer; Problem 7 remains open in general. We prove that for every $D \geq 2$ the $(4D - 3, 5D - 3)$ -benzel has a type-103 tiling, formed by a central triangle of $\frac{(D-1)(D-2)}{2}$ right stones and three congruent sectors, each tiled by parallel bones; for $D \geq 3$ these benzels lie in the open region. We also verify the question, by computer, for every open case with $|a - b| \leq 7$ and for every open case with $\max(a, b) \leq 60$: for each of these benzels a tiling is stored and checked by a short self-contained program. This is a finite verification and says nothing about larger cases. The row $(n, 2n - 3)$ and the window $\max(a, b) \leq 14$ are covered by unrefereed preprints of 2026 (by A. C. Li and by The Clankers), which we credit and do not use. We also record computational evidence that the simplest ring inductions, which add a ring around a smaller benzel, do not reach the cases near the line where the invariant vanishes. This is an unrefereed note.

1 Introduction

1.1 The problem

We use the definitions of Propp [Pro24, §1]. Let $\omega = e^{2\pi i/3}$. The plane is tiled by unit hexagonal cells, and a cell is identified with the triple $(i, j, k) \in \mathbb{Z}^3$, $i + j + k = 1$, for which its centre is $i + j\omega + k\omega^2$. For integers a, b with $2 \leq a \leq 2b$ and $2 \leq b \leq 2a$ (we call such pairs *admissible*) the (a, b) -benzel is the set of cells

$$B(a, b) = \{(i, j, k) : i + j + k = 1, -(a - 1) \leq j - i, k - j, i - k \leq b - 1\}. \quad (1)$$

Let e_1, e_2, e_3 be the unit vectors of $\mathbb{Z}^3$. The five prototiles are the *right stone* $\{p + e_1, p + e_2, p + e_3\}$ with $p \in \mathbb{Z}^3$, $p_1 + p_2 + p_3 = 0$ (a translate of the $(2, 2)$ -benzel $\{e_1, e_2, e_3\}$), the *left stone* $\{q - e_1, q - e_2, q - e_3\}$ with $q_1 + q_2 + q_3 = 2$, and the *vertical, rising and falling bones* $\{c, c + \delta, c + 2\delta\}$ with $\delta = e_2 - e_3$, $e_3 - e_1$ and $e_1 - e_2$ respectively. A *tiling of type 103* of a benzel is a partition into right stones and bones (all three orientations allowed, no left stone), and $T_{103}(a, b)$ is the number of such tilings of $B(a, b)$ [Pro24, §1].

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Conway and Lagarias [CL90] showed that for a simply connected region which can be tiled by stones and bones, the number of cells covered by right stones minus the number covered by left stones is the same for all such tilings. For benzels this *Conway–Lagarias invariant* is [Pro24, §1]

$$I(a, b) = \begin{cases} \frac{1}{2}(3a^2 - 6ab + 3b^2 - a - b), & a + b \equiv 0 \pmod{3}, \\ \frac{1}{2}(-a^2 + 4ab - b^2 - a - b + 2), & a + b \equiv 1 \pmod{3}, \\ \frac{1}{2}(3a^2 - 6ab + 3b^2 + a + b - 2), & a + b \equiv 2 \pmod{3}. \end{cases} \quad (2)$$

In a tiling of type 103 the number of right stones is therefore $I(a, b)/3$, so $I(a, b) < 0$ excludes such tilings; Propp observes this and asks for the converse.

Problem 1.1 (Propp [Pro24, Problem 7]). *Is $T_{103}(a, b) \geq 1$ for every admissible pair (a, b) with $I(a, b) \geq 0$?*

This is the record AMR-123-0003 of the problem corpus UnsolvedMath. Propp’s tables of $T_{103}(a, b)$ for $a, b \leq 10$ [Pro24, §6] are positive wherever $I(a, b) \geq 0$, and his list of problem statuses, dated June 2025, gives Problem 7 as open [Pro25].

1.2 What is known

Published theorems settle Problem 1.1 outside an explicit region (Proposition 2.2 below): for $a + b \equiv 1 \pmod{3}$ by a theorem of Defant, Li, Propp and Young [DLPY23, Theorem 1.1], for $a + b \equiv 2 \pmod{3}$ by the product formula of Defant, Foster, Li, Propp and Young [DFL⁺25, Theorem 1.1], for $b = 2a$ by the coincidence of the $(a, 2a)$ - and the $(a, 2a - 2)$ -benzel [Pro24, §1], and for $I(a, b) = 0$ by the pentagonal number theorem for bone tilings of Kim and Propp [KP23, Theorem 3]. What remains, up to the symmetry $T_{103}(a, b) = T_{103}(b, a)$ [Pro24, §1], is the set

$$\mathcal{R}_7 = \left\{ (d + 3J, 2d + 3J) : d \geq 3, 1 \leq J \leq \frac{(d-2)(d+1)}{2} \right\}, \quad (3)$$

on which $a + b \equiv 0 \pmod{3}$ and $I(a, b) > 0$; for $(a, b) \in \mathcal{R}_7$ a tiling of type 103 has exactly $\frac{d(d-1)}{2} - J$ right stones.

Two preprints of 2026, both unrefereed, treat parts of $\mathcal{R}_7$. A. C. Li [Li26b, Theorem 1.1] gives a proof, which we have not checked, of Propp’s formula $T_{103}(n, 2n - 3) = \frac{(3n+3)(3n-7)!}{(n-5)!(2n-1)!}$ for $n \geq 5$ (Problem 6 of [Pro24]); as this number is positive, it would give type-103 tilings on the row $J = 1$ of $\mathcal{R}_7$. The Clankers [The26] report an exhaustive exact-cover verification of the sign rules of Problems 4 and 7 for $\max(a, b) \leq 14$ and a shell recursion giving $T_{103}(n, 2n - 3) \geq 2 \cdot 3^{n-5}$. Li’s finite-defect theorem [Li26b, Theorem 7.2] assumes that a type-103 tiling exists and does not give existence. We use none of these results.

Problem 4 of [Pro24] (record AMR-123-0002) is the mirror question for left stones and $I(a, b) \leq 0$. It is answered in the affirmative in an unrefereed preprint of A. C. Li [Li26a, Theorem 2.1], by a collar recursion around a translate of the $(a - 3, b - 3)$ -benzel, starting from the bone tilings of Kim and Propp and a few small cases. The computations behind this note also produced a collar induction of the same kind; it is the same idea as Li’s and is not claimed here.

1.3 Results

Theorem 1.2. *For every integer $D \geq 2$ the $(4D - 3, 5D - 3)$ -benzel has a tiling of type 103, by $\frac{(D-1)(D-2)}{2}$ right stones and bones. For $D \geq 3$ this benzel lies in $\mathcal{R}_7$, with $d = D$ and $J = D - 1$; for $D = 2$ it is the $(5, 7)$ -benzel, for which $I = 0$.*

The tiling consists of a triangle of right stones in the centre and three sectors, which are images of each other under rotation by 120° , each tiled by bones of one orientation (Figure 1). The proof (Section 3) is elementary.

Theorem 1.3. *Every $(a, b) \in \mathcal{R}_7$ with $b - a \leq 7$, every $(a, b) \in \mathcal{R}_7$ with $b \leq 60$, and every $(a, a + 8) \in \mathcal{R}_7$ except $(77, 85)$, $(80, 88)$, $(83, 91)$, $(86, 94)$ and $(89, 97)$ has a tiling of type 103.*

The proof is a computation: for each of these 219 benzels a tiling is stored, and a short program checks it (Section 4). Together with Proposition 2.2 and the symmetry $T_{103}(a, b) = T_{103}(b, a)$ this gives:

Corollary 1.4. *The answer to Problem 1.1 is affirmative for all admissible (a, b) with $|a - b| \leq 7$, for all admissible (a, b) with $\max(a, b) \leq 60$, and for $(a, b) = (4D - 3, 5D - 3)$ and $(5D - 3, 4D - 3)$, $D \geq 2$.*

Problem 1.1 remains open in general. The smallest open cases are $(a, 61)$ with $a \in \{32, 35, \dots, 50\}$ (the first of them lies on the row $(n, 2n - 3)$), and for $|a - b| = 8$ the five cases excluded in Theorem 1.3, which have 5, 4, 3, 2, 1 right stones. In Section 5 we record computational evidence that the simplest inductions which pass from a benzel to a larger one by adding a ring (the moves tested there, for small parameters) do not reach the cases near the line $I = 0$; this evidence is not a proof of anything.

2 Coordinates and the open region

For a cell $c = (i, j, k)$ put

$$u = j - i, \quad v = k - j, \quad w = i - k, \quad u + v + w = 0.$$

By (1), $B(a, b) = \{-(a - 1) \leq u, v, w \leq b - 1\}$. We write cells in the coordinates (u, v, w) from now on.

Lemma 2.1. (a) *The map $(i, j, k) \mapsto (u, v, w)$ is a bijection from the cells onto the triples $(u, v, w) \in \mathbb{Z}^3$ with $u + v + w = 0$ and $v \equiv u + 1 \pmod{3}$. In particular $(u, v, w) \equiv (u, u + 1, u - 1) \pmod{3}$: the three coordinates of a cell lie in three different residue classes modulo 3.*

(b) *The bone directions $e_2 - e_3$, $e_3 - e_1$, $e_1 - e_2$ (vertical, rising, falling) change (u, v, w) by $(1, -2, 1)$, $(1, 1, -2)$, $(-2, 1, 1)$.*

(c) *For the right stone $\{p + e_1, p + e_2, p + e_3\}$ put $(U, V, W) = (p_2 - p_1, p_3 - p_2, p_1 - p_3)$, its centre. Its cells are*

$$(U - 1, V, W + 1), \quad (U + 1, V - 1, W), \quad (U, V + 1, W - 1), \quad (4)$$

and the centres of right stones are exactly the triples with $U + V + W = 0$ and $U \equiv V \equiv W \pmod{3}$.

(d) *The rotation $\rho(i, j, k) = (k, i, j)$ (multiplication of the centre by ω) acts as $(u, v, w) \mapsto (w, u, v)$. It maps every benzel onto itself and right stones to right stones, and it maps vertical bones to rising bones, rising to falling and falling to vertical bones.*

Proof. (a) Given (u, v, w) with sum 0, the equations $j = i + u$, $k = j + v$, $i + j + k = 1$ give $3i = 1 - 2u - v$, which has an integer solution if and only if $v \equiv 1 - 2u \equiv u + 1 \pmod{3}$; then $w = -u - v \equiv u - 1$. (b) and the first part of (c) are direct computations. For the second part of (c), $U - V = 3p_2 - (p_1 + p_2 + p_3) = 3p_2$; conversely, if $U \equiv V \equiv W$ and $U + V + W = 0$, then

$p_1 = -(2U + V)/3$, $p_2 = p_1 + U$, $p_3 = p_2 + V$ is an integer solution with $p_1 + p_2 + p_3 = 0$. (d) ρ permutes the coordinates cyclically, so it maps the triples e_1, e_2, e_3 and the differences $e_2 - e_3 \mapsto e_3 - e_1 \mapsto e_1 - e_2 \mapsto e_2 - e_3$ as stated, and the conditions defining $B(a, b)$ are symmetric in u, v, w . $\square$

Write $p_d = \frac{d(3d-1)}{2}$ for the pentagonal numbers.

Proposition 2.2 (the open region). *Let (a, b) be admissible with $a \leq b$ and put $d = b - a$.*

- (a) *If $a + b \not\equiv 0 \pmod{3}$, or $b = 2a$, or $I(a, b) = 0$, then $T_{103}(a, b) \geq 1$, by the published theorems cited below.*
- (b) *If $a + b \equiv 0 \pmod{3}$, then $a \equiv d \pmod{3}$, $I(a, b) = \frac{1}{2}(3d^2 - (a + b))$, and $I(a, b) \geq 0$ if and only if $a \leq p_d$; moreover $I(a, b) = 0$ if and only if $(a, b) = (p_d, p_d + d)$.*
- (c) *Consequently Problem 1.1 is open exactly for the pairs (a, b) and (b, a) with $(a, b) \in \mathcal{R}_7$, see (3). For $(a, b) = (d + 3J, 2d + 3J) \in \mathcal{R}_7$, a type-103 tiling has $\frac{d(d-1)}{2} - J$ right stones.*

Proof. (b) If $a + b \equiv 0$ then $2a + d \equiv 0$, so $a \equiv d \pmod{3}$, and (2) gives $2I = 3(a - b)^2 - (a + b) = 3d^2 - 2a - d$. Hence $I \geq 0$ if and only if $a \leq \frac{3d^2 - d}{2} = p_d$, with equality exactly for $I = 0$.

(a) If $a + b \equiv 1 \pmod{3}$, the (a, b) -benzel has a tiling by right stones only [DLPY23, Theorem 1.1]; it is of type 103. If $a + b \equiv 2 \pmod{3}$, put $n = b + 1 - a \geq 1$ and $k = \frac{2a - b - 1}{3}$; here $b \leq 2a - 1$, because $b = 2a$ gives $a + b \equiv 0$, so k is a nonnegative integer. By [DFL⁺25, Theorem 1.1] the number of tilings by right stones, rising bones and falling bones is $\prod_{j=1}^k \frac{(2j)!(2j+2n-2)!}{(j+n-1)!(j+n+k-1)!}$, which is positive; such a tiling is of type 103. If $b = 2a$, the $(a, 2a)$ -benzel is the $(a, 2a - 2)$ -benzel [Pro24, §1]: both are the set $\{-(a - 1) \leq u, v, w\}$, because a cell with $u, v, w \geq -(a - 1)$ has no coordinate above $2a - 3$ (the coordinates have sum 0, and no two of them are equal by Lemma 2.1(a)). Since $a + (2a - 2) \equiv 1 \pmod{3}$, the first case applies. If $I(a, b) = 0$, then by (b) and (2) $a + b \equiv 0$ and $(a, b) = (p_d, p_d + d) = (\frac{d(3d-1)}{2}, \frac{d(3d+1)}{2})$ with $d \geq 2$ (for $d \leq 1$ the pair is not admissible); this benzel has a tiling by bones [KP23, Theorem 3], which is of type 103 with no stone. (The benzels of [KP23] are the mirror images of Propp's, which does not matter for bone tilings, and the criterion there is symmetric in a and b .)

(c) By (a) and (b), the cases with $a \leq b$ not covered by (a) are those with $a + b \equiv 0$, $b < 2a$ and $0 < I(a, b)$, that is, $a \equiv d \pmod{3}$ and $d < a < p_d$. Writing $a = d + 3J$, the conditions are $J \geq 1$ and $d + 3J \leq p_d - 3$, i.e. $J \leq \frac{(d-2)(d+1)}{2}$; this range is empty for $d \leq 2$. The pairs with $a > b$ reduce to these by $T_{103}(a, b) = T_{103}(b, a)$ [Pro24, §1]: the map $(i, j, k) \mapsto (i, k, j)$ sends (u, v, w) to $(-w, -v, -u)$, hence $B(a, b)$ onto $B(b, a)$, right stones to right stones and bones to bones. Finally a type-103 tiling has $I/3$ right stones, and $\frac{I}{3} = \frac{3d^2 - 2(d+3J) - d}{6} = \frac{d(d-1)}{2} - J$. $\square$

The set $\mathcal{R}_7$ has $\frac{(d-2)(d+1)}{2} = 2, 5, 9, 14, 20, 27$ elements with $b - a = d$ for $d = 3, \dots, 8$, 107 elements with $b \leq 45$ and 206 with $b \leq 60$. Its row $J = 1$ is the diagonal $(n, 2n - 3)$, $n \geq 6$, and its last element in each slice, $J = \frac{(d-2)(d+1)}{2}$, has a single right stone.

3 The family $(4D - 3, 5D - 3)$

Fix an integer $D \geq 2$ and put

$$t = D - 1 \geq 1, \quad a = 4D - 3 = 4t + 1, \quad b = 5D - 3 = 5t + 2,$$

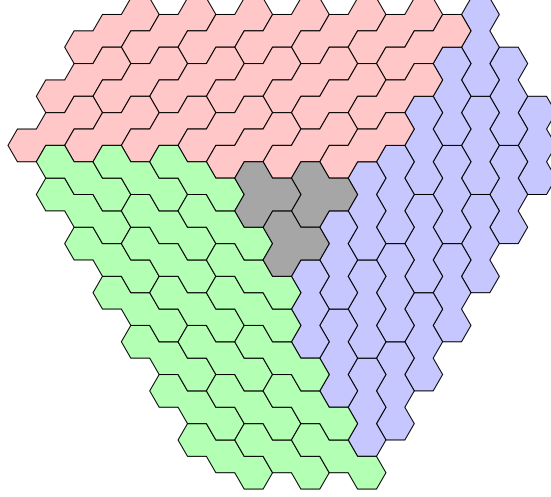


Figure 1: The tiling of Theorem 1.2 for $D = 4$: the $(13, 17)$ -benzel, with the 3 right stones of H (grey) and the sectors V (vertical bones, blue), S (rising bones, red) and F (falling bones, green). Drawn from the construction by `make_figure.py`.

so that $B = B(a, b) = \{-4t \leq u, v, w \leq 5t + 1\}$. Define the following subsets of B (Figure 1):

$$\begin{aligned} H &= \{u, v, w \geq 1 - t\}, \\ V &= \{u \leq -t, w \geq -t\} \setminus \{u = -t, v \leq -t\}, \\ S &= \{v \leq -t, u \geq -t\} \setminus \{v = -t, w \leq -t\}, \\ F &= \{w \leq -t, v \geq -t\} \setminus \{w = -t, u \leq -t\}. \end{aligned}$$

By Lemma 2.1(d), $\rho(B) = B$, $\rho(H) = H$, $\rho(V) = S$, $\rho(S) = F$ and $\rho(F) = V$; for instance, $c \in V$ if and only if $\rho(c) = (w, u, v)$ satisfies the conditions defining S .

Lemma 3.1 (partition). *Every cell of B lies in exactly one of the sets H, V, S, F .*

Proof. Let $c = (u, v, w) \in B$ and let N be the set of its coordinates that are $\leq -t$. As $u + v + w = 0$ and $t \geq 1$, N has at most two elements. If $N = \emptyset$, then $c \in H$ and c lies in none of V, S, F , each of which requires a coordinate $\leq -t$. If $N = \{u\}$, then $v, w > -t$, so $c \in V$ (the excluded set needs $v \leq -t$) and $c \notin S \cup F \cup H$. If $N = \{u, v\}$, then $w = -u - v \geq 2t > -t$; hence $c \in V$ if and only if $u \neq -t$, $c \in S$ if and only if $u = -t$ (the set excluded from S needs $w \leq -t$), and $c \notin F \cup H$. The other four cases are the images of these under ρ and ρ^2 , which preserve B and H and permute V, S, F cyclically. $\square$

A vertical line is a set $L_c = \{u - w = c\}$; by Lemma 2.1(b), $u - w$ is constant and u increases by 1 along the vertical direction.

Lemma 3.2 (one interval per line). *(a) L_c contains cells only if $c \equiv 1 \pmod{3}$, and then one cell for every integer value of u . For such c , the set $V \cap L_c$ consists of the cells of L_c with*

- (i) $c - t \leq u \leq -t$, if $-3t < c \leq 0$; these are $1 - c$ cells;
- (ii) $-4t \leq u \leq -t - 1$, if $-6t < c < -3t$; these are $3t$ cells;
- (iii) $-4t \leq u \leq c + 5t + 1$, if $c < -6t$; these are $\max(0, c + 9t + 2)$ cells;

and $V \cap L_c = \emptyset$ if $c > 0$. In every case $V \cap L_c$ is a set of consecutive cells of L_c , and their number is divisible by 3.

(b) V is a disjoint union of vertical bones, S of rising bones and F of falling bones.

Proof. (a) On L_c we have $w = u - c$ and $v = c - 2u$. By Lemma 2.1(a) the triple is a cell if and only if $c - 2u \equiv u + 1$, that is, $c \equiv 1 \pmod{3}$. Let $c \equiv 1$. The conditions for V are $u \leq -t$ and $w \geq -t$, i.e. $u \geq c - t$, and the cell with $u = -t$ is excluded if and only if $v = c + 2t \leq -t$, i.e. $c \leq -3t$. The conditions for B are $-4t \leq u \leq 5t + 1$, $c - 4t \leq u \leq c + 5t + 1$ and $c - 5t - 1 \leq 2u \leq c + 4t$. If $c > 0$ the conditions $c - t \leq u \leq -t$ are incompatible. Let $c \leq 0$; then $u \leq -t$ and $u \geq c - t$ imply $u \leq 5t + 1$ and $u \geq c - 4t$, so $V \cap L_c$ is given by

$$\max\left(c - t, -4t, \frac{c-5t-1}{2}\right) \leq u \leq \min\left(-t, c + 5t + 1, \frac{c+4t}{2}\right),$$

minus the cell $u = -t$ if $c \leq -3t$. As $3t \equiv 0$ and $c \equiv 1 \pmod{3}$, the values $c = -3t, -6t$ and $-6t - 1$ do not occur. In case (i), $c - t > -4t$ and $\frac{c-5t-1}{2} \leq c - t$ (as $c \geq -3t - 1$), while $c + 5t + 1 > -t$ and $\frac{c+4t}{2} > -t$; so $c - t \leq u \leq -t$, with no excluded cell. In case (ii), $c - t < -4t$ and $\frac{c-5t-1}{2} < -4t$ (as $c < -3t + 1$), the upper bound is $-t$ as in case (i) (as $c > -6t$), and the cell $u = -t$ is excluded; so $-4t \leq u \leq -t - 1$. In case (iii), $c \leq -6t - 2$; the lower bound is $-4t$ as in case (ii), and $c + 5t + 1 \leq -t - 1$ and $c + 5t + 1 \leq \frac{c+4t}{2}$ (as $c \leq -6t - 2$), so $-4t \leq u \leq c + 5t + 1$. The numbers of cells are $1 - c$, $3t$ and $c + 9t + 2$, and each is divisible by 3 because $c \equiv 1 \pmod{3}$.

(b) Cutting each set $V \cap L_c$ into consecutive triples gives vertical bones inside V which partition V . By Lemma 2.1(d), ρ maps V onto S and vertical bones to rising bones, so the images of these bones partition S into rising bones; applying ρ once more gives a partition of F into falling bones. $\square$

Lemma 3.3 (the central triangle). *Let $t \geq 1$ and $r \equiv 2 - t \pmod{3}$. The set $H = \{u, v, w \geq 1 - t\}$ is the disjoint union of the $\frac{t(t-1)}{2}$ right stones whose centres (U, V, W) satisfy $U \equiv V \equiv W \equiv r \pmod{3}$ and $U, V, W \geq 2 - t$.*

For $t \geq 2$, H is the $(t, 2t - 2)$ -benzel (as in the proof of Proposition 2.2(a)), and the lemma is the all-right-stone tiling of [DLPY23, Theorem 1.1]; the following direct proof makes the note self-contained.

Proof. Call a right stone an r -stone if the coordinates of its centre are $\equiv r \pmod{3}$. Order the coordinates cyclically, $u \rightarrow v \rightarrow w \rightarrow u$. By (4), an r -stone contains, for each coordinate x , exactly one cell whose x -coordinate is $\equiv r$: it is obtained from the centre by keeping x , adding 1 to the coordinate following x and subtracting 1 from the coordinate preceding x . Now let c be any cell. By Lemma 2.1(a) exactly one coordinate x of c is $\equiv r$, the following one is $\equiv r + 1$ and the preceding one is $\equiv r - 1$. Hence c lies in exactly one r -stone: its centre is obtained from c by keeping x , subtracting 1 from the following and adding 1 to the preceding coordinate, and this triple has all coordinates $\equiv r$. So the r -stones partition the set of all cells.

Let $c \in H$ and let x be its coordinate $\equiv r$. Then $x \geq 1 - t$ and $x \equiv 2 - t$, so $x \geq 2 - t$; the following coordinate is $\geq 1 - t$ and $\equiv 3 - t$, so it is $\geq 3 - t$, and the corresponding coordinate of the centre is $\geq 2 - t$; the preceding coordinate is $\geq 1 - t$, and that of the centre is $\geq 2 - t$. So the r -stone of c has a centre with all coordinates $\geq 2 - t$. Conversely, by (4) an r -stone whose centre has all coordinates $\geq 2 - t$ has all its cells in H . Hence H is the disjoint union of these stones. Their centres are $(2 - t + 3\alpha, 2 - t + 3\beta, 2 - t + 3\gamma)$ with integers $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma = t - 2$, and there are $\binom{t}{2} = \frac{t(t-1)}{2}$ of them (none if $t = 1$). $\square$

Proof of Theorem 1.2. By Lemma 3.1, $B = H \cup V \cup S \cup F$ with disjoint parts. By Lemma 3.3, H is tiled by $\frac{t(t-1)}{2}$ right stones, and by Lemma 3.2(b), V, S and F are tiled by bones. Together this is a tiling of type 103 of $B(4D - 3, 5D - 3)$ with $\frac{t(t-1)}{2} = \frac{(D-1)(D-2)}{2}$ right stones. (As it must be, this is $I(a, b)/3$: here $a + b = 9t + 3$ and $2I = 3(t + 1)^2 - (9t + 3) = 3t(t - 1)$.) For the position in

$\mathcal{R}_7$: $b - a = D$ and $a = D + 3(D - 1)$, so $d = D$ and $J = D - 1$, and $J \leq \frac{(d-2)(d+1)}{2}$ is equivalent to $D(D - 3) \geq 0$. For $D = 2$, $(5, 7) = (p_2, p_2 + 2)$ and $I = 0$. $\square$

Remark 3.4. (a) The sizes are $|H| = \frac{3t(t-1)}{2}$ and, by Lemma 3.2(a), $|V| = |S| = |F| = \sum_{i=1}^t 3i + t \cdot 3t + \sum_{i=1}^t 3i = 6t^2 + 3t$; the total $\frac{39t^2+15t}{2}$ is the area of the $(4t + 1, 5t + 2)$ -benzel given by Propp's area formula [Pro24, §1].

(b) The program `check_note.py` (Section 6) builds the tiling for $2 \leq D \leq 40$ and checks the partition, the three cases of Lemma 3.2(a), the stones of Lemma 3.3 and the exact cover. The family was first found by a computer search for bone tilings, invariant under ρ , of the complement of a central triangle of right stones, and the construction was first certified for all $D \geq 2$ with the SMT solver Z3 [dMB08]: the partition, the identity $H = \{-(t - 1) \leq u, v, w \leq 2t - 1\}$ and the cutting of every line into whole bones counted from the end of its run were written as formulas of linear integer arithmetic, quantified over all integers $D \geq 2$ and all cells, and Z3 reports that their negations are unsatisfiable (and that the bone claims fail when t is replaced by $D - 2$). These formulas presuppose that every line meets a sector in a set of consecutive cells, which is Lemma 3.2(a); the proof above does not use them.

(c) On the diagonal $b = 2a - 3k$ with $k = D - 1 \geq 2$, the benzel of Theorem 1.2 is the point $a = 4k + 1$, which lies in the range $2k \leq a - 1$ of Li's finite-defect theorem [Li26b, Theorem 7.2]. That theorem assumes that a type-103 tiling exists, and Theorem 1.2 shows that this assumption holds at this point of each such diagonal. We note without further use that the defect number $\binom{k}{2}$ given there for $2a - b = 3k$ (Li's parameter d , not the $d = b - a$ of this note) is the number of right stones in our tiling.

4 Finite verification

Certificates. For every benzel of Theorem 1.3 the package of this note contains a text file `B_a_b.txt` with one tile per line: a letter **R**, **V**, **S** or **F** (right stone, vertical, rising or falling bone) followed by the coordinates (i, j, k) of its three cells. The folder `certificates/original` holds 136 files from the computation in which the results were first obtained: 132 for benzels of $\mathcal{R}_7$ and four for the benzels $(5, 7)$, $(12, 15)$, $(22, 26)$, $(35, 40)$ with $I = 0$, kept as tests. The folder `certificates/verification_run_B` holds 87 files, for benzels of $\mathcal{R}_7$ with $46 \leq b \leq 60$, found in the second independent verification run (Section 6). The tilings were found with SAT solvers (CaDiCaL and Glucose, called through PySAT [IMMS18]), for many benzels after fixing a central triangle of right stones and solving only for bones, and for some by an encoding restricted to tilings invariant under ρ . How a tiling was found plays no role: once checked, each file is a complete proof that its benzel has a tiling of type 103.

The check. The program `check_note.py` recomputes the cell set of $B(a, b)$ from (1) and checks, for each file, that every line lists three cells forming a translate of the prototile named by its letter (the type is computed from the cells alone), that no left stone occurs, that the tiles cover every cell of $B(a, b)$ exactly once, and that $3 \cdot \#\{\text{right stones}\} = I(a, b)$ (which then holds automatically, and serves as a consistency check). It accepts all 223 files and rejects modified copies (a tile removed, a letter changed, a file renamed to another benzel). The checker `verify_tilings.py` of the original computation, which shares no code with it, also accepts all 223 files, and so do the checkers written in the three verification runs.

Proof of Theorem 1.3. The stored files, checked by the program `check_note.py`, are type-103 tilings of the 219 benzels of Table 1. These benzels include every $(a, b) \in \mathcal{R}_7$ with $b - a \leq 7$ or $b \leq 60$, and every $(a, a + 8) \in \mathcal{R}_7$ except the five listed. $\square$

part of $\mathcal{R}_7$	size	certified	not certified
$b - a = 3, 4, 5, 6, 7$	2, 5, 9, 14, 20	all	none
$b - a = 8$	27	22	(77, 85), (80, 88), (83, 91), (86, 94), (89, 97)
$b \leq 45$	107	all	none
$b \leq 60$	206	all	none
all certificates		219	(largest: (74, 82), 5958 cells)

Table 1: The benzels of $\mathcal{R}_7$ covered by the stored certificates. The 219 certified benzels are the 206 with $b \leq 60$ and 13 further ones with $b - a \in \{7, 8\}$.

Proof of Corollary 1.4. Let (a, b) be admissible with $I(a, b) \geq 0$; by the symmetry $T_{103}(a, b) = T_{103}(b, a)$ we may assume $a \leq b$. If $(a, b) \notin \mathcal{R}_7$, Proposition 2.2 gives a tiling of type 103. If $(a, b) \in \mathcal{R}_7$ and $b - a \leq 7$ or $b \leq 60$, Theorem 1.3 gives one, and for $(4D - 3, 5D - 3)$ Theorem 1.2 does. $\square$

For comparison, the window $\max(a, b) \leq 14$ of [The26] contains the five benzels (6, 9), (9, 12), (7, 11), (10, 14), (8, 13) of $\mathcal{R}_7$, and the row $(n, 2n - 3)$ of [Li26b] meets the region of Theorem 1.3 in the 26 benzels with $6 \leq n \leq 31$; these cases are covered here by certificates, independently of those preprints.

5 Evidence on ring inductions near the line $I = 0$

This section records evidence, not proofs. A natural approach to Problem 1.1, and the one that answers Problem 4 [Li26a], is induction: place a translate $B(a, b) + \tau$ of a smaller benzel inside $B(a', b')$ and tile the ring $B(a', b') \setminus (B(a, b) + \tau)$. If $a \leq b \leq 2a - 2$ (the $(n, 2n)$ - and $(n, 2n - 1)$ -benzels are the $(n, 2n - 2)$ -benzel), each of the coordinates u, v, w takes the values $-(a - 1)$ and $b - 1$ on $B(a, b)$; then such a translate exists only if $a' \geq a$ and $b' \geq b$, because the conditions $-(a' - 1) \leq -(a - 1) + \tau_x$ and $b - 1 + \tau_x \leq b' - 1$ for the three coordinate shifts τ_x , which have sum 0, give $3(a - a') \leq 0 \leq 3(b' - b)$. If the smaller benzel has a type-103 tiling, a type-103 tiling of the ring gives one of $B(a', b')$, and the ring then contains exactly $\frac{1}{3}(I(a', b') - I(a, b))$ right stones.

Let $a + b \equiv 0 \pmod{3}$, $b = a + d$, and $(a', b') = (a + x, b + y)$ with $x, y \geq 0$, $x + y \equiv 0 \pmod{3}$ and $(x, y) \neq (0, 0)$. By (2),

$$2(I(a + x, b + y) - I(a, b)) = 3(y - x)(2d + y - x) - (x + y).$$

If $x - y \leq 2d$, this is nonnegative only if $y > x$. (If $x - y > 2d$, the new pair has $b' - a' < -d$; it is the mirror image of a benzel with a larger difference.) In particular the collar of Problem 4, $x = y = 3$, carries one left stone and cannot be used for type 103. The simplest moves with $y > x$ are $(x, y) = (3k + 1, 3k + 2)$, $k \geq 0$; they change (d, J) to $(d + 1, J + k)$, and the ring contains $d - k$ right stones, so $k \leq d$.

With the program `tring_test.py` of the original computation we tested, by SAT, every translation τ with $B(a, b) + \tau \subset B(a + 3k + 1, b + 3k + 2)$, for $k = 0, 1$ with $2 \leq d \leq 5$ and for $k = 2$ with $d = 3, 4$, and for all inner benzels with $0 \leq J \leq 7$ (here $J = (a - d)/3$, and $J = 0$ is $b = 2a$). In every tested case the ring has a type-103 tiling for some τ if and only if $J \leq k + 1$. The third verification run (Section 6) repeated these 80 tests with its own SAT encoding, computing the admissible translations from the cell sets, and found the same pattern. If this pattern persists, such a move reaches only $J + k \leq 2k + 1 \leq 2d + 1$ for the new pair, while the slice $b - a = d + 1$ of $\mathcal{R}_7$ contains values of J up to $\frac{(d-1)(d+2)}{2}$. Thus moves of this kind would stay far from the line $I = 0$, near which a tiling consists

almost entirely of bones. The construction of Section 3 is of a different kind: its three sectors of parallel bones reach from the central triangle to the boundary of the benzel. We do not know whether the family $(4D - 3, 5D - 3)$ can be connected to other parts of $\mathcal{R}_7$ by ring moves.

6 Computations and reproducibility

The package of this note (folder `reproducibility/` of the source archive) contains the following; its `README.md` gives the command lines and running times.

The program of this note, `check_note.py` (standard library only, a few seconds). It checks: (1) the area formula [Pro24, §1] against the number of cells for all admissible $a, b \leq 30$, the coincidence of the $(n, 2n)$ -, $(n, 2n - 1)$ - and $(n, 2n - 2)$ -benzels for $n \leq 30$, the mirror map of the proof of Proposition 2.2, and the classification of Proposition 2.2 for all admissible $a \leq b \leq 120$, including the description (3) of $\mathcal{R}_7$ and the counts 107 and 206; (2) 18 values of T_{103} and 18 of T_{013} from Propp's tables [Pro24, §§4, 6], by exact counting (this fixes which stone is the right one: $T_{103}(3, 3) = 0$ and $T_{013}(3, 3) = 3$); (3) the construction of Theorem 1.2 for $2 \leq D \leq 40$, with Lemma 3.1, the three cases of Lemma 3.2(a), Lemma 3.3 and the exact cover; (4) all 223 certificate files and the statements of Table 1; (5) the identities of Section 5 and of Remark 3.4(c). The program `make_figure.py` draws Figure 1.

The programs with which the results were first obtained (original/): the encoding and the exact-cover counter (checked against Propp's tables of T_{013} and T_{103} for $a, b \leq 10$), the SAT searches that produced `certificates/original`, the checker `verify_tilings.py`, the search that suggested Theorem 1.2, a construction of the family for $D \leq 16$, the Z3 certificates of Remark 3.4(b) with their negative control, and the ring tests of Section 5, with their recorded outputs.

Three independent verification runs, with their programs (and, for the second and third runs, their recorded outputs), in the folders `verification_run_A/`, `verification_run_B/` and `independent_run_2/` (the third run, made on the text of this note); see the next paragraph.

Verification. The results were first obtained with the programs in `original/`: Theorem 1.2 with the Z3 certificates and a construction for $D \leq 16$, and Theorem 1.3 for the 132 benzels of $\mathcal{R}_7$ in `certificates/original`. Three independent verification runs, all AI-assisted, followed; each wrote its own programs from Propp's definitions. The first two examined the computation before this note was written; the third examined the text of the note.

The first run read Propp's definitions and Problems 4–7 in the arXiv source of [Pro24] and checked the reduction to $\mathcal{R}_7$ against the statements of [DLPY23, DFL⁺25, KP23] in their arXiv sources. It re-derived the proof of Theorem 1.2, including the case analysis of Lemma 3.2(a), and gave the direct proof of Lemma 3.3 used here. Its own exact-cover counter reproduced Propp's tables of T_{103} and T_{013} for $a, b \leq 7$; it re-ran the Z3 certificates, built the tiling of Theorem 1.2 with its own program for $2 \leq D \leq 30$, and accepted all 136 original certificate files with its own checker.

The second run gave a proof of Theorem 1.2 without a solver, along the same lines. Its own programs reproduced Propp's tables of T_{103} , T_{013} and T_{102} for $a, b \leq 10$ and confirmed the invariant formula on 468 tilings by stones and bones found at random. It built the tiling of Theorem 1.2 for $2 \leq D \leq 40$ and accepted the 136 original files with its own checker, which rejected every corrupted file it was tested on (a tile removed, duplicated or shifted, two tiles mixed, a phone, a tiling of another benzel, a tiling with left stones). It then searched, by SAT, for type-103 tilings of the 87 benzels of $\mathcal{R}_7$ with $46 \leq b \leq 60$ that had no certificate; it found one for each of them (these are the files of `certificates/verification_run_B`) and no case without a tiling. Both runs compared the statements quoted from [Li26b, Li26a, The26] with these texts.

The third run read the sources again: Propp's definitions, tables and Problems 4 and 7 in the arXiv source of [Pro24], his status page, the statements of the theorems of [DLPY23, DFL⁺25, KP23]

used in Proposition 2.2 in their arXiv sources, and the statements of [Li26b, Li26a, The26] that are quoted here. It checked every proof of Sections 2 and 3 line by line, including the three cases of Lemma 3.2(a). Its own programs reproduced the classification of Proposition 2.2 for $a \leq b \leq 120$, 18 values each of T_{103} and T_{013} by exact counting, and the sign pattern of Propp’s whole table of T_{103} ; built the tiling of Theorem 1.2 for $2 \leq D \leq 30$, cutting each sector along its own lines, and checked it with its own checker; and accepted all 223 certificate files with that checker, which rejected each corrupted file it was tested on. It recomputed the coverage of Table 1, repeated the ring tests of Section 5 with its own SAT encoding, and re-ran the programs of the package from the source archive, with the recorded outputs. Its programs and outputs are in `independent_run_2/`.

All corrections required by the first two runs were made: the published version of [Pro24] is cited; the proof that each sector meets each line in consecutive cells (Lemma 3.2(a)), which the Z3 formulas presuppose, was added, and so was the direct proof of Lemma 3.3; the row $(n, 2n - 3)$ is described as covered only by unrefereed preprints; the condition $x - y \leq 2d$ was added in Section 5; and the 87 certificates of the second run were added after the checkers of this package and of the first run had accepted them. The corrections required by the third run were also made: the ring evidence is stated at its actual scope, the remark on translates in Section 5 is restricted to $a \leq b \leq 2a - 2$, the parameter of [Li26b, Theorem 7.2] is named $2a - b$ in Remark 3.4(c), and the descriptions of the preprints, of what was read and of the version records were made more precise. The program of this note, `check_note.py`, passes all its checks. No proof in Sections 2 and 3 depends on a computation; Theorem 1.3 is a computation.

Scope and priority. The note gives a partial answer to Problem 7 of [Pro24]: Theorems 1.2 and 1.3 and Corollary 1.4. The problem remains open in general: among the benzels of $\mathcal{R}_7$ with $b \geq 61$ and $b - a \geq 8$, only the family of Theorem 1.2 and eight benzels with $b - a = 8$ are settled here. The problem, the definitions, the tables and the formulas for the area and the invariant are Propp’s [Pro24]; the invariant is due to Conway and Lagarias [CL90]; the reduction to $\mathcal{R}_7$ in Proposition 2.2 combines theorems of [DLPY23, DFL⁺25, KP23] with remarks of [Pro24], and Lemma 3.3 is a special case of [DLPY23, Theorem 1.1]. The row $(n, 2n - 3)$ of $\mathcal{R}_7$ and the window $\max(a, b) \leq 14$ are covered by the unrefereed preprints [Li26b, The26], and Problem 4 is answered in the unrefereed preprint [Li26a]; we report their statements, have not checked their proofs, and use none of them. Our own collar induction for Problem 4 is the same idea as that of [Li26a] and is not claimed. Only the family of Theorem 1.2 and the certificates of Theorem 1.3 for the benzels outside the window $\max(a, b) \leq 14$ and outside the row $(n, 2n - 3)$ are, to our knowledge, not in print.

What was read: of [Pro24], the arXiv version v4 (the published version was not compared), its Sections 1–6, which contain the definitions, the tables and Problems 1–7; of [DLPY23, DFL⁺25, KP23], the statements of the theorems used, in the arXiv sources; of [Li26b] (version v6) and [Li26a] (version 1.0.2), the full texts of the PDF files; of [The26], the L^AT_EX source; of [BCL25], the abstract and the opening of its section on Problems 8–11 (tilings without vertical bones), in the arXiv source; and Propp’s status page [Pro25]. We did not read [CL90]; what we say about it is taken from [Pro24, §1]. The proofs of the cited theorems were not checked.

The searches were made on 18 and 19 October 2026: queries to the arXiv API (benzel, benzels, tribone, “stones and bones”, trihex, “Conway–Lagarias invariant”, trimer tilings, trimer covers), to OpenAlex (benzel and benzel tilings, and the works citing [Pro24, DLPY23, DFL⁺25, KP23, BCL25]), to the Zenodo API (benzel), to zbMATH (benzel, tribone), to the OEIS (benzel), to Crossref, and three web searches; the Semantic Scholar service refused the automated requests (HTTP status 429), so its citation lists were not used. No printed answer to the question was found in the sources searched, and the family of Theorem 1.2 was not found in [Pro24, DLPY23, DFL⁺25, KP23, BCL25, Li26b, Li26a, The26]. A search that finds nothing is not a proof of novelty; we make no priority claim.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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