

An Algorithm for the Exponent in Mosher’s Theorem on Schottky Subgroups of Mapping Class Groups

Alper Ferudun*

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Abstract

Let S be a closed oriented surface of genus at least 2. By a theorem of Mosher (1997), for pairwise independent pseudo-Anosov mapping classes $\varphi_1, \dots, \varphi_n$ of S there is an integer M such that for all integers $\alpha_1, \dots, \alpha_n \geq M$ the classes $\varphi_i^{\alpha_i}$ freely generate a subgroup H whose surface group extension Γ_H is word hyperbolic. Problem 3.3 of Mosher’s chapter in Farb’s problem book (2006) asks whether there is an algorithm which produces M from $\varphi_1, \dots, \varphi_n$. We answer yes, as a promise problem and modulo published theorems which we list. An explicit search, which reads off finitely many exact distances in the curve graph $\mathcal{C}(S)$, halts on every admissible input and outputs an integer M such that for all $\alpha_1, \dots, \alpha_n \geq M$, chosen independently of each other, the powers $\varphi_i^{\alpha_i}$ freely generate a free group of rank n whose orbit map into $\mathcal{C}(S)$ is a quasi-isometric embedding; hence H is convex cocompact and Γ_H is word hyperbolic. On inputs that violate the promise the search may run forever, but an output, if there is one, is always correct. Correctness rests on an explicit four-point constant $\delta = 340$ for $\mathcal{C}(S)$, derived from the constant 17 of Hensel, Przytycki and Webb, a local-to-global lemma for broken sequences and a lemma on periodic sequences; termination uses the positive translation length of Masur and Minsky, the Morse lemma and Mosher’s theorem itself. For $n \geq 3$ no M depending only on S exists. The novelty is modest. Every ingredient was in print, and the closest prior work, Kim (2020), gives a semi-algorithm which certifies convex cocompactness of one given subgroup by the same local-to-global method in $\mathcal{C}(S)$; what we add is a written, complete proof with a procedure that is uniform in the exponents, and experts may regard the answer as folklore. The constants are crude ($\delta = 340$; M is huge and far from optimal), and the algorithm has not been implemented: no computation in a curve graph was run. The only computations are tests in model spaces (a tree, finite graphs, Cayley graphs of free products of cyclic groups, the hyperbolic plane and hyperbolic 3-space), which are not part of the proof; the test in a tree, where $\delta = 0$, checks only the bookkeeping of the procedure. This is an unrefereed note.

1 Introduction

1.1 The question

We follow the conventions of Mosher’s chapter [Mos06, §§1–3]. The surface S is closed and oriented, of genus $g \geq 2$, and $\mathrm{MCG}(S) = \mathrm{Homeo}(S)/\mathrm{Homeo}_0(S)$; orientation-reversing classes are included, and we write $\mathrm{Mod}(S)$ for the subgroup of index two of orientation-preserving classes. For $p \in S$ the group $\mathrm{MCG}(S, p)$ is identified with $\mathrm{Aut}(\pi_1(S, p))$, and for a subgroup $G < \mathrm{MCG}(S)$ the *surface group extension* $\Gamma_G < \mathrm{MCG}(S, p)$ is the preimage of G , so that $1 \rightarrow \pi_1 S \rightarrow \Gamma_G \rightarrow G \rightarrow 1$ [Mos06, §1]. A pseudo-Anosov class φ acts on the Thurston boundary of Teichmüller space with a repelling fixed

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>. Unrefereed preprint. CC BY 4.0.

point $\xi_-(\varphi)$ and an attracting fixed point $\xi_+(\varphi)$, every other orbit tending to $\xi_{\mp}(\varphi)$ under backward and forward iteration. Two pseudo-Anosov classes φ, ψ are *independent* if $\{\xi_{\pm}(\varphi)\} \cap \{\xi_{\pm}(\psi)\} = \emptyset$, equivalently, as the source states, if the two pairs are different [Mos06, §3]. A subgroup is *Schottky* if it is free and convex cocompact in the sense of Farb and Mosher [Mos06, §2], [FM02].

Theorem 3.1 of [Mos06] [Mos97]: if $\varphi_1, \dots, \varphi_n \in \text{MCG}(S)$ are pairwise independent pseudo-Anosov classes, then there is $M > 0$ such that for integers $\alpha_1, \dots, \alpha_n \geq M$ the classes $\varphi_1^{\alpha_1}, \dots, \varphi_n^{\alpha_n}$ freely generate a subgroup H with Γ_H word hyperbolic. By Theorem 2.1 of [Mos06] (due to [FM02]) H is then Schottky [Mos06, Theorem 3.2]. Noting that it is unclear how to use the theorem to write down a single example, Mosher writes that it would be interesting to make it “more effective”, and asks:

Problem 3.3 [Mos06, p. 263]. “Does there exist an algorithm which produces the integer M in Theorem 3.1, given $\varphi_1, \dots, \varphi_n$?”

He suggests train track expansions of pseudo-Anosov homeomorphisms as one approach, and mentions that Hamidi-Tehrani’s thesis [HT97] (and a preprint cited there) contains an algorithm producing an M for which the powers are free generators, without the hyperbolicity of Γ_H . The problem is record AMR-109-0231 of the UnsolvedMath collection [ula26], where it is listed as open.

1.2 The answer

Theorem A (modulo the published theorems listed in Section 3). *There is an algorithm with the following property. Its input is the genus $g \geq 2$, an integer $n \geq 1$ and n words in a fixed finite generating set of $\text{MCG}(S)$, for definiteness Lickorish’s Dehn twist generators of $\text{Mod}(S)$ together with one fixed orientation-reversing class, representing classes $\varphi_1, \dots, \varphi_n$ which are promised to be pairwise independent and pseudo-Anosov. On every such input it halts and outputs an integer $M \geq 1$ such that for all integers $\alpha_1, \dots, \alpha_n \geq M$, chosen independently of each other, the classes $\varphi_i^{\alpha_i}$ freely generate a free group H of rank n whose orbit map $H \rightarrow \mathcal{C}(S)$, $h \mapsto hx$, into the curve graph is a quasi-isometric embedding. Hence H is convex cocompact (Schottky) and Γ_H is word hyperbolic. On inputs which violate the promise the algorithm may not halt; whenever it halts, its output has the stated property (Remark 5.3).*

Any other finite description of the input from which such words can be computed, for example automorphisms of $\pi_1 S$ given on generators, would do as well. The question does not ask for the least M , for an M depending only on S , or for an efficient algorithm, and we give none of these; for $n \geq 3$ an M depending only on S does not exist (Remark 2.1). For $n = 1$ the question is trivial: every nonzero power of a pseudo-Anosov class generates a Schottky cyclic group (by the positive translation length [MM99] and step (v) of the proof of Theorem 5.1), so $M = 1$ works.

The algorithm (Section 5) searches for a finite certificate made of exact distances in $\mathcal{C}(S)$. Correctness for *all* tuples $\alpha_i \geq M$ (Theorem 5.1) follows from the lemmas of Section 4, with explicit constants, and from [KL08, FM02]; termination (Theorem 6.1) uses non-explicit constants, among them the M of Mosher’s theorem, which enter only the proof that the search stops.

1.3 What is new and what is not

Every ingredient of the proof was in print. The closest prior work is H. Kim’s paper [Kim20]: its Theorem A(i) is a partial algorithm which, for one subgroup of the mapping class group given by finitely many generators, halts if and only if the subgroup is stable, that is, convex cocompact [DT15], [Kim20, Theorem 2.2]. It works, as we do, with an orbit in $\mathcal{C}(S)$, a local-to-global principle with computable constants, and the distance algorithms of [Lea02, Sha12, Web15, BMM16] [Kim20, §2.1].

Problem 3.3 asks for one M valid for the infinitely many tuples $\alpha_i \geq M$, whereas Kim’s procedure certifies one subgroup; run tuple by tuple it never produces such an M . What this note adds is the part which is uniform in the exponents (Lemma 4.3, Step 2 of the algorithm and the bookkeeping of Theorem 5.1), and a written, complete proof. Kim does not mention Mosher’s problem. The non-effective template of the argument is Kent and Leininger’s proof of their Theorem 1.4 [KL08] (a common exponent, constants from existence statements). Experts may regard the answer as folklore, or as a routine consequence of [Kim20] and [KL08] together with the periodic-sequence lemma; we claim no priority. Related work (Fujiwara’s uniform bound for $n = 2$, Runnels’ explicit common exponent, and others) is discussed in Section 8, together with the searches made.

2 Reading of the question

We read Problem 3.3 as a promise problem. The input consists of g, n and words for $\varphi_1, \dots, \varphi_n$ as in Theorem A, promised to be pairwise independent pseudo-Anosov classes; the algorithm must halt on every such input and output some integer M with the property of Theorem 3.1 of [Mos06], for all tuples $\alpha_1, \dots, \alpha_n \geq M$ at once, the α_i being independent of each other. We neither use nor prove that the promise is decidable (see Remark 5.3). Any M with the stated property is “an integer M in Theorem 3.1”; the question does not refer to the particular constant of the proof in [Mos97]. Since the universal quantifier ranges over infinitely many tuples, the semi-decidability of convex cocompactness for one finitely generated subgroup [Kim20] does not answer the question by itself. Nearby variants which are not the question: freeness only (the algorithm of [HT97] mentioned in [Mos06]); an M depending only on S , which for $n = 2$ is provided by Fujiwara’s theorem [Fuj15, Theorem 16] (for pseudo-Anosov a, b in his $\text{Mod}(S)$ with no commuting nonzero powers, the group $\langle a^k, b^l \rangle$ is free of rank two and convex cocompact for all $k, l \geq M(S)$, so that its extension group is word hyperbolic by (FM) of Section 3), and which does not exist for $n \geq 3$ by the following remark; and the optimal exponent.

Remark 2.1 (no bound depending only on S for $n \geq 3$). Let φ_1, φ_2 be independent pseudo-Anosov classes, let $K \geq 1$ and $\varphi_3 := \varphi_1^K \varphi_2 \varphi_1^{-K}$, a pseudo-Anosov class with $\xi_{\pm}(\varphi_3) = \varphi_1^K(\xi_{\pm}(\varphi_2))$.

(a) φ_3 and φ_1 are independent. If $\varphi_1^K(\xi) \in \{\xi_{\pm}(\varphi_1)\}$ for some $\xi \in \{\xi_{\pm}(\varphi_2)\}$, then $\xi \in \varphi_1^{-K}(\{\xi_{\pm}(\varphi_1)\}) = \{\xi_{\pm}(\varphi_1)\}$, contradicting the independence of φ_1 and φ_2 .

(b) φ_3 and φ_2 are independent. If the two fixed pairs met, they would be equal, by the “equivalently” clause of the definition of independence in [Mos06, §3]. Then φ_1^K maps the pair $\{\xi_{\pm}(\varphi_2)\}$ to itself and the pseudo-Anosov class φ_1^{2K} fixes $\xi_{+}(\varphi_2)$. By the source–sink dynamics the only fixed points of φ_1^{2K} on the Thurston boundary are $\xi_{\pm}(\varphi_1)$, which again contradicts the independence of φ_1 and φ_2 .

So $\varphi_1, \varphi_2, \varphi_3$ are pairwise independent. But for $\alpha_1 = \alpha_2 = \alpha_3 = K$ and $s_i := \varphi_i^K$ we have $s_3 = \varphi_1^K \varphi_2^K \varphi_1^{-K}$, hence $s_3 s_1 s_2^{-1} s_1^{-1} = 1$, and $s_3 s_1 s_2^{-1} s_1^{-1}$ is a nontrivial reduced word. Thus s_1, s_2, s_3 are not free generators, and every valid M for this triple is larger than K . As K is arbitrary, no M depending only on S works for $n = 3$. For $n \geq 4$ add the classes $\varphi_1^{K_j} \varphi_2 \varphi_1^{-K_j}$ with distinct integers $K_j > K$; (a), and (b) applied with K replaced by a nonzero difference $K_j - K_{j'}$ after conjugation by a power of φ_1 , show that all these classes are pairwise independent; at the common exponent K the first three powers are not free generators, so again every valid M is larger than K . The same phenomenon for Dehn twists is pointed out by Runnels [Run21, p. 3 of the arXiv version], and Fujiwara’s Example 18 [Fuj15] makes the analogous point for the constant of his Theorem 17. For $n = 2$ the uniform bound $M(S)$ of [Fuj15, Theorem 16] exists; we do not use it.

3 Published results used

The curve graph $\mathcal{C}(S)$ has as vertices the isotopy classes of essential simple closed curves on S , two vertices being joined by an edge when the classes have disjoint representatives. We write d for its path metric (edges of length 1) and $|\mathcal{C}(S)|$ for the metric graph, a geodesic space. For closed S of genus $g \geq 2$ the graph is connected [MM99, Lemma 2.1 of the arXiv version]. The group $\text{MCG}(S)$ acts on $\mathcal{C}(S)$ by isometries. For points a, b, w of a metric space, the Gromov product is $\langle a, b \rangle_w = \frac{1}{2}(d(w, a) + d(w, b) - d(a, b))$. The proof of Theorem A uses the following published results as black boxes.

- (HPW) *Hensel, Przytycki and Webb* [HPW15, Theorem 1.1]: if $\mathcal{C}(S)$ is connected, it is 17-hyperbolic. Here a connected graph is k -hyperbolic if “all of its triangles formed by geodesic edge-paths are k -centred”, a triangle being k -centred at a vertex c if c is at distance at most k from each of its three sides [HPW15, p. 1]. Closed surfaces are covered: at the end of the proof the curve graph of a closed surface is treated as a 1-Lipschitz retract of the curve graph of the once-punctured surface. (The journal version is titled “1-slim triangles and uniform hyperbolicity for arc graphs and curve graphs”, the arXiv version, which we read, “Slim unicorns and uniform hyperbolicity for arc graphs and curve graphs”.)
- (MM) *Masur and Minsky* [MM99]: $|\mathcal{C}(S)|$ is Gromov hyperbolic; and there is $c(S) > 0$ such that $d(h^m \gamma, \gamma) \geq c(S)|m|$ for every pseudo-Anosov h , every vertex γ and every $m \in \mathbb{Z}$. The second statement is Proposition 3.6 of the arXiv version (math/9804098), which we read; in the journal version it is numbered 4.6 according to Kent and Leininger, who cite it as Theorem 4.6 of [MM99] [KL08, proof of Theorem 1.4]. We did not compare the journal version.
- (Morse) *Stability of quasi-geodesics* [BH99, III.H, Theorem 1.7]: in a δ' -hyperbolic geodesic space, for all $\lambda \geq 1$, $\varepsilon \geq 0$ there is $R = R(\delta', \lambda, \varepsilon)$ such that every (λ, ε) -quasi-geodesic (not necessarily continuous, [BH99, I.8.22]) is within Hausdorff distance R of every geodesic segment joining its endpoints.
- (KL) *Kent and Leininger* [KL08, Theorems 1.3 and 7.1]: a finitely generated subgroup $G < \text{Mod}(S)$ is convex cocompact if and only if an orbit map $G \rightarrow \mathcal{C}(S)$ is a quasi-isometric embedding; Theorem 7.1 is the direction “if”. Kent and Leininger work in $\text{Mod}(S) = \pi_0(\text{Homeo}^+(S))$ and define convex cocompactness by the conditions of Farb and Mosher [KL08, §1, Theorem 1.1 there]; their S is any oriented complete hyperbolic surface of finite area, so closed surfaces are included. The same characterisation was found independently by Hamenstädt [Ham05].
- (FM) *Farb and Mosher* [FM02], in the form of [Mos06, Theorem 2.1]: for finitely generated $G < \text{MCG}(S)$, if Γ_G is word hyperbolic then G is convex cocompact, and the converse holds when G is free.
- (Mos) *Mosher’s theorem* [Mos06, Theorem 3.1], [Mos97], as stated in Section 1. It is used only to prove that the search terminates.
- (DIST) *Computability*. There is an algorithm which, given $g \geq 2$, a fixed curve x and two words w, w' in the generators of Theorem A, computes $d(wx, w'x)$. Two published ingredients enter. First, the action of the generators on a finite encoding of curves is computable: Penner [Pen84] gives explicit formulas for the action of the Dehn twist generators of Lickorish on the Dehn–Thurston coordinates of simple closed curves on a closed surface; if mapping classes are given as automorphisms of $\pi_1 S$ and curves as conjugacy classes, the action is substitution. Second, the distance between two curves is computable, by the algorithms of Leasure [Lea02], Shackleton

[Sha12], Webb [Web15] and Birman, Margalit and Menasco [BMM16], and in polynomial time by Bell and Webb [BW16]. The main theorem of [BMM16] is stated for closed surfaces (of genus $g \geq 2$), and Shackleton writes that his algorithm “applies to all surfaces”. We do not discuss the routine translation between the encodings of curves used in these papers; Kim relies on the same kind of computation [Kim20, §2.1, first proof of Theorem A(i)].

The proofs of these results were not checked by us.

4 Hyperbolic geometry with explicit constants

Lemma 4.1 (centred triangles give the four-point condition). *Let Γ be a connected graph in which every triangle formed by geodesic edge-paths is k -centred. Then for all vertices x, y, z, w ,*

$$\langle x, z \rangle_w \geq \min(\langle x, y \rangle_w, \langle y, z \rangle_w) - 20k.$$

In particular, by (HPW), the vertex set of $\mathcal{C}(S)$ satisfies this with $20 \cdot 17 = 340$ in place of $20k$.

Throughout we work in the metric graph $|\Gamma|$; a geodesic edge-path between vertices is a geodesic segment. For a vertex c and a geodesic edge-path σ , the distance $d(c, \sigma)$ is attained at a vertex of σ , since on an edge uv the function $t \mapsto \min(d(c, u) + t, d(c, v) + 1 - t)$ is minimal at an endpoint. Three elementary facts, valid in every metric space, are used repeatedly:

- (I1) $\langle a, b \rangle_c + \langle c, b \rangle_a = d(a, c)$;
- (I2) if q lies on a geodesic segment $[a, b]$, then $\langle a, b \rangle_w \leq d(w, q)$ for every w ,
since $d(a, b) = d(a, q) + d(q, b) \geq d(w, a) - d(w, q) + d(w, b) - d(w, q)$;
- (I3) $|\langle a, b \rangle_w - \langle a', b \rangle_w| \leq d(a, a')$.

Proof. A triangle with two equal corners is 0-centred at the repeated corner, so degenerate triangles cause no exception below.

Step 1. Let (w, x, y) be a triangle with geodesic edge-path sides s_{wx}, s_{wy}, s_{xy} , centred at c , and let $a \in s_{wx}, b \in s_{wy}, p \in s_{xy}$ be vertices with $d(c, a), d(c, b), d(c, p) \leq k$. Then $2\langle x, y \rangle_w = [d(w, a) + d(a, x)] + [d(w, b) + d(b, y)] - [d(x, p) + d(p, y)]$. Since $d(a, p), d(b, p) \leq 2k$ we have $|d(a, x) - d(x, p)| \leq 2k$ and $|d(b, y) - d(p, y)| \leq 2k$, and $|d(w, a) - d(w, c)|, |d(w, b) - d(w, c)| \leq k$. Hence $|\langle x, y \rangle_w - d(w, c)| \leq 3k$ and

$$d(w, s_{xy}) \leq d(w, p) \leq d(w, c) + k \leq \langle x, y \rangle_w + 4k. \quad (1)$$

Step 2 (insize). Let (x, y, z) be a triangle with sides s_{xy}, s_{yz}, s_{xz} , centred at c , and let $a \in s_{xy}, b \in s_{yz}, e \in s_{xz}$ be vertices within k of c , hence pairwise within $2k$. Let $i_{xy} \in s_{xy}$ and $i_{xz} \in s_{xz}$ be the points at distance $\langle y, z \rangle_x$ from x , and $i_{yz} \in s_{yz}$ the point at distance $\langle x, z \rangle_y$ from y (they may be midpoints of edges). From $2\langle y, z \rangle_x = d(x, a) + d(a, y) + d(x, e) + d(e, z) - d(y, b) - d(b, z)$ we get

$$2\langle y, z \rangle_x - 2d(x, a) = [d(x, e) - d(x, a)] + [d(a, y) - d(y, b)] + [d(e, z) - d(b, z)],$$

each bracket in $[-2k, 2k]$; so $d(i_{xy}, a) = |\langle y, z \rangle_x - d(x, a)| \leq 3k$. The same identity with a and e exchanged gives $d(i_{xz}, e) \leq 3k$, and the analogous identity for $2\langle x, z \rangle_y$ gives $d(i_{yz}, b) \leq 3k$. Hence the three internal points are pairwise within $3k + 2k + 3k = 8k$.

Step 3 (thinness). In the triangle (x, y, z) let q be a vertex of s_{xz} with $t := d(x, q) \leq \langle y, z \rangle_x$, and let q' be the vertex of s_{xy} with $d(x, q') = t$ (it exists, as t is an integer and $t \leq \langle y, z \rangle_x \leq d(x, y)$). Then $d(q, q') \leq 16k$. Indeed q lies on the subsegment $[x, i_{xz}]$ of s_{xz} , so

$$d(q, y) \leq d(q, i_{xz}) + d(i_{xz}, i_{xy}) + d(i_{xy}, y) \leq (\langle y, z \rangle_x - t) + 8k + (d(x, y) - \langle y, z \rangle_x) = d(x, y) - t + 8k,$$

hence $t - 4k \leq \langle q, y \rangle_x \leq t$. Apply Step 2 to the triangle (x, q, y) with sides $s_{xz}|_{[x,q]}$, s_{xy} and any geodesic edge-path from q to y : its internal points $j \in [x, q]$ and $j' \in s_{xy}$ at distance $\langle q, y \rangle_x$ from x satisfy $d(j, j') \leq 8k$, while $d(q, j) = d(q', j') = t - \langle q, y \rangle_x \leq 4k$. So $d(q, q') \leq 16k$.

Step 4 (slimness). Every vertex q of s_{xz} is within $16k$ of a vertex of s_{xy} or of s_{yz} : since $\langle y, z \rangle_x + \langle x, y \rangle_z = d(x, z) = d(x, q) + d(q, z)$ by (I1), either $d(x, q) \leq \langle y, z \rangle_x$ (Step 3) or $d(z, q) \leq \langle x, y \rangle_z$ (Step 3 with x and z exchanged).

Step 5. Given x, y, z, w , choose geodesic edge-paths s_{xy}, s_{yz}, s_{xz} and let q be a vertex of s_{xz} nearest to w . By (1) for the triangle (w, x, z) , $d(w, q) \leq \langle x, z \rangle_w + 4k$. By Step 4 there is a vertex q' of s_{xy} or of s_{yz} with $d(q, q') \leq 16k$, and by (I2) $\langle x, y \rangle_w \leq d(w, q')$ or $\langle y, z \rangle_w \leq d(w, q')$, while $d(w, q') \leq \langle x, z \rangle_w + 20k$. $\square$

From now on X is the vertex set of $\mathcal{C}(S)$ and $\delta := 340$, so that for all $a, b, c, e, w \in X$

$$\langle a, c \rangle_w \geq \min(\langle a, b \rangle_w, \langle b, c \rangle_w) - \delta, \quad \langle a, e \rangle_w \geq \min(\langle a, b \rangle_w, \langle b, c \rangle_w, \langle c, e \rangle_w) - 2\delta, \quad (2)$$

the second form following from two applications of the first. Any larger δ also satisfies (2), and everything below holds for any isometric action on a metric space satisfying (2).

Lemma 4.2 (local-to-global for broken sequences). *Let $y_0, \dots, y_m \in X$ ($m \geq 1$), $C \geq 0$ and $L > 2C + 2\delta$, with $d(y_{k-1}, y_k) \geq L$ for $1 \leq k \leq m$ and $\langle y_{k-1}, y_{k+1} \rangle_{y_k} \leq C$ for $1 \leq k \leq m-1$. Then $\langle y_0, y_{k+1} \rangle_{y_k} \leq C + \delta$ for $1 \leq k \leq m-1$, and*

$$d(y_0, y_m) \geq m(L - 2C - 2\delta) + 2(C + \delta) > 0.$$

Proof. We show $Q(k)$: $\langle y_0, y_k \rangle_{y_{k-1}} \leq C + \delta$, for $1 \leq k \leq m$. $Q(1)$ holds since the left side is 0. Assume $Q(k)$ with $k \leq m-1$. By (2) at y_k , $C \geq \langle y_{k-1}, y_{k+1} \rangle_{y_k} \geq \min(A, B) - \delta$ with $A = \langle y_{k-1}, y_0 \rangle_{y_k}$ and $B = \langle y_0, y_{k+1} \rangle_{y_k}$. By (I1) and $Q(k)$, $A = d(y_k, y_{k-1}) - \langle y_k, y_0 \rangle_{y_{k-1}} \geq L - C - \delta$. If the minimum were A , then $C \geq L - C - 2\delta$, contradicting $L > 2C + 2\delta$. So $B \leq C + \delta$, which is $Q(k+1)$. Finally $d(y_0, y_{k+1}) = d(y_0, y_k) + d(y_k, y_{k+1}) - 2\langle y_0, y_{k+1} \rangle_{y_k}$ by the definition of the Gromov product; summing over $k = 1, \dots, m-1$, $d(y_0, y_m) \geq mL - 2(m-1)(C + \delta) = m(L - 2C - 2\delta) + 2(C + \delta)$. $\square$

Lemma 4.3 (periodic sequences). *Let f be an isometry of X , $x \in X$, $P \geq 1$, $z_k := f^{kP}x$ ($k \in \mathbb{Z}$), $l := d(x, f^P x)$ and $c := \langle f^{-P}x, f^P x \rangle_x$, and assume $l > 2c + 3\delta$. Then for all integers $k_1 \leq k_2 \leq k_3$:*

$$(a) \ d(z_{k_1}, z_{k_3}) \geq (k_3 - k_1)\lambda, \text{ where } \lambda := l - 2c - 2\delta > \delta;$$

$$(b) \ \langle z_{k_1}, z_{k_3} \rangle_{z_{k_2}} \leq c + 2\delta.$$

Since $d(f^{-P}x, f^P x) = d(x, f^{2P}x)$, we have $c = l - \frac{1}{2}d(x, f^{2P}x)$, and the hypothesis reads $d(x, f^{2P}x) > l + 3\delta$.

Proof. Consecutive distances are $d(z_k, z_{k+1}) = l$ and, applying f^{-kP} , $\langle z_{k-1}, z_{k+1} \rangle_{z_k} = c$. So both $(z_{k_1}, \dots, z_{k_3})$ and the reversed sequence satisfy the hypotheses of Lemma 4.2 with $L = l$ and $C = c$, and (a) follows (it is trivial if $k_1 = k_3$). For (b) we may assume $k_1 < k_2 < k_3$, the product being 0 otherwise. Lemma 4.2 for the forward sequence gives $\langle z_{k_1}, z_{k_2+1} \rangle_{z_{k_2}} \leq c + \delta$, and for the reversed sequence it gives $\langle z_{k_3}, z_{k_2} \rangle_{z_{k_2+1}} \leq c + \delta$ (this product is 0 if $k_2 + 1 = k_3$). By (I1), $\langle z_{k_3}, z_{k_2+1} \rangle_{z_{k_2}} = l - \langle z_{k_2}, z_{k_3} \rangle_{z_{k_2+1}} \geq l - c - \delta$. By (2) at z_{k_2} ,

$$c + \delta \geq \langle z_{k_1}, z_{k_2+1} \rangle_{z_{k_2}} \geq \min(\langle z_{k_1}, z_{k_3} \rangle_{z_{k_2}}, l - c - \delta) - \delta.$$

Since $l - c - 2\delta > c + \delta$, the minimum is the first term, and $\langle z_{k_1}, z_{k_3} \rangle_{z_{k_2}} \leq c + 2\delta$. $\square$

5 The algorithm and its correctness

The algorithm. The input is as in Theorem A. Fix a curve x (a vertex of $\mathcal{C}(S)$) and $\delta = 340$. All distances below are exact distances in $\mathcal{C}(S)$ between explicit curves, computed by (DIST); Gromov products are computed from them. We write $\varphi_i^m x$ for the curve $\varphi_i^m(x)$.

Step 1 (periods). For each i , search $P = 1, 2, 3, \dots$ and stop at the first P with

$$l := d(x, \varphi_i^P x) > 2c + 3\delta, \quad \text{where } c := \langle \varphi_i^{-P} x, \varphi_i^P x \rangle_x.$$

Record $P_i := P$, $l_i := l$, $c_i := c$, $\lambda_i := l_i - 2c_i - 2\delta$ ($> \delta$) and $R_i := \max_{0 \leq r < P_i} d(x, \varphi_i^r x)$.

Step 2 (pairs of ends). An *end* is a pair (i, η) with $\eta \in \{1, -1\}$. For each pair of ends $e = (i, \eta)$, $e' = (j, \eta')$ with $i < j$, search $N = 1, 2, 3, \dots$ and stop at the first N with

$$d(x, u) > G + c_i + 4\delta \quad \text{and} \quad d(x, v) > G + c_j + 4\delta, \quad \text{where } u := \varphi_i^{\eta NP_i} x, \quad v := \varphi_j^{\eta' NP_j} x, \quad G := \langle u, v \rangle_x.$$

Record $N_{e,e'} := N$ and $G_{e,e'} := G$, and put $N_{e',e} := N_{e,e'}$, $G_{e',e} := G_{e,e'}$.

Step 3 (output). Put

$$C := \max\left(\max_i (c_i + 2\delta + 2R_i), \max_{\text{pairs } e, e'} (G_{e,e'} + 2\delta + R_i + R_j)\right), \quad L := 2C + 2\delta + 1,$$

$$K_i := \max\left(\{N_{e,e'} : e \text{ or } e' \text{ has index } i\} \cup \{[(L + R_i)/\lambda_i]\}\right), \quad \text{and output } M := \max_i P_i K_i.$$

Theorem 5.1 (correctness). *If the algorithm halts with output M , then for all integers $\alpha_1, \dots, \alpha_n \geq M$ the classes $s_i := \varphi_i^{\alpha_i}$ freely generate a free group H of rank n , the orbit map $H \rightarrow \mathcal{C}(S)$, $h \mapsto hx$, is a quasi-isometric embedding, and Γ_H is word hyperbolic; hence H is Schottky by [Mos06, Theorems 2.1, 3.2]. In particular M is an integer as in Theorem 3.1 of [Mos06].*

Proof. Fix $\alpha_i \geq M$ and write $\alpha_i = k_i P_i + r_i$ with $0 \leq r_i < P_i$, so that $k_i = \lfloor \alpha_i / P_i \rfloor \geq K_i \geq 1$. Put $z_k^i := \varphi_i^{k P_i} x$; by the choice of P_i , Lemma 4.3 applies to $f = \varphi_i$, $P = P_i$.

(i) *Long translations.* By Lemma 4.3(a), $d(z_0^i, z_{k_i}^i) \geq k_i \lambda_i$, and $d(\varphi_i^{\alpha_i} x, z_{k_i}^i) = d(\varphi_i^{r_i} x, x) \leq R_i$. Hence $d(x, \varphi_i^{\alpha_i} x) \geq k_i \lambda_i - R_i \geq L$, as $k_i \geq K_i \geq (L + R_i)/\lambda_i$.

(ii) *Same index, opposite directions.* The point $\varphi_i^{-\alpha_i} x$ is within $d(x, \varphi_i^{-r_i} x) = d(\varphi_i^{r_i} x, x) \leq R_i$ of $z_{-k_i}^i$. Lemma 4.3(b) with $(k_1, k_2, k_3) = (-k_i, 0, k_i)$ gives $\langle z_{-k_i}^i, z_{k_i}^i \rangle_x \leq c_i + 2\delta$, so by (I3) $\langle \varphi_i^{-\alpha_i} x, \varphi_i^{\alpha_i} x \rangle_x \leq c_i + 2\delta + 2R_i \leq C$.

(iii) *Different indices.* Let $e = (i, \eta)$, $e' = (j, \eta')$ with $i \neq j$, $N = N_{e,e'}$ and $G = G_{e,e'}$. Put $u_k := \varphi_i^{\eta k P_i} x$, $v_l := \varphi_j^{\eta' l P_j} x$, $k := k_i$ and $l := k_j$; then $k, l \geq N$. The sequence $(u_k)_{k \geq 0}$ is (z_k^i) if $\eta = 1$ and (z_{-k}^i) if $\eta = -1$, so Lemma 4.3(b) gives $\langle x, u_k \rangle_{u_N} \leq c_i + 2\delta$, whence by (I1) and the stopping rule of Step 2

$$\langle u_N, u_k \rangle_x = d(x, u_N) - \langle x, u_k \rangle_{u_N} > (G + c_i + 4\delta) - c_i - 2\delta = G + 2\delta.$$

Likewise $\langle v_N, v_l \rangle_x > G + 2\delta$. The second form of (2) gives

$$G = \langle u_N, v_N \rangle_x \geq \min(\langle u_N, u_k \rangle_x, \langle u_k, v_l \rangle_x, \langle v_l, v_N \rangle_x) - 2\delta,$$

and the minimum can be neither the first nor the third term; so $\langle u_k, v_l \rangle_x \leq G + 2\delta$. As $d(\varphi_i^{\eta \alpha_i} x, u_k) = d(\varphi_i^{\eta r_i} x, x) \leq R_i$ and similarly for j , (I3) gives

$$\langle \varphi_i^{\eta \alpha_i} x, \varphi_j^{\eta' \alpha_j} x \rangle_x \leq G + 2\delta + R_i + R_j \leq C.$$

(iv) *Ping-pong.* Let $w = t_1 t_2 \dots t_m$ ($m \geq 1$) be a nontrivial reduced word in the letters $s_i^{\pm 1}$, and $y_k := t_1 \dots t_k x$. Then $d(y_{k-1}, y_k) = d(x, t_k x) = d(x, \varphi_i^{\alpha_i} x) \geq L$ by (i), for $t_k = s_i^{\pm 1}$; and, translating

by $(t_1 \cdots t_k)^{-1}$, $\langle y_{k-1}, y_{k+1} \rangle_{y_k} = \langle t_k^{-1}x, t_{k+1}x \rangle_x$. If t_k and t_{k+1} have different indices, this is a product as in (iii), with η the opposite of the sign of t_k and η' the sign of t_{k+1} . If they have the same index then, the word being reduced, $t_{k+1} = t_k$, and the product is the one in (ii). In both cases it is at most C . Since $L = 2C + 2\delta + 1$, Lemma 4.2 gives $d(x, wx) \geq m(L - 2C - 2\delta) = m > 0$. So w acts nontrivially, the natural map from the free group on $s_1, \dots, s_n$ onto H is injective, and $|w| \leq d(x, wx) \leq |w| \max_i d(x, s_i x)$ for the word length $|w|$. As $d(hx, h'x) = d(x, h^{-1}h'x)$, the orbit map is a quasi-isometric embedding of H with its word metric.

(v) *Convex cocompactness and hyperbolicity.* Let $H^+ := H \cap \text{Mod}(S)$. Then $[H : H^+] \leq 2$, so H^+ is a finitely generated free group, the inclusion $H^+ \rightarrow H$ is a quasi-isometry, and the orbit map of H^+ into $\mathcal{C}(S)$ is a quasi-isometric embedding. By (KL), $H^+ < \text{Mod}(S)$ is convex cocompact. Convex cocompactness in (KL) is the notion of Farb and Mosher (it is defined in [KL08, §1] by their conditions), and (FM) is applied to the free group $H^+ < \text{Mod}(S)$ of the closed surface S , as in [Mos06, Theorem 2.1]; so the chain from (KL) to (FM) concerns one and the same notion. By (FM), free case, Γ_{H^+} is word hyperbolic. Finally Γ_{H^+} and Γ_H are the preimages of $H^+ < H$ in $\text{MCG}(S, p)$, so $[\Gamma_H : \Gamma_{H^+}] = [H : H^+] \leq 2$; hence Γ_H is quasi-isometric to Γ_{H^+} and word hyperbolic. By (FM) again, H is convex cocompact, that is, Schottky. $\square$

Remark 5.2 (where the hypotheses are used). The promise (pseudo-Anosov, pairwise independent) is used only for termination (Section 6). Steps (i)–(iv) use only the stopping rules of Steps 1 and 2 and (2), so they hold for any isometric action on a space satisfying (2) and for any valid δ . The hypothesis $g \geq 2$ enters through (HPW), (DIST), (KL) and (FM).

Remark 5.3 (inputs which violate the promise). If the classes are not pseudo-Anosov or not pairwise independent, a search of Step 1 or Step 2 need not stop; for instance, if $\varphi_2 = \varphi_1$ the Gromov product in Step 2 grows with N . If the algorithm halts, Theorem 5.1 applies, since its proof never uses the promise; so the algorithm never outputs a wrong M . We do not prove that the promise is decidable. It seems possible to remove the promise, but we have neither written this out nor checked it: if Step 1 stops for φ_i , then Lemma 4.3(a) gives φ_i positive stable translation length on $\mathcal{C}(S)$, which should force φ_i to be pseudo-Anosov [MM99]; if the whole algorithm halts, the conclusion of Theorem 5.1 rules out common nonzero powers of two inputs; and the failure of the promise seems to be semi-decidable, since the Nielsen–Thurston type is computable [BH95, KM15, BW16] and, by a theorem attributed to McCarthy which we did not look up, dependent pseudo-Anosov classes have common nonzero powers. Running both searches side by side would then decide the promise.

6 Termination

Theorem 6.1 (termination). *On every input satisfying the promise, each search in Steps 1 and 2 stops; hence the algorithm halts. Together with Theorem 5.1 this proves Theorem A and answers Problem 3.3 of [Mos06] affirmatively.*

Proof. Write $D_i := d(x, \varphi_i x)$.

Step 1 stops. The class φ_i^2 is an orientation-preserving pseudo-Anosov class, so by (MM) $d(x, \varphi_i^{2q} x) \geq c(S)|q|$ for all $q \in \mathbb{Z}$. Writing $m = 2q + \rho$ with $\rho \in \{0, 1\}$, so that $|q| \geq (|m| - 1)/2$, we get $d(x, \varphi_i^m x) \geq c(S)|q| - d(x, \varphi_i^\rho x) \geq \frac{c(S)}{2}|m| - \frac{c(S)}{2} - D_i$; translating, $d(\varphi_i^a x, \varphi_i^b x) \geq \frac{c(S)}{2}|a - b| - \frac{c(S)}{2} - D_i$, and trivially $d(\varphi_i^a x, \varphi_i^b x) \leq D_i|a - b|$. So $t \mapsto \varphi_i^{\lfloor t \rfloor} x$ ($t \in \mathbb{R}$) is a (λ, ε) -quasi-geodesic for suitable constants, passing through x at $t = 0$. By (Morse), applied in the hyperbolic geodesic space $|\mathcal{C}(S)|$ (MM), there is R_0 with $d(x, [\varphi_i^{-P} x, \varphi_i^P x]) \leq R_0$ for every $P \geq 1$ and every geodesic segment between these points, so by (I2) $c(P) := \langle \varphi_i^{-P} x, \varphi_i^P x \rangle_x \leq R_0$ for all P , while $d(x, \varphi_i^P x) \geq \frac{c(S)}{2}P - \frac{c(S)}{2} - D_i$ tends to infinity. So the condition of Step 1 holds for all large P .

Step 2 stops. Fix ends $e = (i, \eta)$, $e' = (j, \eta')$ with $i \neq j$. We show

$$B := \sup_{k, l \geq 1} \langle \varphi_i^{\eta k P_i} x, \varphi_j^{\eta' l P_j} x \rangle_x < \infty.$$

Given this, the stopping rule holds for all large N : $G \leq B$, while $d(x, \varphi_i^{\eta N P_i} x) \geq N \lambda_i$ and $d(x, \varphi_j^{\eta' N P_j} x) \geq N \lambda_j$ by Lemma 4.3(a) tend to infinity.

Proof that $B < \infty$. By (Mos) applied to the independent pair φ_i, φ_j there is M_{ij} such that the conclusion holds for all exponents $\geq M_{ij}$; take an even $a_* \geq M_{ij}$. Then $s := \varphi_i^{a_*}$ and $t := \varphi_j^{a_*}$ freely generate a free group H_2 with Γ_{H_2} word hyperbolic, and $s, t \in \text{Mod}(S)$ since a_* is even. By (FM), H_2 is convex cocompact, so by (KL) the orbit map $H_2 \rightarrow \mathcal{C}(S)$, $h \mapsto hx$, is a (λ', ε') -quasi-isometric embedding for the word metric in $\{s, t\}$. For integers $a, b \geq 0$ put $h_r := s^{\eta|r|}$ for $-a \leq r \leq 0$ and $h_r := t^{\eta'r}$ for $0 \leq r \leq b$. Since s, t is a free basis, the word $h_r^{-1} h_{r'}$ is reduced of length $|r - r'|$ (for $r \leq 0 \leq r'$ it is $s^{-\eta|r|} t^{\eta'r'}$). So $r \mapsto h_r x$ is a (λ', ε') -quasi-geodesic from $s^{\eta a} x$ to $t^{\eta' b} x$ through $h_0 x = x$, and (Morse) with (I2) gives $\langle s^{\eta a} x, t^{\eta' b} x \rangle_x \leq R_1$ for all $a, b \geq 0$, with R_1 independent of a, b . Finally, for $m \geq 1$ write $m = qa_* + r$ with $0 \leq r < a_*$; then $\varphi_i^{\eta m} x = s^{\eta q} \varphi_i^{\eta r} x$ is within $d(\varphi_i^{\eta r} x, x) \leq R_i^* := \max_{0 \leq r < a_*} d(x, \varphi_i^r x)$ of $s^{\eta q} x$, and similarly for j . By (I3), $B \leq R_1 + R_i^* + R_j^*$. $\square$

Remark 6.2. (a) The output depends on the input words only through finitely many exact distances in $\mathcal{C}(S)$. The constants $c(S)$, R_0 , R_1 , M_{ij} and a_* of the proof above are not explicit, but they enter only the proof that the searches stop, never the output.

(b) The bound $B < \infty$ can also be seen through the Gromov boundary of $\mathcal{C}(S)$, as in Kent and Leininger's proof of their Theorem 1.4, where independence makes the endpoints of the orbit rays distinct [KL08, proof of Theorem 1.4]; the proof above needs only the results listed in Section 3.

7 Computations

No proof in this note depends on a computation. The algorithm has not been implemented, and no computation in a curve graph was run. The reproducibility package contains the following programs; all of them are tests, not proofs. The recorded outputs were reproduced byte for byte when this note was written, and again, from the released source archive, by the third verification run (Section 8).

`check_note.py` (written with this note; standard library; about two seconds) checks the finite and exact parts of the proofs: the constants of Lemma 4.1, (I1)–(I3) on random graph metrics, the algebra of Lemma 4.2, the identity $c = l - \frac{1}{2}d(x, f^{2P}x)$ of Lemma 4.3 in a free group acting on its tree, the bookkeeping of Theorem 5.1, the case split of step (iv) over all 4,680 reduced words of length 2 to 5 in three letters, the relation of Remark 2.1 in the free group of rank two (hence in every group), and the arithmetic of the proof of Theorem 6.1; it also tests Lemma 4.1 by brute force on 14 small graphs. All nine checks pass.

`toy_tree_check.py` (the program with which the procedure was first tested; standard library; about one second) runs Steps 1–3 for the free group $F_2 = \langle a, b \rangle$ acting on its Cayley tree, where $\delta = 0$, and again with the deliberately too large constants $\delta = 1, 3$, which (2) allows: nine inputs with $n \leq 3$, among them the triples $a, b, a^K b a^{-K}$ of Remark 2.1 for $K = 2, 4, 7$, so 27 runs. For each output M it checks (i)–(iii) of the proof of Theorem 5.1 for all exponents in a window $[M, M + W]$ ($W = 2$ for $n \leq 2$, $W = 1$ for $n = 3$) and free generation by Stallings folding. The recorded output has 28 lines (a header and 27 runs): no violation, no tuple which fails to be free, the output exceeds K for the triples of Remark 2.1, and at exponent K the folded rank is 2. Since $\delta = 0$ in a tree, this test checks only the bookkeeping of Steps 1–3 and of Theorem 5.1; it cannot detect an error in any coefficient of δ .

Three programs of the verification runs A and B test the terms in δ in metrics which are not trees. `lemma42_finite_test2.py` (run A; numpy; about one second) tests Lemma 4.2 in eight finite graphs (grids, a ladder, cylinders and necklaces of cycles) with their exact four-point constant $d_4 \in \{1, 2, 3\}$ as δ , for $C \in \{0, \frac{1}{2}, 1, \frac{3}{2}\}$ and $L = 2C + 2d_4 + 1$: neither conclusion fails on the 200,524 admissible sequences with $m = 2, 3$, while the control with $\delta = d_4/2$ gives 41,400 violations. `vB_lemma41_graphs.py` (run B; numpy and networkx; about two seconds) confirms Lemma 4.1, (1) and the slimness bound on 53 graphs with exact constants (largest ratio of four-point to centring constant: 2) and tests Lemma 4.2; of its controls, the false bound $k^*/4$ for the four-point constant is refuted on 45 of the 53 graphs and Lemma 4.2 with δ replaced by 0 (hypothesis $L > 2C$ only) fails in 337 of 3,063 sequences, while its third control, which drops δ from the conclusion of Lemma 4.2 alone, found no violation. `vB_h2_model.py` (run B; mpmath; about 15 seconds) runs Steps 1–3 for elements of $\mathrm{PSL}(2, \mathbb{Z})$ acting on the hyperbolic plane with $\delta = \log 2$ and $\delta = 1$ (22 inputs, 44 runs) and checks (i)–(iii) on windows of exponents and the conclusion of step (iv) for all reduced words of length at most 5 ($n = 2$) or 4 ($n = 3$): no violation; a deliberately wrong M is refuted, and for dependent pairs (g, g^2) , (g, g^{-1}) the search of Step 2 runs to its cap, as Remark 5.3 predicts.

The third verification run wrote five further programs, in the folder `independent_run_2/` of the package. The first, `r2_constants.py` (sympy; under a second), checks the algebra of Lemmas 4.1 to 4.3 and of Theorems 5.1 and 6.1 symbolically or exhaustively on ranges and recounts the 4,680 reduced words: eleven checks pass. `r2_lemma41_42_graphs.py` (numpy and networkx; about eight seconds) computes on 37 graphs the exact centring constant k^* and the worst case of every intermediate bound in the proof of Lemma 4.1: the largest ratios to k^* are 2 (Step 1), 2 (inequality (1)), 4 (insize), 4 (Step 3), 2 (slimness) and 2 (four-point constant), against the proved 3, 4, 8, 16, 16, 20, and false smaller bounds are refuted on 6 to 37 of the graphs; on these and seven larger graphs Lemma 4.2 holds for all 409,246 admissible sequences with $\delta = d_4$, while $\delta = d_4 - \frac{1}{2}$ and $\delta = d_4/2$ give 10,076 and 10,676 violations. `r2_freeprod_model.py` (standard library; about five minutes) and `r2_h3_model.py` (mpmath; about two seconds) run Steps 1–3 and check (i)–(iv) and the intermediate inequalities of Lemma 4.3 and of step (iii) for free products of cyclic groups on their Cayley graphs ($\delta = 1, 2$; 50 runs) and for $\mathrm{PSL}(2, \mathbb{Z}[i])$ acting on hyperbolic 3-space ($\delta = \log 2, 1$; 20 runs): no violation; the wrong value $M = K$ for the triples of Remark 2.1 is refuted, and dependent pairs make Step 2 run to its cap. These two models, however, hardly exercise the terms in δ : along the orbits tested the excess of the Gromov products of Lemma 4.3(b) over c is 0 in the free products (`r2_freeprod_excess.py`) and at most 0.03δ in hyperbolic 3-space, and runs with smaller coefficients of δ , or with $\delta = 0$, are not refuted in the free products. So they check the bookkeeping with $\delta > 0$, not the coefficients of δ ; of all the tests, only those of Lemma 4.2 in finite graphs detect a too small δ . All these programs are evidence about the lemmas in model spaces; they say nothing about curve graphs.

8 Related work, verification, scope and priority

Related work. Kim’s semi-algorithm [Kim20, Theorem A(i)] and its relation to this note are described in the introduction; besides the curve-graph method, Kim gives a second proof for closed surfaces through the extension group, a complete algorithm for undistorted subgroups (Theorem A(ii)) and a partial algorithm for Morse subgroups (Theorem A(iii)); none of these concerns exponents. Kent and Leininger [KL08, Theorem 1.4] prove Mosher’s theorem for one common exponent, with constants from existence statements; our argument is an effective version of theirs which also allows unequal exponents. Fujiwara [Fuj15, Theorem 16] proves that for $n = 2$ there is a bound $M(S)$ depending only on S beyond which the two powers freely generate a convex cocompact group (Section 2); the bound is not made effective there, there is no algorithm and no statement for $n \geq 3$, and his Theorem 14,

for acylindrical actions, gives a bound depending on the constants of the action. Since Shackleton’s abstract announces a computable version of Bowditch’s acylindricity theorem [Sha12] and Webb computes stable lengths [Web15], the bound for $n = 2$ might be computable along these lines; we have not checked this. Runnels [Run21, Theorem 4.1] gives an explicit common exponent N such that the N -th and higher equal powers of an irredundant collection of pure mapping classes generate the expected right-angled Artin group, undistorted after enlarging N ; combined with the theorem of Bestvina, Bromberg, Kent and Leininger that undistorted purely pseudo-Anosov groups are convex cocompact [BBKL20], this gives at most a common exponent, not an M for independent exponents, and no algorithm. Anderson, Aramayona and Shackleton [AAS07] quantify freeness only, as does the algorithm of Hamidi-Tehrani [HT97] named in [Mos06]. Mangahas [Man13] constructs pseudo-Anosov elements of short word length and quotes Theorem 3.1 of [Mos06] and Fujiwara’s bound; it was read only in part, at the stage at which the result was found, and we found no algorithm for M in it.

Verification. The result was first obtained with a written proof and the toy test of Section 7. Three independent verification runs, all AI-assisted, followed; each wrote its own programs. Runs A and B examined the first write-up. Run A compared the statement with the source, re-derived Lemmas 4.1 to 4.3, Theorem 5.1, Remark 2.1 and Theorem 6.1 line by line, read the cited statements of [HPW15, KL08, MM99, Fuj15, Run21], and tested the lemmas in finite graphs and in Cayley graphs of free products of cyclic groups. It confirmed every statement and required four corrections of presentation: the warning that the tree test cannot see the coefficients of δ , the journal numbering for (MM), the scope of (DIST) with a reference for the action on curves, and the link between the notions of (KL) and (FM). Run B re-derived the same proofs, checked the black boxes against the texts or abstracts and the bibliography against Crossref, re-ran the toy program (identical output), tested Steps 1–3 in the hyperbolic plane and repeated the literature search. It found no mathematical error, found Kim’s paper [Kim20], which the first write-up had missed, and required that it be cited as the closest prior work with the difference stated, and that a line count be corrected. The third run examined the text of this note. It read the source and the cited texts again (the chapter, [Kim20], [HPW15], [KL08], [MM99], [Fuj15], [Run21] and the introduction of [BMM16] in the arXiv versions, and the abstracts of [Sha12, Web15, BW16]), re-derived every lemma and theorem with all their constants, wrote the five programs described in Section 7 before re-running the package from the released archive (all outputs identical), checked all digital object identifiers of the bibliography in Crossref and repeated the literature search. It found no mathematical error and required corrections of presentation only: a more precise account of the controls of one program, the description of the tests in the abstract, the statement of Fujiwara’s theorem in Section 2, two further sources not seen, and the record of its own checks. All corrections were made.

Scope and priority. The note answers Problem 3.3 of [Mos06] affirmatively, as a promise problem, modulo the published theorems of Section 3. It gives neither the least M nor an efficient algorithm; the constants are crude ($\delta = 340$, and M is huge), and the algorithm has not been implemented. The question is Mosher’s [Mos06], on his theorem [Mos97] (see also [KL08, Theorem 1.4], where it is attributed to [FM02]). Every ingredient was in print: the certificate method for a single subgroup in Kim’s paper (arXiv 2019, journal 2020) [Kim20], the orbit-map characterisation of convex cocompactness in [KL08, Ham05], the link with hyperbolicity of Γ_H in [FM02], the constant 17 in [HPW15], the translation length in [MM99], the Morse lemma in [BH99] and the distance algorithms in [Lea02, Sha12, Web15, BMM16, BW16]. Experts may regard the answer as folklore, or as a routine consequence of [Kim20] and [KL08] plus the periodic-sequence lemma. What the note adds is a written, complete proof of the affirmative answer, with an explicit procedure which is uniform in the exponents (Lemma 4.3 and Step 2) and explicit, crude constants. We claim no priority.

What was read: Sections 1–3 of [Mos06] in the open copy of the book (<https://www.math.uchicago.edu/~farb/papers/mcgbook.pdf>, PDF pages 268–270, printed pages 261–263; the chapter occupies printed pages 261–273 there, while Crossref lists pages 245–256), to whose printed pages we refer; the arXiv versions of [HPW15, KL08, MM99, Fuj15, Run21] in the parts cited; the arXiv version 1 of [Kim20], its introduction and Section 2; the introduction of the arXiv version of [BMM16]; the abstracts of [Sha12, BW16, Ham05, BBKL20, AAS07]; and the review of [Pen84] in zbMATH. We did not see [Lea02, HT97], the preprint of Hamidi-Tehrani cited in [Mos06], or Mosher’s paper on train tracks which [Mos06] suggests as an approach, and we did not re-read [BH99]; [FM02] is quoted through [Mos06, Theorem 2.1], [DT15] through [Kim20, Theorem 2.2], and [BH95, KM15] are cited only in the unchecked part of Remark 5.3. Journal versions were not compared, and the proofs of the cited theorems were not checked.

The searches were made on 20 October 2026: queries to the arXiv API, the lists of works citing [Mos06] in OpenAlex (5 works, none about Problem 3.3) and zbMATH (one citing document, not relevant), the Semantic Scholar lists of works citing [Mos97] (52), [KL08] (134) and [Fuj15] (20), screened by title, bibliographic checks in Crossref, and five web searches. The third verification run repeated the search on the same day (queries to the arXiv API, zbMATH, the Semantic Scholar lists of works citing [Mos97], [KL08] and [Kim20], Crossref and one web search); it confirmed the single citing document in zbMATH and the counts 52 and 134, and OpenAlex confirmed the count of 5 citing works but refused to return the list, which that run therefore did not re-check. No paper was found which states an algorithm for M or answers Problem 3.3; no printed answer to the question was found in these sources. A search that finds nothing is not a proof of novelty.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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