

On Propp’s Problem 12: The Sum of the Entries of the Inverse Kasteleyn–Percus Matrix of the Aztec Diamond

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Abstract

Let K_n be the Kasteleyn–Percus matrix of the Aztec diamond of order n : the adjacency matrix between its black and its white squares in which the entry of every vertical domino whose white square is on top is replaced by -1 . In Problem 12 of his list of open problems in the enumeration of matchings (1996, revised 1998; the sign convention is fixed in the version of 1999, where the problem is still open), J. Propp asked for a proof that the sum of the entries of K_n^{-1} is $\frac{1}{2}(n-1)(n+3) - 2^{n-1} + 2$, a formula he had checked for $n \leq 8$. We prove it for every $n \geq 1$. The matrix K_n maps the constant function $\frac{1}{2}$ on the white squares to the constant 1, except on one edge of the diamond; in coordinates rotated by 45° the remaining equation has an explicit solution whose values are the Krawtchouk coefficients $-\frac{1}{2}[\xi^m](1-\xi)^k(1+\xi)^{n-k}$, and two transpositions reduce the entry sum to $\frac{n(n+1)}{2} + \frac{n}{2} - \frac{2^n-1}{2} = \frac{(n+1)^2}{2} - 2^{n-1}$. The argument also gives every row sum and every column sum of K_n^{-1} as an explicit Krawtchouk coefficient. The problem and the formula are Propp’s; the matrix and the determinant method are due to Kasteleyn and Percus; the Aztec diamond and the number $2^{n(n+1)/2}$ of its domino tilings to Elkies, Kuperberg, Larsen and Propp; explicit formulas for the individual entries of the inverse, as sums of products of Krawtchouk coefficients, to Helfgott (1998, posted 2000), whose coupling function differs from the inverse by an explicit sign factor, extended to different weights for horizontal and vertical dominoes by Chhita, Johansson and Young. These works do not compute the entry sum or the row sums. What the note adds is an elementary proof of Propp’s formula and the row and column sums. Computations show that the entry sum depends on the choice of the Kasteleyn signs, so that the convention of the 1999 text is essential. This is an unrefereed note.

1 Introduction

1.1 The problem

For $n \geq 1$ the *Aztec diamond* A_n of order n [EKLP92a, EKLP92b] is the union of the $2n(n+1)$ closed unit squares whose centres (x, y) lie in $(\mathbb{Z} + \frac{1}{2})^2$ and satisfy $|x| + |y| \leq n$. We call a square *black* if $x + y$ is odd and *white* if $x + y$ is even; adjacent squares have different colours, and there are $n(n+1)$ squares of each colour (Lemma 2.1). Let $\mathcal{B}$ and $\mathcal{W}$ be the sets of black and white squares. A *domino* is a pair of squares with a common side; it is *vertical* if the centres are (x, y) and $(x, y+1)$, and then the square with centre $(x, y+1)$ is the one *on top*.

The matrix of the problem is the *Kasteleyn–Percus matrix* $K = K_n \in \{0, 1, -1\}^{\mathcal{B} \times \mathcal{W}}$ of A_n in the

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convention of [Pro99]:

$$K(b, w) = \begin{cases} -1 & \text{if } \{b, w\} \text{ is a vertical domino whose white square } w \text{ is on top,} \\ 1 & \text{if } b \text{ and } w \text{ are adjacent otherwise,} \\ 0 & \text{if } b \text{ and } w \text{ are not adjacent.} \end{cases} \quad (1)$$

So K_n is the bipartite adjacency matrix of A_n (rows: black squares, columns: white squares) in which the entry of every vertical domino with its white square on top is replaced by -1 . It is invertible (Proposition 2.2), and $K_n^{-1} \in \mathbb{Q}^{\mathcal{W} \times \mathcal{B}}$.

Problem 1.1 (Propp [Pro98a, Problem 12], [Pro99, Problem 12]). *Show that the sum of the entries of K_n^{-1} is $\frac{1}{2}(n-1)(n+3) - 2^{n-1} + 2$.*

The 1998 text describes the matrix as the Kasteleyn matrix in which “every other vertical domino has its sign flipped” [Pro98a, Problem 12]; the 1999 text puts the -1 on “every vertical domino with its white square on top”, relative to a fixed checkerboard colouring [Pro99]. The remaining choices do not matter (Lemma 2.3), and the 1998 wording admits a second reading, which gives other sums and is excluded by Propp’s data (Remark 2.4). Propp states that the formula works for $n = 1$ through $n = 8$; he prints no values. In the progress paragraph of 1999 he reports that Helfgott “has solved a similar problem” with the main result of his thesis [Hel00], that Problem 12 can probably be proved in the same way, and that Helfgott’s sign convention gives other signs and “a different sum” for the inverse; no proof is given. Propp’s progress report of September 1998 [Pro98b] has no entry for Problem 12, and Lai’s collection of new problems [Lai24] does not include the still-open problems of the 1999 list.

1.2 Results

Theorem 1.2. *For every $n \geq 1$,*

$$S_n := \sum_{w \in \mathcal{W}} \sum_{b \in \mathcal{B}} K_n^{-1}(w, b) = \frac{(n+1)^2}{2} - 2^{n-1} = \frac{(n-1)(n+3)}{2} - 2^{n-1} + 2.$$

n	1	2	3	4	5	6	7	8	9	10	11	12
S_n	1	$\frac{5}{2}$	4	$\frac{9}{2}$	2	$-\frac{15}{2}$	-32	$-\frac{175}{2}$	-206	$-\frac{903}{2}$	-952	$-\frac{3927}{2}$

Table 1: The values of the formula. For $n \leq 8$ they are also obtained by exact inversion of K_n (Section 7).

The proof gives more. To state it we use the coordinates

$$s := n + \epsilon x + y, \quad t := n + \epsilon x - y, \quad \epsilon := (-1)^{n+1}, \quad (2)$$

of the square with centre (x, y) . They are integers in $[0, 2n]$; s is even and t is odd for black squares, and s is odd and t is even for white squares (Lemma 4.1). For $0 \leq k \leq n$ and $m \in \mathbb{Z}$ let

$$\kappa_n(m, k) := [\xi^m] (1 - \xi)^k (1 + \xi)^{n-k} \quad (3)$$

be the coefficient of ξ^m . Up to normalisation these are the values of the Krawtchouk polynomials; $\kappa_n(m, k)$ is the number $\text{kr}(m, n, k)$ of Helfgott [Hel00, Main Result 1].

Corollary 1.3. *For every $n \geq 1$, the row sum of K_n^{-1} at the white square w with coordinates (s, t) and the column sum at the black square b with coordinates (s, t) are*

$$\sum_{b' \in \mathcal{B}} K_n^{-1}(w, b') = \frac{1 - \kappa_n\left(\frac{s+1}{2}, \frac{t}{2}\right)}{2}, \quad \sum_{w' \in \mathcal{W}} K_n^{-1}(w', b) = \frac{1 - \kappa_n\left(\frac{t+1}{2}, \frac{s}{2}\right)}{2}.$$

In particular, all row sums and all column sums of K_n^{-1} lie in $\frac{1}{2}\mathbb{Z}$.

1.3 The idea of the proof

The signs of Problem 12 fit the constant function: the entry -1 sits exactly on the dominoes formed by a black square and its northern neighbour. Hence K_n maps the constant $\frac{1}{2}$ on $\mathcal{W}$ to the constant 1 on $\mathcal{B}$, except on the n black squares of one edge Σ_S of the diamond, where it gives 0 (Lemma 3.1). So $K_n^{-1}\mathbf{1}_{\mathcal{B}} = \frac{1}{2}\mathbf{1}_{\mathcal{W}} + \varphi$ with $K_n\varphi = \mathbf{1}_{\Sigma_S}$, and the same observation for K_n^T reduces S_n to the sum of φ over the n white squares of an adjacent edge Σ_N (Proposition 3.2). In the coordinates (2), K_n is a difference operator with constant coefficients on a square, with Σ_S and Σ_N on two of its sides, and $\varphi(s, t) = -\frac{1}{2}\kappa_n\left(\frac{s+1}{2}, \frac{t}{2}\right)$ by two one-line polynomial identities (Proposition 4.3); on Σ_N the values are $-\frac{1}{2}\binom{n}{m}$, $m = 1, \dots, n$. The only input from the literature is the invertibility of K_n , which Problem 1.1 presupposes (Proposition 2.2).

1.4 Related work and credit

Kasteleyn [Kas61] counted dimer arrangements on the square lattice with signed Pfaffians and treated general planar graphs in his exposition [Kas67], and Percus [Per69] gave a determinant version for bipartite graphs; Propp calls the resulting matrices Kasteleyn–Percus matrices [Pro99]. The Aztec diamond and the number $2^{n(n+1)/2}$ of its domino tilings are due to Elkies, Kuperberg, Larsen and Propp [EKLP92a, EKLP92b]. Minors of the inverse of a Kasteleyn matrix give the probabilities of local patterns in a random tiling [Ken97], [Hel00, Section 2], which is Propp’s motivation for Problem 12. Helfgott [Hel00] (a Brandeis thesis of 1998, posted in 2000) computed every entry of the inverse Kasteleyn matrix of A_n as 2^{-n} times a sum of products of two Krawtchouk coefficients, with a sign factor that his coupling function omits (Remark 6.2); Chhita, Johansson and Young [CJY15] extended this to different weights for horizontal and vertical dominoes, and Chhita and Young [CY14] computed the inverse for three weightings by recurrence relations. None of these works computes the entry sum or the row sums of the inverse, or mentions Problem 12. Our proof does not use Helfgott’s formula; it solves one linear system, whose solution has single Krawtchouk coefficients as values. Scope, credits and the searches behind these statements are described in Section 7.

2 The matrix

2.1 Coordinates

For a square c of A_n with centre (x, y) let $E(c)$, $W(c)$, $N(c)$, $S(c)$ be the squares with centres $(x+1, y)$, $(x-1, y)$, $(x, y+1)$, $(x, y-1)$ (east, west, north, south), when they belong to A_n , and put

$$p := x + y + n, \quad q := x - y + n.$$

Lemma 2.1. *(a) The map $(x, y) \mapsto (p, q)$ is a bijection from the squares of A_n onto the integer points of $[0, 2n]^2$ with $p + q$ odd. A square is black if and only if $p \equiv n + 1 \pmod{2}$. There are $n(n+1)$ black and $n(n+1)$ white squares.*

- (b) In these coordinates $E(c) = (p+1, q+1)$, $W(c) = (p-1, q-1)$, $N(c) = (p+1, q-1)$ and $S(c) = (p-1, q+1)$, and each of them belongs to A_n if and only if its coordinates lie in $[0, 2n]^2$.
- (c) The four edges $\{p=0\}$ (south-west), $\{p=2n\}$ (north-east), $\{q=0\}$ (north-west) and $\{q=2n\}$ (south-east) of A_n consist of n squares each, and no square lies on two of them. A square on $\{p=0\}$ has exactly the neighbours E and N ; on $\{p=2n\}$ exactly W and S ; on $\{q=0\}$ exactly E and S ; on $\{q=2n\}$ exactly W and N . Every other square has all four neighbours.
- (d) The squares on $\{p=0\}$ and on $\{p=2n\}$ are black if n is odd and white if n is even; the squares on $\{q=0\}$ and on $\{q=2n\}$ are white if n is odd and black if n is even.

Proof. (a) Since $x, y \in \mathbb{Z} + \frac{1}{2}$, the numbers p, q are integers and $p+q = 2x+2n$ is odd. As $|x|+|y| = \max(|x+y|, |x-y|)$, the condition $|x|+|y| \leq n$ is equivalent to $0 \leq p, q \leq 2n$. Conversely, for integers p, q with $p+q$ odd, $x = \frac{p+q}{2} - n$ and $y = \frac{p-q}{2}$ lie in $\mathbb{Z} + \frac{1}{2}$. The square is black if and only if $x+y = p-n$ is odd. If n is odd, the black squares are the points with p even ($n+1$ values) and q odd (n values); if n is even, those with p odd (n values) and q even ($n+1$ values). The white squares are counted in the same way. (b) is a substitution. (c) A square on $\{p=0\}$ has q odd, so $1 \leq q \leq 2n-1$; then $E = (1, q+1)$ and $N = (1, q-1)$ lie in $[0, 2n]^2$, while W and S have $p = -1$. The other three edges are treated in the same way, and each edge has n squares because the free coordinate is odd and lies in $[1, 2n-1]$. A square on two edges would have $p+q \in \{0, 2n, 4n\}$, which is even. (d) By (a), the squares with $p \in \{0, 2n\}$ are black if and only if $n+1$ is even. On $\{q=0\}$ and $\{q=2n\}$ the coordinate p is odd, so these squares are black if and only if $n+1$ is odd. $\square$

2.2 The matrix and its determinant

For functions u on $\mathcal{W}$ and v on $\mathcal{B}$, (1) reads

$$(K_n u)(b) = u(Eb) + u(Wb) + u(Sb) - u(Nb), \quad (K_n^T v)(w) = v(Ew) + v(Ww) + v(Nw) - v(Sw), \quad (4)$$

where a term is omitted when the neighbour does not exist. The second formula holds because $w = N(b)$ if and only if $b = S(w)$.

Proposition 2.2 (Kasteleyn, Percus; Elkies, Kuperberg, Larsen, Propp). $|\det K_n| = 2^{n(n+1)/2}$. In particular K_n is invertible.

Proof. Consider the plane graph G_n whose vertices are the centres of the squares of A_n and whose edges join the centres of the two squares of a domino. Every edge of G_n is a side of a unit square spanned by the centres of a 2×2 block of squares of A_n (for a vertical domino with centres (x, y) , $(x, y+1)$, add the column $x - \text{sgn}(x)$; for a horizontal domino, add the row $y - \text{sgn}(y)$). These unit squares are the squares $[i - \frac{1}{2}, i + \frac{1}{2}] \times [j - \frac{1}{2}, j + \frac{1}{2}]$ with $(i, j) \in \mathbb{Z}^2$ and $|i|+|j| \leq n-1$, and their union is star-shaped with respect to the origin. Hence the bounded faces of G_n are exactly these $n^2 + (n-1)^2$ unit squares. The block of squares around such a face contains two vertical dominoes; their top squares are horizontally adjacent, hence of different colours, so exactly one of the four dominoes of the block carries the entry -1 in K_n . Thus the product of the signs around every bounded face is -1 , which is Kasteleyn's condition for faces of length four, and by Kasteleyn's theorem [Kas61, Kas67] in the determinant form of Percus [Per69] the number $|\det K_n|$ is the number of domino tilings of A_n . This number is $2^{n(n+1)/2}$ [EKLP92a, EKLP92b]. $\square$

Problem 1.1 presupposes the invertibility of K_n . Below it is used only to identify explicit solutions of $K_n u = f$ and $K_n^T v = g$ with $K_n^{-1} f$ and $(K_n^T)^{-1} g$. The program of this note also computes $|\det K_n| = 2^{n(n+1)/2}$ exactly for $n \leq 8$ (Section 7).

2.3 Conventions

Lemma 2.3. *The sum of the entries of the inverse is the same for K_n , for its transpose, for the matrix K'_n with the entry -1 on the vertical dominoes whose white square is at the bottom, and for the two matrices obtained from K_n and K'_n by exchanging the two colours.*

Proof. The entry sum of $(M^\top)^{-1} = (M^{-1})^\top$ is that of M^{-1} , and the entry sum of $(PMQ)^{-1} = Q^{-1}M^{-1}P^{-1}$ is that of M^{-1} for permutation matrices P, Q . The rotation $(x, y) \mapsto (-x, -y)$ maps A_n onto itself, preserves the parity of $x + y$ and hence the colours, and maps a vertical domino with its white square on top to one with its white square at the bottom; so $K'_n = PK_nQ$ for permutation matrices P, Q . Exchanging the colours turns “white square on top” into “black square on top”, which is “white square at the bottom” for the original colours, and exchanges rows and columns; so it turns K_n into $K_n'^\top$ and K'_n into $K_n^\top$. $\square$

Remark 2.4 (the wording of 1998). In every row and in every column of A_n the vertical dominoes with their white square on top are every other one, so K_n fits the description “every other vertical domino has its sign flipped” of [Pro98a]. So does the matrix with -1 on all vertical dominoes of every other column. That matrix also satisfies Kasteleyn’s condition of Proposition 2.2, but the entry sums of its inverse are $1, \frac{5}{2}, 4, 6, 9, \frac{25}{2}, 16, 20$ for $n = 1, \dots, 8$, for either choice of the columns (Remark 6.1); they differ from Propp’s formula for $n = 4, \dots, 8$. This reading is therefore excluded by the 1999 text and by Propp’s statement that the formula works for $n \leq 8$. By Lemma 2.3, the choices left open in the 1999 convention do not affect the sum.

3 Reduction to a boundary block

Let Σ_S be the set of black squares on the edges $\{p = 0\}$ and $\{q = 2n\}$, and Σ_N the set of white squares on the edges $\{p = 2n\}$ and $\{q = 0\}$. By Lemma 2.1(d),

$$\Sigma_S = \begin{cases} \{p = 0\} & (n \text{ odd}), \\ \{q = 2n\} & (n \text{ even}), \end{cases} \quad \Sigma_N = \begin{cases} \{q = 0\} & (n \text{ odd}), \\ \{p = 2n\} & (n \text{ even}), \end{cases}$$

and each of these sets has n squares. Thus Σ_S is the south-west edge and Σ_N the north-west edge if n is odd, and Σ_S is the south-east edge and Σ_N the north-east edge if n is even; in both cases the two edges meet at a corner of A_n . We write $\mathbf{1}_X$ for the indicator function of a set X of squares.

Lemma 3.1. $K_n(\frac{1}{2}\mathbf{1}_W) = \mathbf{1}_B - \mathbf{1}_{\Sigma_S}$ and $K_n^\top(\frac{1}{2}\mathbf{1}_B) = \mathbf{1}_W - \mathbf{1}_{\Sigma_N}$.

Proof. By (4), $K_n(\frac{1}{2}\mathbf{1}_W)(b)$ is $\frac{1}{2}$ times the number of the neighbours Eb, Wb, Sb that exist, minus $\frac{1}{2}$ if Nb exists. By Lemma 2.1(c) this is $\frac{1}{2}(3 - 1) = 1$ for a square with four neighbours, $\frac{1}{2}(1 + 1) = 1$ on $\{p = 2n\}$ (neighbours W, S) and on $\{q = 0\}$ (E, S), and $\frac{1}{2}(1 - 1) = 0$ on $\{p = 0\}$ (E, N) and on $\{q = 2n\}$ (W, N). The black squares among the last are those of Σ_S . In the same way $K_n^\top(\frac{1}{2}\mathbf{1}_B)(w)$ is $\frac{1}{2}$ times the number of the existing neighbours EW, Ww, Nw , minus $\frac{1}{2}$ if Sw exists: this is 1 for a square with four neighbours, 1 on $\{p = 0\}$ and on $\{q = 2n\}$, and 0 on $\{p = 2n\}$ (W, S) and on $\{q = 0\}$ (E, S), whose white squares form Σ_N . $\square$

Proposition 3.2. *Let $\varphi := K_n^{-1}\mathbf{1}_{\Sigma_S}$, a function on $\mathcal{W}$. Then*

$$S_n = \frac{n(n+1)}{2} + \frac{n}{2} + \mathbf{1}_{\Sigma_N}^\top K_n^{-1} \mathbf{1}_{\Sigma_S} = \frac{n(n+1)}{2} + \frac{n}{2} + \sum_{w \in \Sigma_N} \varphi(w).$$

Proof. By Lemma 3.1, $K_n^{-1}\mathbf{1}_B = \frac{1}{2}\mathbf{1}_W + K_n^{-1}\mathbf{1}_{\Sigma_S}$, and, transposing $\mathbf{1}_W = K_n^\top(\frac{1}{2}\mathbf{1}_B) + \mathbf{1}_{\Sigma_N}$, also $\mathbf{1}_W^\top K_n^{-1} = \frac{1}{2}\mathbf{1}_B^\top + \mathbf{1}_{\Sigma_N}^\top K_n^{-1}$. Hence

$$S_n = \mathbf{1}_W^\top K_n^{-1}\mathbf{1}_B = \frac{1}{2}|\mathcal{W}| + \mathbf{1}_W^\top K_n^{-1}\mathbf{1}_{\Sigma_S} = \frac{1}{2}|\mathcal{W}| + \frac{1}{2}|\Sigma_S| + \mathbf{1}_{\Sigma_N}^\top K_n^{-1}\mathbf{1}_{\Sigma_S},$$

and $|\mathcal{W}| = n(n+1)$, $|\Sigma_S| = n$. $\square$

4 The explicit solution

4.1 Corner coordinates

Lemma 4.1. *Let (s, t) be the coordinates (2).*

- (a) $(s, t) = (p, q)$ if n is odd and $(s, t) = (2n - q, 2n - p)$ if n is even. The map $c \mapsto (s, t)$ is a bijection from the squares of A_n onto the integer points of $[0, 2n]^2$ with $s + t$ odd; the black squares are those with s even (and t odd), the white squares those with s odd (and t even).
- (b) For every square c with coordinates (s, t) , $\mathbf{N}(c) = (s + 1, t - 1)$, $\mathbf{S}(c) = (s - 1, t + 1)$ and $\{\mathbf{E}(c), \mathbf{W}(c)\} = \{(s + 1, t + 1), (s - 1, t - 1)\}$, and each of them belongs to A_n if and only if its coordinates lie in $[0, 2n]^2$.
- (c) Σ_S is the set of squares with $s = 0$, and Σ_N is the set of squares with $t = 0$.

Proof. (a) For odd n , $\epsilon = 1$ and $(s, t) = (x + y + n, x - y + n) = (p, q)$; for even n , $\epsilon = -1$ and $(s, t) = (n - x + y, n - x - y) = (2n - q, 2n - p)$. The map $(p, q) \mapsto (2n - q, 2n - p)$ is the reflection of $[0, 2n]^2$ in its anti-diagonal; it preserves $[0, 2n]^2$ and the parity of $p + q$. By Lemma 2.1(a) a square is black if and only if $p \equiv n + 1 \pmod{2}$: for odd n this means that $s = p$ is even, and for even n that p is odd, that is, $t = 2n - p$ is odd and s is even. (b) The map $(x, y) \mapsto (s, t)$ is affine with linear part $(x, y) \mapsto (\epsilon x + y, \epsilon x - y)$; so $\mathbf{N}$ and $\mathbf{S}$ change (s, t) by $\pm(1, -1)$, and $\mathbf{E}$, $\mathbf{W}$ change it by $\pm(\epsilon, \epsilon)$. The existence criterion follows from Lemma 2.1(b), since $(s, t) \in [0, 2n]^2$ if and only if $(p, q) \in [0, 2n]^2$. (c) For odd n , $\Sigma_S = \{p = 0\} = \{s = 0\}$ and $\Sigma_N = \{q = 0\} = \{t = 0\}$; for even n , $\Sigma_S = \{q = 2n\} = \{s = 0\}$ and $\Sigma_N = \{p = 2n\} = \{t = 0\}$. $\square$

We identify a square with its coordinates (s, t) and read a function on squares as 0 at points outside $[0, 2n]^2$. Since $\mathbf{E}$ and $\mathbf{W}$ enter (4) in the same way, Lemma 4.1(b) gives, for functions u on $\mathcal{W}$, v on $\mathcal{B}$, black squares (s, t) and white squares (s', t') ,

$$(K_n u)(s, t) = u(s + 1, t + 1) + u(s - 1, t - 1) + u(s - 1, t + 1) - u(s + 1, t - 1), \quad (5)$$

$$(K_n^\top v)(s', t') = v(s' + 1, t' + 1) + v(s' - 1, t' - 1) + v(s' + 1, t' - 1) - v(s' - 1, t' + 1). \quad (6)$$

4.2 Krawtchouk coefficients

Let $\kappa_n(m, k)$ be as in (3); it vanishes unless $0 \leq m \leq n$. For $0 \leq k \leq n - 1$ and $m \in \mathbb{Z}$ put $\lambda(m, k) := \kappa_{n-1}(m, k) = [\xi^m](1 - \xi)^k(1 + \xi)^{n-1-k}$.

Lemma 4.2. *For $0 \leq k \leq n - 1$ and every $m \in \mathbb{Z}$,*

$$\kappa_n(m, k) + \kappa_n(m, k + 1) = 2\lambda(m, k), \quad \kappa_n(m + 1, k + 1) - \kappa_n(m + 1, k) = -2\lambda(m, k), \quad (7)$$

and $\lambda(0, k) = 1$, $\lambda(n, k) = 0$.

Proof. Put $R(\xi) := (1 - \xi)^k(1 + \xi)^{n-1-k}$. Then $(1 - \xi)^k(1 + \xi)^{n-k} = (1 + \xi)R(\xi)$ and $(1 - \xi)^{k+1}(1 + \xi)^{n-k-1} = (1 - \xi)R(\xi)$, so their sum is $2R(\xi)$ and the second minus the first is $-2\xi R(\xi)$. Taking the coefficient of ξ^m in the sum and of ξ^{m+1} in the difference gives (7). Finally $R(0) = 1$, and R has degree $n - 1$. $\square$

4.3 The solution

Proposition 4.3. *The function φ on $\mathcal{W}$ defined by*

$$\varphi(s, t) := -\frac{1}{2} \kappa_n\left(\frac{s+1}{2}, \frac{t}{2}\right) \quad (s \text{ odd}, 1 \leq s \leq 2n-1; t \text{ even}, 0 \leq t \leq 2n)$$

satisfies $K_n \varphi = \mathbf{1}_{\Sigma_S}$. Hence $\varphi = K_n^{-1} \mathbf{1}_{\Sigma_S}$.

Proof. Let b be a black square, with coordinates $(s, t) = (2m, 2k+1)$, where $0 \leq m \leq n$ and $0 \leq k \leq n-1$. Since $0 \leq t-1$ and $t+1 \leq 2n$, the points $(s+1, t \pm 1)$ lie in $[0, 2n]^2$ if and only if $m \leq n-1$, and the points $(s-1, t \pm 1)$ if and only if $m \geq 1$. By definition,

$$\begin{aligned} \varphi(s+1, t+1) &= -\frac{1}{2} \kappa_n(m+1, k+1), & \varphi(s+1, t-1) &= -\frac{1}{2} \kappa_n(m+1, k), \\ \varphi(s-1, t-1) &= -\frac{1}{2} \kappa_n(m, k), & \varphi(s-1, t+1) &= -\frac{1}{2} \kappa_n(m, k+1). \end{aligned}$$

By (5) and Lemma 4.2, with $[\cdot]$ the indicator of a condition,

$$\begin{aligned} (K_n \varphi)(b) &= -\frac{1}{2} [m \leq n-1] (\kappa_n(m+1, k+1) - \kappa_n(m+1, k)) - \frac{1}{2} [m \geq 1] (\kappa_n(m, k) + \kappa_n(m, k+1)) \\ &= [m \leq n-1] \lambda(m, k) - [m \geq 1] \lambda(m, k). \end{aligned}$$

This is 0 for $1 \leq m \leq n-1$, it is $\lambda(0, k) = 1$ for $m = 0$, and it is $-\lambda(n, k) = 0$ for $m = n$ (for $n = 1$ only the last two cases occur). Since $m = 0$ means $s = 0$, that is, $b \in \Sigma_S$ (Lemma 4.1(c)), we get $K_n \varphi = \mathbf{1}_{\Sigma_S}$; and K_n is invertible (Proposition 2.2). $\square$

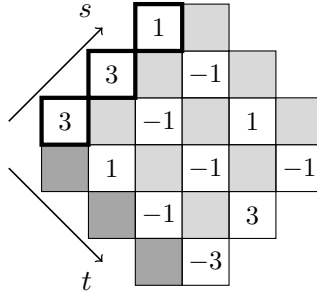


Figure 1: The Aztec diamond A_3 ; black squares are shaded. Each white square (s, t) carries the integer $\kappa_3(\frac{s+1}{2}, \frac{t}{2}) = -2\varphi(s, t)$ of Proposition 4.3. The dark squares form Σ_S ($s = 0$), the white squares with thick borders form Σ_N ($t = 0$), where the values are $\binom{3}{1}, \binom{3}{2}, \binom{3}{3}$. At every black square b , the values at Eb, Wb, Sb minus the value at Nb add up to -2 if $b \in \Sigma_S$ and to 0 otherwise. The arrows show the directions in which s and t increase. For even n , x is replaced by $-x$ in (2), and Σ_S, Σ_N lie on the two eastern edges.

5 Proofs of the theorem and of the corollary

Proof of Theorem 1.2. By Lemma 4.1(c), Σ_N consists of the white squares $(s, 0)$ with $s = 1, 3, \dots, 2n-1$. By Proposition 4.3,

$$\varphi(s, 0) = -\frac{1}{2} \kappa_n(m, 0) = -\frac{1}{2} [\xi^m] (1 + \xi)^n = -\frac{1}{2} \binom{n}{m}, \quad m = \frac{s+1}{2} \in \{1, \dots, n\},$$

so that $\sum_{w \in \Sigma_N} \varphi(w) = -\frac{1}{2}(2^n - 1)$. By Proposition 3.2,

$$S_n = \frac{n(n+1)}{2} + \frac{n}{2} - \frac{2^n - 1}{2} = \frac{n^2 + 2n + 1}{2} - 2^{n-1} = \frac{(n+1)^2}{2} - 2^{n-1},$$

and $\frac{(n-1)(n+3)}{2} + 2 = \frac{n^2 + 2n + 1}{2}$. $\square$

In particular, the block of K_n^{-1} with rows in Σ_N and columns in Σ_S has the entry sum $\frac{1}{2} - 2^{n-1}$.

Proof of Corollary 1.3. The rows of K_n^{-1} are indexed by $\mathcal{W}$. By Lemma 3.1 and Proposition 4.3, the row sum at the white square $w = (s, t)$ is

$$(K_n^{-1} \mathbf{1}_B)(w) = \frac{1}{2} + \varphi(w) = \frac{1}{2} (1 - \kappa_n(\frac{s+1}{2}, \frac{t}{2})).$$

The column sum at a black square b is $(\mathbf{1}_W^T K_n^{-1})(b) = \frac{1}{2} + \psi(b)$, where $\psi := (K_n^T)^{-1} \mathbf{1}_{\Sigma_N}$, again by Lemma 3.1. We claim that $\psi(s, t) = \varphi(t, s)$ for every black square (s, t) . The reflection $\sigma(s, t) := (t, s)$ maps $[0, 2n]^2$ onto itself and the white squares onto the black squares (Lemma 4.1(a)). Let $u := \psi \circ \sigma$, a function on $\mathcal{W}$. For a black square $b = (s, t)$, (5) and (6) at the white square $\sigma(b) = (t, s)$ give

$$(K_n u)(b) = \psi(t+1, s+1) + \psi(t-1, s-1) + \psi(t+1, s-1) - \psi(t-1, s+1) = (K_n^T \psi)(t, s) = \mathbf{1}_{\Sigma_N}(t, s),$$

which is 1 if $s = 0$ and 0 otherwise (Lemma 4.1(c)). So $K_n u = \mathbf{1}_{\Sigma_S}$, hence $u = \varphi$ by Proposition 4.3 and the invertibility of K_n , that is, $\psi(t, s) = \varphi(s, t)$ for every white square (s, t) . Therefore the column sum at the black square $b = (s, t)$ is $\frac{1}{2} + \varphi(t, s) = \frac{1}{2} (1 - \kappa_n(\frac{t+1}{2}, \frac{s}{2}))$. Since the numbers $\kappa_n(m, k)$ are integers, all these sums lie in $\frac{1}{2}\mathbb{Z}$. $\square$

Remark 5.1 (a second evaluation). Proposition 4.3 also gives S_n without Proposition 3.2. By Lemma 3.1, $S_n = \mathbf{1}_W^T K_n^{-1} \mathbf{1}_B = \frac{n(n+1)}{2} + \sum_{w \in \mathcal{W}} \varphi(w)$, and

$$\sum_{w \in \mathcal{W}} \varphi(w) = -\frac{1}{2} \sum_{k=0}^n \sum_{m=1}^n \kappa_n(m, k) = -\frac{1}{2} \sum_{k=0}^n \left(\left[(1-\xi)^k (1+\xi)^{n-k} \right]_{\xi=1} - 1 \right) = -\frac{1}{2} (2^n - (n+1)),$$

because the sum of the coefficients of a polynomial is its value at 1, the constant coefficient here is 1, and the value at 1 is 2^n for $k = 0$ and 0 for $k \geq 1$. This gives $S_n = \frac{n(n+1)}{2} - \frac{2^n - n - 1}{2} = \frac{(n+1)^2}{2} - 2^{n-1}$ again. The two evaluations use different parts of φ : the values on Σ_N , and the values on all of $\mathcal{W}$.

6 Remarks: the choice of signs, and Helfgott's formula

The two remarks of this section report computations and readings of the literature; they are not used in the proofs.

Remark 6.1 (the entry sum depends on the signs). Any placement of -1 on a set of dominoes such that every 2×2 block of squares contains an odd number of them gives a matrix K' with $|\det K'| = 2^{n(n+1)/2}$, as in Proposition 2.2, but in general the entry sum of K'^{-1} is not S_n . The program of this note computes, exactly for $n \leq 8$: (a) for -1 on all vertical dominoes of every other column (either choice of the columns), the second reading of Remark 2.4, the sums $1, \frac{5}{2}, 4, 6, 9, \frac{25}{2}, 16, 20$; (b) for the signs that Helfgott [Hel00, Section 2.1] calls the Kasteleyn–Wilson matrix ($+1$ on horizontal dominoes, $(-1)^k$ on a vertical domino, where k is the number of squares to its left in its lower row, or in its upper row), the sums $1, \frac{5}{2}, \frac{11}{2}, 8, 13, \frac{31}{2}, \frac{45}{2}, 23$ in both readings. In both cases it finds diagonal matrices D, D' with entries ± 1 such that $K' = DK_n D'$ (a gauge transformation), and the entry sum of $K'^{-1} = D' K_n^{-1} D$, computed from the exact inverse of K_n , agrees with the direct computation. So

the entry sum is not an invariant of the gauge class, and Theorem 1.2 concerns the convention of [Pro99], up to Lemma 2.3. The convention enters the proof through Lemma 3.1: at a black square with four neighbours, the constant $\frac{1}{2}$ gives the value 1 only if exactly one of its four dominoes carries -1 . For the signs of Problem 12 this holds at every such square; it fails at every such square for (a), and for (b) at the black squares of the row just above (lower-row reading) or just below (upper-row reading) the horizontal axis, as the program checks for $n \leq 8$.

Remark 6.2 (Helfgott’s formula). Helfgott [Hel00, Proposition 9] gives the entry of the inverse of his Kasteleyn matrix of A_n at a black square $(w_0 + d_0, w_1)$ and a white square $(w_0, w_1 + d_1)$ of his coordinates, for $d_0, d_1 > 0$, as $(-1)^{d_0+d_1+w_1} 2^{-n}$ times a sum over j of products of two Krawtchouk coefficients, of degrees n and $n-1$; the other cases are treated in [Hel00, Corollary 10]. The coupling function of his main result [Hel00, Theorem 11] omits the sign $(-1)^{d_0+d_1+w_1}$, which does not affect minors up to a common sign. Since $d_0 + d_1 + w_1$ is the first coordinate of the black square minus the first coordinate of the white square plus the second coordinate of the white square, this sign is the product of a sign of the black square and a sign of the white square, that is, a gauge as in Remark 6.1. The matrix of [Hel00, Proposition 9] carries the signs that [Hel00, Section 2.1] introduces as more convenient and uses for the Aztec diamond from its Section 2.4 on: $+1$ on horizontal dominoes and $(-1)^k$ on a vertical domino, where k is the number of vertical dominoes between the same two rows to its left. In every pair of adjacent rows of A_n the lower square of the leftmost vertical domino has $x + y \equiv n \pmod{2}$, and both these signs and the signs of Problem 12 alternate along the row; so Helfgott’s signs are those of Problem 12 for the colouring of Section 1 when n is even and for the other colouring when n is odd, and by Lemma 2.3 the entry sum of the inverse of his matrix is S_n for every n . The third verification run (Section 7) read the coordinates of [Hel00, Figure 17] from the source of that figure (white squares (x, y) with $1 \leq x \leq n$, $1 \leq y \leq n+1$, black squares (x', y') with $1 \leq x' \leq n+1$, $1 \leq y' \leq n$, adjacent if and only if $x' - x \in \{0, 1\}$ and $y - y' \in \{0, 1\}$) and found, in exact arithmetic for $n \leq 7$: with the sign factor, Proposition 9 and Corollary 10 reproduce every entry of the inverse of his matrix (the text fixes the vertical direction of the board in the figure but not its left and right; for one of the two possibilities the agreement is exact, for the other it holds up to a gauge); without it, the coupling function has the same multiset of absolute values and the entry sums $1, \frac{1}{2}, 4, \frac{11}{2}, 16, \frac{57}{2}, 64$ for $n = 1, \dots, 7$. So the omitted sign factor is exactly the gauge between his coupling function and K_n^{-1} . Propp’s remark on “a different sum” [Pro99] fits the coupling function and the Kasteleyn–Wilson matrix of Remark 6.1(b); which convention of the thesis of 1998 it refers to cannot be decided from the texts. Summing Helfgott’s formula over all entries, with the identification above, would give another proof of Theorem 1.2, the route anticipated by Propp; we have not carried it out. By contrast, φ in Proposition 4.3 is the sum of the n columns of K_n^{-1} indexed by Σ_S , and its values are single Krawtchouk coefficients.

7 Computations and reproducibility

No theorem, proposition, lemma or corollary of this note depends on a computation. The program `check_note.py` of the reproducibility package (Python 3, standard library, exact integer and rational arithmetic, about two seconds) builds K_n from the definition (1) and checks: (A) Lemma 2.1, the condition on the 2×2 blocks in the proof of Proposition 2.2, and Lemma 4.1; (B) Lemma 3.1; (C) Lemma 4.2; (D) $K_n(2\varphi) = 2 \cdot \mathbf{1}_{\Sigma_S}$ for the vector of Proposition 4.3, and the equation for ψ in the proof of Corollary 1.3; (E) the values of φ on Σ_N , the formula of Theorem 1.2 through Proposition 3.2, and Remark 5.1; all of (A)–(E) for $n \leq 30$; (F) for $n \leq 8$, the exact inverse of K_n : $K_n K_n^{-1} = I$, $|\det K_n| = 2^{n(n+1)/2}$, the entry sum, the block sum $\frac{1}{2} - 2^{n-1}$ of Proposition 3.2, and the row and column sums of Corollary 1.3; (G) Lemma 2.3 and (H) the computations of Remarks 2.4 and 6.1, for $n \leq 8$; (N) four negative controls (a changed value of φ , the factors $1 - \xi$ and $1 + \xi$ exchanged,

the explicit vector tested against the matrix with -1 on the dominoes with the white square at the bottom, and the wrong formula $\frac{(n+1)^2}{2} - 2^n$, all detected. All 38 checks pass. The claim of Figure 1 was checked in exact arithmetic. The package also contains the programs and outputs of the third verification run (folder `independent_run_2`), among them the tracing of the faces of G_n for Proposition 2.2, the check of the labels of Figure 1 and the comparison with Helfgott’s formula of Remark 6.2.

Verification. The result was first obtained with a written proof and exact programs: S_n for $n \leq 8$, the vectors φ and ψ against the exact inverse for $n \leq 9$, the equations $K_n(2\varphi) = 2 \cdot \mathbf{1}_{\Sigma_S}$ and $K_n^T(2\psi) = 2 \cdot \mathbf{1}_{\Sigma_N}$ for $n \leq 40$, and S_n in floating point for $n \leq 26$. Two independent verification runs, both AI-assisted, followed; each re-derived the proof step by step and wrote its own programs, and neither read the other’s report. The first run checked the statement against both versions of Propp’s text; it computed S_n exactly for $n \leq 9$ for the signs of Problem 12 and their mirror image, in both colourings, and for the alternate-columns signs ($|\det| = 2^{n(n+1)/2}$ in all 72 cases), and it verified the explicit vectors, the two edges and Lemma 3.1 for $n \leq 60$, the vectors against the exact solutions for $n \leq 8$, and Lemma 4.2 symbolically for $n \leq 15$. The second run built the matrix independently, computed S_n exactly for $n \leq 12$ in both colourings (certified by the exact residual and the determinant), summed the exact inverse for $n \leq 8$, verified the explicit vectors on its own matrix for $n \leq 60$, compared S_n with the formula modulo two primes for $n \leq 40$, computed the sums for other signs and for random gauges, tested Helfgott’s formula, ran negative controls, and repeated the search of the literature. The second run reproduced the recorded outputs of the original programs byte for byte, and the first obtained results consistent with them. Neither run found a mathematical error. Both required that the account of the 1998 wording mention the alternate-columns reading (Remark 2.4); the first also required that the values of Table 1 not be attributed to Propp, who prints none, and suggested stating the geometric fact behind Kasteleyn’s condition (Proposition 2.2); the second required complete bibliographic data, and its comparison of Helfgott’s coupling function with K_n^{-1} (the same absolute values, other signs, for $n \leq 7$) was reported with its caveat. All corrections were made. An observation on the individual entries of the block in Proposition 3.2, checked for $n \leq 11$ but not proved, is not included.

A third run, also AI-assisted, examined the final text, including the parts added when the note was written (Remark 5.1, the gauges and the last sentences of Remark 6.1, Remark 6.2, Figure 1 and the program of this note). It read Problem 12 in both versions of Propp’s text and the progress paragraph again, Sections 1 and 2 of [Hel00] with the source of his Figure 17, and the abstract of [CJY15]; it re-derived every lemma, proposition and proof from the text as written, including the geometric part of Proposition 2.2, and found no error. Its own programs, in exact arithmetic, built K_n in another encoding and checked Lemmas 2.1, 3.1, 4.1 and 4.2, Proposition 4.3, Corollary 1.3 and Remark 5.1 for $n \leq 40$, the exact inverse for $n \leq 8$, S_n for $n \leq 10$ in both colourings with the -1 on top or at the bottom and for the transposes, Lemma 4.2 symbolically for $n \leq 12$, the faces of the plane graph of Proposition 2.2 by tracing them for $n \leq 25$, the sums, gauges and the analysis of the interior squares in Remarks 2.4 and 6.1 for $n \leq 8$, the labels of Figure 1 against the source of the figure, and Proposition 9, Corollary 10 and the coupling function of [Hel00] against the inverse of his matrix for $n \leq 7$ (Remark 6.2); it re-ran the program of this note from the release archive with identical output, verified the bibliography with Crossref and repeated the search of the literature. It required the account of Helfgott’s signs in Remark 6.2 and the credit to [Hel00] in the abstract to be made precise, the citation of [Kas67] to be added, and the date and this paragraph to be corrected; all corrections were made. Its programs and outputs are part of the reproducibility package.

Scope and priority. The note proves the formula of Problem 12 for every $n \geq 1$, for the matrix of [Pro99]; by Lemma 2.3 the answer does not depend on the colouring, on the choice between top and bottom, or on the choice of rows and columns. Other choices of the Kasteleyn signs give other sums (Remark 6.1) and are not covered. The problem and the formula are Propp’s, who stated the formula from computations for $n \leq 8$ [Pro98a, Pro99]. The matrix is Kasteleyn’s and Percus’s [Kas61, Per69], and the Aztec diamond and the count $2^{n(n+1)/2}$ are due to Elkies, Kuperberg, Larsen and Propp [EKLP92a, EKLP92b]. Explicit formulas for the entries of the inverse Kasteleyn matrix of the Aztec diamond, as sums of products of Krawtchouk coefficients, are due to Helfgott [Hel00], whose coupling function differs from the inverse by an explicit sign factor (Remark 6.2), and were generalised by Chhita, Johansson and Young [CJY15]; neither computes the entry sum or the row sums, and neither mentions Problem 12. What the note adds is an elementary proof of Propp’s identity by an explicit solution of $K_n u = \mathbf{1}_B$, and the row and column sums of K_n^{-1} (Corollary 1.3). Propp expected the identity to follow from Helfgott’s work, and specialists may regard it as a routine consequence of the known formulas; no priority is claimed. Indeed, by Remark 6.2, Proposition 9 and Corollary 10 of [Hel00] with their sign factor give the entries of K_n^{-1} itself in Helfgott’s coordinates and colouring (checked for $n \leq 7$), so that Theorem 1.2 is equivalent to a double-sum identity for his formula, which we have not evaluated. No printed answer to the question was found in the sources searched, listed below; a search that finds nothing is not a proof of novelty.

What was read: [Pro98a] and [Pro99] in their arXiv L^AT_EX sources (Problem 12, the description of the matrix, the progress paragraph of 1999 and the description of the results of Kasteleyn and Percus; the printed version of [Pro99] was not compared); [Pro98b] completely; [Hel00] in its arXiv version, completely by the second verification run and, for this note, Sections 1 and 2.1, Proposition 9, Corollary 10 and Theorem 11, and by the third run also Section 2.4 (the definition of the Aztec diamond in coordinates, Lemma 1 and the sign computation) and the source of Figure 17; [CJY15, CY14, Lai24] in their arXiv versions, searched in full text for sums of entries of the inverse and for Propp’s problems (the Problem 12 of [Lai24] is a different problem). [Kas61, Kas67, Per69, EKLP92a, EKLP92b, Ken97] are cited for background and were not read for this note. The searches (17 and 18 October 2026) used the arXiv interface (inverse Kasteleyn matrices, the Aztec diamond with Kasteleyn or Krawtchouk, Kasteleyn–Percus, Propp and Helfgott), zbMATH Open (including the works citing [Pro99]), Crossref, OpenAlex, Semantic Scholar (the 145 works citing [Pro99], filtered by keywords in their titles and abstracts) and seven web searches. Not available to us: Propp’s handbook chapter [Pro15], of which no open copy was found and which could not be read; Helfgott’s thesis in its original form of 1998; and the lists of works citing [Pro98a] in Semantic Scholar and OpenAlex and of works citing [Pro99] in OpenAlex, which could not be retrieved.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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