

Unbounded Multiplicity in the Dilatation Spectrum of Every Term of the Johnson Filtration: An Observation on a Question of Farb

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Abstract

Farb asked (Question 7.6 of *Some problems on mapping class groups and moduli space*) whether $\text{spec}(\mathcal{I}_g(k))$, the set of the logarithms of the dilatations of the pseudo-Anosov elements of the k -th term of the Johnson filtration of the mapping class group Mod_g , has bounded multiplicity for $k \geq 3$; the multiplicity of a value is the number of conjugacy classes of Mod_g which realise it. We observe that the question is answered in the negative: for every $g \geq 2$, $k \geq 1$ and N there are N pseudo-Anosov classes in $\mathcal{I}_g(k)$ with one dilatation which are pairwise non-conjugate in Mod_g , also up to inversion and in the extended mapping class group. This is an observation: the negative answer follows from the mechanism which the source itself gives for the Torelli group and the Johnson kernel (Thurston's representation of the group generated by the twists about two filling curves, equal traces of non-conjugate elements, the two curves chosen separating), applied to a two-generator subgroup of a deep term of the lower central series of that group, which the source itself places in $\mathcal{I}_g(k)$, with a reference to Farb, Leininger and Margalit. What the note adds is the comparison of conjugacy in the Fuchsian subgroup with conjugacy in the mapping class group, which the source leaves implicit also for $k \leq 2$, written out with proofs, and explicit instances, which are computer-assisted, in exact integer arithmetic: for example, 64 classes in $\mathcal{I}_2(4)$ have one dilatation λ , where $\lambda + \lambda^{-1}$ is an integer with 5116 digits. The answer is apparently not recorded in the literature accessible to us, and it may well be known to specialists. Question 7.7 of the source, on the simple length spectrum, is not answered. This is an unrefereed note.

1 Introduction

1.1 The question

Let Σ_g be a closed oriented surface of genus $g \geq 2$ and Mod_g its mapping class group. In his problem list [Far06, §7], Farb considers the set $\text{spec}(\text{Mod}_g)$ of the numbers $\log \lambda(f)$, where f runs through the pseudo-Anosov classes of Mod_g and $\lambda(f)$ is the dilatation, and for a subgroup $H < \text{Mod}_g$ the subset $\text{spec}(H)$ coming from the pseudo-Anosov elements of H . He asks [Far06, Question 7.6]: “Does $\text{spec}(\mathcal{I}_g(k))$ have bounded multiplicity for $k \geq 3$?” Here $\mathcal{I}_g(k)$ is the k -th term of the Johnson filtration. The source defines multiplicity only for $H = \text{Mod}_g$: the set $\text{spec}(\text{Mod}_g)$ has unbounded multiplicity if for every N some value is $\log \lambda(f_i)$ for at least N pseudo-Anosov classes f_i which are different conjugacy classes of Mod_g . We read the question accordingly: the *multiplicity* of a value r of $\text{spec}(H)$ is the number of conjugacy classes of Mod_g which contain a pseudo-Anosov element of H with $\log \lambda = r$. This is the reading under which the source's argument for $\mathcal{I}_g$ and $\mathcal{I}_g(2)$ makes sense: it passes from a subgroup to a larger group, which is legitimate only if the count is monotone in H .

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If one counted conjugacy classes of H instead, every value of $\text{spec}(\mathcal{I}_g(k))$, $k \geq 1$, would have infinite multiplicity for a trivial reason, because $\mathcal{I}_g(k)$ is normal of infinite index and the centraliser of a pseudo-Anosov class is virtually cyclic. Theorem 1.1 below also covers classes counted up to inversion and conjugacy in the extended mapping class group $\text{Mod}_g^\pm$. The question is record AMR-109-0082 of the UnsolvedMath collection [ula26], where it is listed as open. Only the first sentence of the record is Question 7.6; the second belongs to Question 7.7 (simple length spectrum), *not* answered here.

Notation. We write $S = \Sigma_g$, $\text{Mod}(S) = \text{Mod}_g$, $\text{Mod}^\pm(S) = \text{Mod}_g^\pm$, $\pi = \pi_1(S)$, and for a group G we put $\gamma_1(G) = G$, $\gamma_{i+1}(G) = [G, \gamma_i(G)]$, $[x, y] = xyx^{-1}y^{-1}$. The source numbers the lower central series from zero and defines $\mathcal{I}_g(k)$ as the kernel of $\text{Mod}_g \rightarrow \text{Out}(\Gamma/\Gamma_k)$ for $\Gamma = \pi$ [Far06, §1.2, (5)]; in our notation $\Gamma_k = \gamma_{k+1}(\pi)$, so $\mathcal{I}(k) = \mathcal{I}_g(k) = \ker(\text{Mod}(S) \rightarrow \text{Out}(\pi/\gamma_{k+1}(\pi)))$. Thus $\mathcal{I}(1)$ is the Torelli group and, by Johnson, $\mathcal{I}(2)$ is generated by the twists about separating curves [Far06, §1.2]. Sections, statements and formulas of the cited papers are numbered as in their arXiv versions, which are the versions we read (see the end of Section 8).

What the source says is known [Far06, §7.2], in our words. First, $\text{spec}(\text{Mod}_g)$ has unbounded multiplicity, because moduli space contains isometrically embedded hyperbolic surfaces of finite area, whose length spectra have unbounded multiplicity (the source gives no reference; this is a theorem of Randol [Ran80], which uses trace identities of Horowitz [Hor72]). Secondly, for two curves a, b which fill Σ_g , Thurston's representation of the group $\langle T_a, T_b \rangle$ generated by the two Dehn twists into $\text{PSL}(2, \mathbb{R})$ is injective with discrete image, every element is pseudo-Anosov or a power of a twist, and $\text{spec}(\langle T_a, T_b \rangle)$ is, with the qualifier *essentially*, the length spectrum of the quotient of $\mathbb{H}^2$ by the image; hence it has unbounded multiplicity. Thirdly, a and b can be chosen separating, and therefore $\text{spec}(\mathcal{I}_g)$ and $\text{spec}(\mathcal{I}_g(2))$ have unbounded multiplicity. The question asks what happens for $k \geq 3$. Elsewhere, in the proof of [Far06, Theorem 5.10], the source notes, with a reference to [FLM08], that an element of the k -th level of the lower central series of a free group generated by the twists about two separating curves lies in $\mathcal{I}_g(k)$ (see the proof of [FLM08, Proposition 4.8]).

1.2 The observation

Theorem 1.1. *Let $g \geq 2$, $k \geq 1$ and $N \geq 1$. There are pseudo-Anosov classes $f_1, \dots, f_N \in \mathcal{I}_g(k)$ with the same dilatation such that for $i \neq j$ the class f_i is conjugate neither to f_j nor to f_j^{-1} in $\text{Mod}_g^\pm$. In particular the f_i are pairwise non-conjugate in Mod_g , and $\text{spec}(\mathcal{I}_g(k))$ has unbounded multiplicity for every $k \geq 1$. The f_i can be chosen in $\gamma_d(\langle T_a, T_b \rangle) \subset \mathcal{I}_g(2d)$, $d = \max(k, 2)$, where a and b are separating simple closed curves which fill Σ_g .*

Thus Question 7.6 is answered in the negative for every $g \geq 2$ and every k . *This is an observation:* the mechanism is the one which the source indicates for $k \leq 2$, applied to a two-generator subgroup of the d -th term of the lower central series of $\langle T_a, T_b \rangle$; that such subgroups lie in $\mathcal{I}_g(k)$ is noted in the source and is the construction in the proof of [FLM08, Proposition 4.8]. The source leaves implicit how conjugacy in $\langle T_a, T_b \rangle$ compares with conjugacy in Mod_g . This step, needed for $k \leq 2$ as well, is what the note supplies. First, if two affine pseudo-Anosov classes of a square-tiled flat structure are conjugate in $\text{Mod}_g^\pm$, then their derivatives are conjugate in $\text{PGL}(\Lambda)$, where Λ is the lattice of the holonomy vectors of saddle connections and $\text{PSL}(\Lambda) \cong \text{PSL}(2, \mathbb{Z})$ (Lemma 4.4); this rests on the uniqueness of the invariant Teichmüller disc. Secondly, a conjugacy class of $\text{PSL}(2, \mathbb{Z})$ meets a finitely generated subgroup G in a number of conjugacy classes of G which is bounded in terms of G (Lemma 6.1 and Corollary 6.2); this is needed because the image of $\langle T_a, T_b \rangle$ has infinite index in $\text{PSL}(\Lambda)$. We use a finitely generated subgroup of $\gamma_d(\langle T_a, T_b \rangle)$, and not γ_d itself, because the hyperbolic surface of γ_d is a regular cover with infinite deck group of that of $\langle T_a, T_b \rangle$: every length occurs on it with infinite multiplicity for a trivial reason, while all deck translates of a closed geodesic give one class of Mod_g ; and Lemma 6.1 needs finite generation.

What is not answered. Question 7.7 of the source (the simple length spectrum) is not answered; the closed Teichmüller geodesics of our classes are not simple (Remark 7.2). The growth of the multiplicities is not determined: the proof gives values $r \rightarrow \infty$ with multiplicity at least $cr^{0.63}$ (Remark 7.1). Each multiplicity is finite, if the theorem of Arnoux–Yoccoz and Ivanov [AY81, Iva90] is used in the form that there are only finitely many conjugacy classes of pseudo-Anosov classes of Mod_g with dilatation below a given bound; the source quotes this theorem as the discreteness of $\text{spec}(\text{Mod}_g)$ [Far06, §7.1], and we did not read the two papers. Whether the classes f_i are primitive is not treated in general (Remark 7.1).

2 The Johnson filtration

For a group G and $k \geq 1$ let $A_k(G)$ be the set of the automorphisms ϕ of G with $\delta_\phi(x) := \phi(x)x^{-1} \in \gamma_{k+1}(G)$ for all $x \in G$. The identities

$$\delta_\phi(zw) = \delta_\phi(z) \cdot z \delta_\phi(w) z^{-1}, \quad \delta_\phi(z^{-1}) = z^{-1} \delta_\phi(z)^{-1} z \quad (1)$$

show that for a normal subgroup M the set $\{z : \delta_\phi(z) \in M\}$ is a subgroup; so membership in $A_k(G)$ can be tested on generators of G . Also $A_k(G)$ is a group, since $\delta_{\phi\psi}(x) = \phi(\delta_\psi(x)) \delta_\phi(x)$. We use the standard inclusion

$$[\gamma_i(G), \gamma_j(G)] \subset \gamma_{i+j}(G), \quad (2)$$

which follows by induction on i from the three subgroups lemma: $[[G, \gamma_i], \gamma_j] \subset [[\gamma_i, \gamma_j], G] \cdot [[\gamma_j, G], \gamma_i] \subset [\gamma_{i+j}, G] \cdot [\gamma_{j+1}, \gamma_i] \subset \gamma_{i+j+1}$.

Lemma 2.1. (a) If $\phi \in A_k(G)$ and $x \in \gamma_m(G)$, then $\phi(x)x^{-1} \in \gamma_{m+k}(G)$. (b) $[A_k(G), A_l(G)] \subset A_{k+l}(G)$. (c) $[\mathcal{I}(k), \mathcal{I}(l)] \subset \mathcal{I}(k+l)$. Consequently $\gamma_j(H) \subset \mathcal{I}(2j)$ for every subgroup $H \subset \mathcal{I}(2)$.

Proof. Write $\gamma_i = \gamma_i(G)$ and $\delta = \delta_\phi$. (a) Induction on m ; the case $m = 1$ is the definition. By (1) it suffices to treat the generators $[x, y]$ of γ_{m+1} , where $x \in G$ and $y \in \gamma_m$. Put $\phi(x) = ax$ and $\phi(y) = by$ with $a \in \gamma_{k+1}$ and, by induction, $b \in \gamma_{m+k}$. Modulo γ_{m+k+1} the element b is central, so $\phi([x, y]) = [ax, by] \equiv [ax, y] = a[x, y]a^{-1} \cdot [a, y]$. Here $[a, y] \in \gamma_{m+k+1}$ by (2), and $a[x, y]a^{-1} = [a, [x, y]][x, y]$ with $[a, [x, y]] \in \gamma_{m+k+2}$. Hence $\delta([x, y]) \in \gamma_{m+k+1}$.

(b) Let $\phi \in A_k(G)$, $\psi \in A_l(G)$, $y \in G$, $\phi(y) = ay$, $\psi(y) = by$ with $a \in \gamma_{k+1}$, $b \in \gamma_{l+1}$. By (a), $\phi(b) = b'b$ and $\psi(a) = a'a$ with $a', b' \in \gamma_{k+l+1}$. Then $\phi\psi(y)(\psi\phi(y))^{-1} = b'b ay \cdot y^{-1}b^{-1}a^{-1}a'^{-1} = b'[b, a]a'^{-1} \in \gamma_{k+l+1}$. With $y = \phi^{-1}\psi^{-1}(x)$ this says that $[\phi, \psi](x)x^{-1} \in \gamma_{k+l+1}$.

(c) A class f lies in $\mathcal{I}(k)$ if and only if it is represented by an automorphism in $A_k(\pi)$: if a representative induces on $\pi/\gamma_{k+1}(\pi)$ the conjugation by the class of $x \in \pi$, compose it with the inverse of the conjugation by x . For $f \in \mathcal{I}(k)$, $h \in \mathcal{I}(l)$ take representatives $\phi \in A_k(\pi)$, $\psi \in A_l(\pi)$; then $[\phi, \psi] \in A_{k+l}(\pi)$ represents $[f, h]$. Finally $\gamma_{j+1}(H) = [H, \gamma_j(H)] \subset [\mathcal{I}(2), \mathcal{I}(2j)] \subset \mathcal{I}(2j+2)$. $\square$

This is classical; that the Johnson filtration is central is quoted from Bass and Lubotzky in the introduction of [FLM08] and is used in the same way in the proofs of [FLM08, Proposition 4.8] and [AT17, Theorem 5.1].

Lemma 2.2. The Dehn twists about a separating simple closed curve c lie in $\mathcal{I}(2)$.

Proof. Let $S = S_1 \cup_c S_2$ and take the base point on c , so that π is generated by $\pi_1(S_1)$ and $\pi_1(S_2)$. A twist about c is represented by a homeomorphism supported in a collar of c inside S_2 . It fixes the loops in S_1 and maps a loop y in S_2 to $c^e y c^{-e}$, $e = \pm 1$. So $\delta(x) = 1$ for $x \in \pi_1(S_1)$ and $\delta(y) = [c^e, y]$ for $y \in \pi_1(S_2)$. The loop c bounds S_2 , so it is a product of commutators and lies in $\gamma_2(\pi)$; hence $\delta(y) \in \gamma_3(\pi)$, and by (1) this representative lies in $A_2(\pi)$. $\square$

3 Filling pairs of separating curves

Let n be even and let τ, τ' be fixed-point-free involutions of $\mathbb{Z}/n$. Cut the cylinder $C = (\mathbb{R}/n\mathbb{Z}) \times [0, 1]$ into the unit squares $Q_i = [i, i+1] \times [0, 1]$. Glue the top edge of Q_i to that of $Q_{\tau(i)}$ by the half-turn $x \mapsto i + \tau(i) + 1 - x$, and the bottom edge of Q_i to that of $Q_{\tau'(i)}$ by $x \mapsto i + \tau'(i) + 1 - x$. The result $S(n, \tau, \tau')$ is a closed connected oriented surface with a flat metric with cone points, tiled by n unit squares; the transition maps of its natural charts are $z \mapsto \pm z + c$. Let a be the curve $\{y = \frac{1}{2}\}$ and b the union of the vertical segments $\{x = i + \frac{1}{2}\}$.

Lemma 3.1. (i) *The corner points $(i, 1)$ of C fall into the classes given by the cycles of $i \mapsto \tau(i) + 1$, the corner points $(i, 0)$ into those given by the cycles of $i \mapsto \tau'(i) + 1$. A class with m points is a point of cone angle $m\pi$. The genus is $(n + 2 - c - c')/2$, where c, c' are the numbers of cycles.* (ii) *The curve a is a separating simple closed curve. The set b is a single closed curve if and only if $\langle \tau, \tau' \rangle$ is transitive on $\mathbb{Z}/n$; in that case b is separating if and only if $\tau(i) \not\equiv i$ and $\tau'(i) \not\equiv i$ modulo 2 for all i .* (iii) *The complement of $a \cup b$ consists of open discs around the corner classes, with $2m$ sides for a class with m points. So $a \cup b$ fills, and there is a bigon if and only if $\tau(i) = i - 1$ or $\tau'(i) = i - 1$ for some i .*

Proof. (i) The half-turn of the top edge of Q_i sends its end points $i, i+1$ to $\tau(i) + 1, \tau(i)$. Each corner point of C carries the angle π . There are $c + c'$ vertices, $2n$ edges and n faces. (ii) No top edge is glued to a bottom edge, so $S \setminus a$ has the two components “above” and “below”. The vertical segment of Q_i is continued by those of $Q_{\tau(i)}$ and $Q_{\tau'(i)}$, which gives the first statement. Let b be a single curve and orient it; because the gluings are half-turns, its direction (up or down) changes at every passage through a top or bottom edge. The right half of Q_i and the left half of Q_{i+1} lie on the same side of b exactly if the directions in Q_i and Q_{i+1} differ, and halves glued along top or bottom edges always lie on the same side. So b separates if and only if the direction alternates with i ; since it also alternates along b , this is the parity condition. (iii) The $2m$ quarter squares at a corner class with m points form the open star of that vertex. A class with one point is a fixed point of one of the two maps of (i). $\square$

Call (n, τ, τ') *admissible* if b is a single separating curve and there is no bigon. Then all cone angles are at least 2π , so the flat structure is that of a holomorphic quadratic differential q without poles, and by the bigon criterion a and b are in minimal position. The sum of $m - 2$ over the corner classes is $2n - 2(c + c') = 4g - 4$, so for $g \geq 2$ there is a cone point of angle at least 3π ; and $n = 2g - 2 + c + c' \geq 2g \geq 4$. Call an intersection point of two curves *good* if the four quadrants at the point lie in four different complementary regions. In an admissible model every intersection point is good. Indeed, by Lemma 3.1(ii) the maps $i \mapsto \tau(i) + 1$ and $i \mapsto \tau'(i) + 1$ preserve the parity, so the corners $(i, 1)$ and $(i+1, 1)$ lie in different classes, and so do $(i, 0)$ and $(i+1, 0)$; a top corner and a bottom corner are never in one class. So the four corners of Q_i lie in four different classes, and by Lemma 3.1(iii) the four quadrants at the centre of Q_i lie in the regions around them.

Seeds. For odd c let $\tau_c(i) = i + c(-1)^i$ on $\mathbb{Z}/2h$, and let $M_2 = S(8, \tau_3, \tau_5)$, $M_3 = S(12, \tau_3, \tau_5)$. For M_2 the images of $0, \dots, 7$ are $\tau = (3, 6, 5, 0, 7, 2, 1, 4)$ and $\tau' = (5, 4, 7, 6, 1, 0, 3, 2)$. The cycles of $i \mapsto \tau(i) + 1$ are $(04)(26)(1753)$, those of $i \mapsto \tau'(i) + 1$ are $(0642)(15)(37)$. So the genus is 2, there are two cone points of angle 4π and no bigon; the parities are as required, and b passes through the squares $0, 3, 6, 1, 4, 7, 2, 5$. Hence M_2 is admissible. For M_3 the cycles are $(048)(2610)(1119753)$ and $(06)(28)(410)(195)(3117)$, the genus is 3, the cone angles are 6π and four times 3π , and b passes through $0, 3, 10, 1, 8, 11, 6, 9, 4, 7, 2, 5$; so M_3 is admissible.

Lemma 3.2 (Sum at good points). *Let (S, a, b) and (S', a', b') be closed oriented surfaces of genus g and g' with transverse separating simple closed curves all of whose complementary regions are discs*

with at least four sides, and let p, p' be good intersection points. Then the surface of genus $g + g'$ carries a pair of curves with the same properties and with $|a \cap b| + |a' \cap b'| - 2$ intersection points.

Proof. Remove small open discs $\Delta \ni p, \Delta' \ni p'$ which meet the curves in two diameters. On $\partial\Delta$ let a_+, b_+, a_-, b_- be the four points of $a \cup b$ in the cyclic order given by the orientation, and similarly on $\partial\Delta'$. Glue $S \setminus \Delta$ and $S' \setminus \Delta'$ by a homeomorphism $\partial\Delta \rightarrow \partial\Delta'$ with $a_\pm \mapsto a'_\mp$ and $b_\pm \mapsto b'_\pm$; it reverses the cyclic order, as is needed for an oriented sum. The remaining arcs form two transverse simple closed curves a'', b'' . The arc of $\partial\Delta$ from a_+ to a_- through b_+ is glued to the arc of $\partial\Delta'$ through b'_+ , so the two sides of a are glued to the two sides of a' , and a'' separates; the same holds for b'' . As p is good, the regions $R_1, \dots, R_4$ at p are different, and $R_i \setminus \Delta$ is a disc which meets $\partial\Delta$ in one arc α_i ; the same holds for the regions R'_i at p' , numbered so that α_i is glued to α'_i . The new regions are the other old ones and the four discs $(R_i \setminus \Delta) \cup_{\alpha_i} (R'_i \setminus \Delta')$, with two sides fewer than R_i and R'_i together, so at least six. $\square$

Proposition 3.3. (a) A triple (S, a, b) as in Lemma 3.2 with n intersection points is of the form $S(n, \tau, \tau')$, with its curves a and b , for an admissible triple (n, τ, τ') . (b) For every $g \geq 2$ there is an admissible triple (n, τ, τ') such that $S(n, \tau, \tau')$ has genus g .

Proof. (a) The cell decomposition dual to $a \cup b$ has one square for each intersection point. The squares along a form a cylinder. Each remaining edge is dual to an arc of b and occurs twice on the boundary of this cylinder, in the squares of the two end points of the arc; these are different, because the separating curve a meets b in an even number of points. Because a separates, such an arc leaves a and returns to it on the same side, so top edges are glued to top edges and bottom edges to bottom edges, and the gluings are half-turns because S is oriented. By Lemma 3.1 the triple is admissible. (b) Start with M_2 or M_3 and add copies of M_2 by Lemma 3.2. By (a) every surface obtained is an admissible model, so its intersection points are good and the step can be repeated. $\square$

The existence of such pairs is known: [AT17, Lemma 2.2] proves it by an induction of the same kind, from seeds in genus 2 and 3 which are taken as given there, and [Jef24, Theorem 1.2] gives square-tiled surfaces with one horizontal and one vertical cylinder whose core curves are both separating. The examples of [FLM08, §2.3] are separating *multicurves*.

4 The two affine twists

Fix $g \geq 2$ and an admissible model $S = S(n, \tau, \tau')$ of genus g with its quadratic differential q ; recall that $n \geq 4$. Let $\Sigma \neq \emptyset$ be the set of the cone points of angle at least 3π , let $\text{Aff}^+(q)$ be the group of the orientation-preserving homeomorphisms of S which preserve Σ and are affine in the natural charts on $S \setminus \Sigma$, and $D: \text{Aff}^+(q) \rightarrow \text{PSL}(2, \mathbb{R})$ the derivative. Define $\text{Aff}^\pm(q)$ and $D: \text{Aff}^\pm(q) \rightarrow \text{PGL}(2, \mathbb{R}) = \text{GL}(2, \mathbb{R})/\{\pm 1\}$ in the same way without the condition on the orientation. Let $\mathbb{D} \subset \mathcal{T}(S)$ be the Teichmüller disc of q , the set of the marked structures $M \cdot q$, $M \in \text{SL}(2, \mathbb{R})$, obtained by composing the natural charts with M . We use three facts of [Lei04, §3.2], stated there for closed surfaces with marked points. (L1) Every element of the stabiliser $\text{Stab}(\mathbb{D}) \subset \text{Mod}(S)$ is represented by an affine automorphism of q , and the derivative defines a homomorphism $\text{Stab}(\mathbb{D}) \rightarrow \text{PSL}(2, \mathbb{R})$ (text before Theorem 3.1); in particular an affine automorphism which is isotopic to the identity has derivative ± 1 . (L2) If the derivative of $\phi \in \text{Stab}(\mathbb{D})$ is hyperbolic, then ϕ is pseudo-Anosov with dilatation $\exp(L/2)$, where L is the translation length of the derivative on $\mathbb{H}^2$ (Theorem 3.1(3), attributed to Thurston, Kra and Veech). (L3) A pseudo-Anosov class stabilises exactly one Teichmüller disc (Theorem 3.2, attributed to Bers and stated without proof). This follows from the theorem of Bers [Ber78], which we did not read, that a pseudo-Anosov class has positive translation length and a unique invariant

Teichmüller geodesic: the class acts as a hyperbolic isometry on every disc which it stabilises, with this geodesic as axis, and a Teichmüller geodesic lies in only one disc.

Lemma 4.1. (i) *The shear $(x, y) \mapsto (x + ny, y)$ of the cylinder C defines $\tau_a \in \text{Aff}^+(q)$ with $D(\tau_a) = A = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, and τ_a is a Dehn twist about a . In the same way the squares along b form a vertical cylinder of circumference n and width 1, whose shear defines $\tau_b \in \text{Aff}^+(q)$ with $D(\tau_b) = B = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$, a Dehn twist about b . (ii) If $\phi \in \text{Aff}^+(q)$ and $D\phi$ is hyperbolic, then the class of ϕ is pseudo-Anosov, and its dilatation λ satisfies $\lambda + \lambda^{-1} = |\text{tr } D\phi|$.*

Proof. (i) The shear is the identity on $y = 0$ and the translation by n on $y = 1$, so it respects the gluings. Because b is a single separating curve, the same holds for the cylinder formed by the squares along b : its natural coordinates differ from those of the squares by maps $z \mapsto \pm z + c$, and no edge of one of its boundary circles is glued to an edge of the other. (ii) The class of an affine automorphism stabilises $\mathbb{D}$, and the homomorphism of (L1) maps it to the derivative. So this is (L2), since $2 \cosh(L/2) = |\text{tr } D\phi|$; the invariant foliations are the linear foliations of q in the eigendirections of $D\phi$. $\square$

The flat structure and the two affine twists are Thurston's construction for the pair a, b , in the description of [Lei04, §5.1]: there the cell division dual to the two curves is made of Euclidean rectangles, which are equal squares for a pair of single curves, and the two twists, taken with the same handedness, are affine with derivatives $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ -\mu & 1 \end{pmatrix}$, where $\mu = n$ in our case. The rotation by $\pi/2$ conjugates A into B^{-1} , so the two shears have opposite handedness: if T_c denotes the left-handed twist, then $\tau_a = T_a^{-1}$ and $\tau_b = T_b$. Only the group $\langle T_a, T_b \rangle = \langle \tau_a, \tau_b \rangle$ occurs below. (For $n = 8$ the derivative of $\tau_a \tau_b^{-1} = (T_b T_a)^{-1}$ has trace -62 , as in [FLM08, §2.3].)

Lemma 4.2 (Ping-pong). *Let $n \geq 3$. The matrices A and B generate a free subgroup of $\text{PSL}(2, \mathbb{R})$ with basis A, B , and every element which is not conjugate in $\langle A, B \rangle$ to a power of A or of B is hyperbolic. In particular every non-trivial element of the commutator subgroup of $\langle A, B \rangle$ is hyperbolic.*

Proof. On the circle $\mathbb{R} \cup \{\infty\}$ let $J_A = \{|x| \geq \frac{n}{2}\} \cup \{\infty\}$ and $J_B = \{|x| \leq \frac{2}{n}\}$; these closed arcs are disjoint because $n > 2$. For $k \neq 0$ the map $A^k(x) = x + kn$ sends J_B into $\{|x| \geq n - \frac{2}{n}\}$, which lies in the interior of J_A , and $B^k(x) = 1/(kn + \frac{1}{x})$ sends J_A into $\{|x| \leq n/(n^2 - 2)\}$, which lies in the interior of J_B ; both inclusions use $n^2 > 4$. Every non-trivial element of the free group on α, β is conjugate to a non-zero power of a letter or to a word $w = \alpha^{m_1} \beta^{k_1} \dots \alpha^{m_r} \beta^{k_r}$ with $r \geq 1$ and all exponents non-zero. For such a word, $w(A, B)$ maps J_A into its interior and $w(A, B)^{-1}$ maps J_B into its interior. So $w(A, B) \neq 1$, and it has a fixed point in J_A and another one in J_B ; an element of $\text{PSL}(2, \mathbb{R})$ other than 1 with two fixed points on the circle is hyperbolic. Non-zero powers of A and B are not trivial either. A non-trivial element of the commutator subgroup has exponent sum zero in both letters, so it is not conjugate to a power of a letter. (For $n = 2$ the group is still free, but AB^{-1} is parabolic.) $\square$

This is known. In [Lei04, §6] the group generated by $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ -\mu & 1 \end{pmatrix}$ is shown to be free for $\mu \geq 2$ (Lemma 6.3), and for a filling pair with $\mu > 2$ every element of $\langle T_a, T_b \rangle$ other than the conjugates of powers of the two twists is pseudo-Anosov (Proposition 6.4); the proof of [FLM08, Proposition 4.8] uses this.

Let F_2 be the free group on α, β , and $\rho: F_2 \rightarrow \text{PSL}(2, \mathbb{R})$, $\alpha \mapsto A$, $\beta \mapsto B$, and $\Theta: F_2 \rightarrow \text{Mod}(S)$, $\alpha \mapsto [\tau_a]$, $\beta \mapsto [\tau_b]$. Then ρ is injective, $\Theta(w)$ is the class of the affine automorphism $w(\tau_a, \tau_b)$ with derivative $\rho(w)$, and for every $w \neq 1$ in $[F_2, F_2]$ the class $\Theta(w)$ is pseudo-Anosov with $\lambda + \lambda^{-1} = |\text{tr } \rho(w)|$. By Lemmas 2.1 and 2.2,

$$\Theta(\gamma_j(F_2)) = \gamma_j(\langle T_a, T_b \rangle) \subset \mathcal{I}(2j). \quad (3)$$

Lemma 4.3 (The lattice of saddle connections). *A saddle connection is a straight segment which joins two points of Σ and has no point of Σ in its interior; its holonomy vector is defined up to sign. Let $\Lambda \subset \mathbb{R}^2$ be the group generated by the holonomy vectors of all saddle connections. Then Λ is a lattice contained in $\mathbb{Z}^2$, and*

$$D(\text{Aff}^+(q)) \subset \text{PSL}(\Lambda), \quad D(\text{Aff}^\pm(q)) \subset \text{PGL}(\Lambda),$$

where $\text{GL}(\Lambda) = \{M \in \text{GL}(2, \mathbb{R}) : M\Lambda = \Lambda\} \cong \text{GL}(2, \mathbb{Z})$, $\text{PGL}(\Lambda) = \text{GL}(\Lambda)/\{\pm 1\}$ and $\text{PSL}(\Lambda) = \text{SL}(\Lambda)/\{\pm 1\} \cong \text{PSL}(2, \mathbb{Z})$.

Proof. The points of Σ are corners of the unit squares, and the developing map of the tiling maps corners to $\mathbb{Z}^2$, because the charts of adjacent squares differ by maps $z \mapsto \pm z + c$ with $c \in \mathbb{Z}^2$; so $\Lambda \subset \mathbb{Z}^2$. Follow the horizontal edges from a point $\sigma \in \Sigma$ straight through regular corners. This continuation is injective on directed edges, so the path cannot enter a cycle which avoids its first edge, and its first edge starts at the singular point σ ; hence the path ends in Σ and is a saddle connection with holonomy $(m, 0)$, $m \geq 1$. In the same way there is one with holonomy $(0, m')$, so Λ has rank 2. An element of $\text{Aff}^\pm(q)$ permutes Σ and the saddle connections and transforms holonomy vectors by its derivative M ; applied to the element and to its inverse this gives $M\Lambda = \Lambda$. $\square$

For the seeds, $\Lambda(M_2) = \{(x, y) \in \mathbb{Z}^2 : x + y \text{ even}\}$ and $\Lambda(M_3) = \mathbb{Z}^2$. Indeed, in M_2 the singular corners are the top corners with odd x and the bottom corners with even x ; they and the translation parts $(8, 0)$, $(i + \tau(i) + 1, 2)$, $(i + \tau'(i) + 1, 0)$ of the transition maps have even coordinate sum, and the diagonal of Q_i for even i and the anti-diagonal for odd i are saddle connections with holonomy $(1, 1)$ and $(-1, 1)$. In M_3 all top corners and the odd bottom corners are singular, so the top edges and the vertical edges at odd i are saddle connections.

Lemma 4.4 (Conjugacy is detected in the plane). *Let $\phi_1, \phi_2 \in \text{Aff}^+(q)$ have hyperbolic derivatives. (i) If the classes $[\phi_1]$, $[\phi_2]$ are conjugate in $\text{Mod}(S)$, then $D\phi_1$, $D\phi_2$ are conjugate in $\text{PSL}(\Lambda)$. (ii) If they are conjugate in $\text{Mod}^\pm(S)$, then $D\phi_1$, $D\phi_2$ are conjugate in $\text{PGL}(\Lambda)$.*

Proof. The classes $[\phi_i]$ lie in $\text{Stab}(\mathbb{D})$ and are pseudo-Anosov (Lemma 4.1).

(i) Let $h[\phi_1]h^{-1} = [\phi_2]$ with $h \in \text{Mod}(S)$. Then $h(\mathbb{D})$ is the Teichmüller disc of the transported structure, and it is stabilised by $[\phi_2]$; by (L3), $h(\mathbb{D}) = \mathbb{D}$. By (L1), h is represented by some $\psi \in \text{Aff}^+(q)$, and $D\psi D\phi_1 D\psi^{-1} = D\phi_2$, with $D\psi \in \text{PSL}(\Lambda)$ by Lemma 4.3.

(ii) Let h be orientation-reversing, H a homeomorphism in the class h , and $\kappa(x, y) = (x, -y)$. The charts $\kappa \circ \zeta \circ H^{-1}$, ζ a natural chart of q , define a half-translation structure q^H compatible with the orientation, and $H\phi_1 H^{-1}$ is an affine automorphism of q^H with derivative $\kappa(D\phi_1)\kappa$, in the class $[\phi_2]$. So $[\phi_2]$ stabilises the disc of q^H , which is $\mathbb{D}$ by (L3). Hence there are $M_0 \in \text{SL}(2, \mathbb{R})$ and a homeomorphism ψ isotopic to the identity which is conformal from the complex structure of q^H to that of $M_0 \cdot q$. Then q^H and $\psi^*(M_0 \cdot q)$ are holomorphic quadratic differentials on one Riemann surface X with the same Teichmüller disc, and such differentials are proportional over $\mathbb{C}$: a point $Y \neq X$ of the disc lies on a Teichmüller ray from X for each of them, after a rotation of the differentials; by Teichmüller's uniqueness theorem the extremal map $X \rightarrow Y$ is unique, so the two initial differentials q_1, q_2 satisfy $q_1/|q_1| = q_2/|q_2|$, and a meromorphic function with positive values is constant. Therefore $q^H = \psi^*(M_1 \cdot q)$ with $M_1 \in \text{GL}^+(2, \mathbb{R})$. In charts, this says that $\psi \circ H$ is an orientation-reversing affine automorphism of q in the class h , with derivative $N = \pm M_1^{-1}\kappa$, which lies in $\text{PGL}(\Lambda)$ by Lemma 4.3. The affine automorphisms $(\psi H)\phi_1(\psi H)^{-1}$ and ϕ_2 of q are isotopic, so by (L1) their derivatives agree: $N D\phi_1 N^{-1} = D\phi_2$. $\square$

5 Words with equal traces

Lemma 5.1. (a) For every word w in $x^{\pm 1}, y^{\pm 1}$ there is a polynomial $P_w \in \mathbb{Z}[t_1, t_2, t_3]$ with $\text{tr } w(X, Y) = P_w(\text{tr } X, \text{tr } Y, \text{tr } XY)$ for all $X, Y \in \text{SL}(2, \mathbb{C})$. (b) If w^R denotes w read backwards, then $\text{tr } w^R(X, Y) = \text{tr } w(X, Y)$ for all $X, Y \in \text{SL}(2, \mathbb{C})$.

Proof. (a) Put $x = \text{tr } X$, $y = \text{tr } Y$, $z = \text{tr } XY$. By the theorem of Cayley and Hamilton, $X^2 = xX - 1$ and $X^{-1} = x - X$, the same for Y , and by polarisation $YX = -XY + xY + yX + (z - xy)$. So the set of the matrices $p_0 + p_1X + p_2Y + p_3XY$, with polynomials p_i in x, y, z with integer coefficients, is closed under right multiplication by $X^{\pm 1}, Y^{\pm 1}$, by universal formulas (for example $XY \cdot X = Y + zX - y$), and the trace of such a matrix is $2p_0 + p_1x + p_2y + p_3z$. (b) $w^R(X, Y)$ is the inverse of $w(X^{-1}, Y^{-1})$, inverse matrices in $\text{SL}(2, \mathbb{C})$ have the same trace, and $(\text{tr } X^{-1}, \text{tr } Y^{-1}, \text{tr } X^{-1}Y^{-1}) = (\text{tr } X, \text{tr } Y, \text{tr } XY)$. $\square$

Part (a) is classical: [KLSS07, Lemma 5.1] states it as well known, probably going back to Klein, with a reference to Horowitz [Hor72] for a proof, and it is quoted from [Hor72] in the proof of [Mas00, Lemma 4.1]. Part (b) is [KLSS07, Proposition 5.3], with the same proof; Anderson's survey [And03] is named there for a discussion of such identities.

Let σ_0, σ_1 be the substitutions (endomorphisms of the free monoid on x, y)

$$\sigma_0: x \mapsto xyx^2y^2, y \mapsto y; \quad \sigma_1: x \mapsto y^2x^2yx, y \mapsto y,$$

and for $m \geq 0$ and $\varepsilon \in \{0, 1\}^m$ let $w_\varepsilon = \sigma_{\varepsilon_1} \circ \dots \circ \sigma_{\varepsilon_m}(x)$, a positive word with 3^m letters x and $\frac{3}{2}(3^m - 1)$ letters y . For $m = 1$ the following lemma says that $\text{tr}(XYX^2Y^2) = \text{tr}(Y^2X^2YX)$.

Lemma 5.2. For each m the 2^m words w_ε satisfy: (a) $\text{tr } w_\varepsilon(X, Y)$ does not depend on ε , for all $X, Y \in \text{SL}(2, \mathbb{C})$; (b) in the free group on x, y they are pairwise non-conjugate, and no w_ε is conjugate to the inverse of any $w_{\varepsilon'}$.

Proof. Let W_m be the set of the w_ε and R the reversal. Both $R \circ \sigma_0$ and $\sigma_1 \circ R$ reverse products and agree on the letters; so $R \circ \sigma_0 = \sigma_1 \circ R$, hence $R \circ \sigma_1 = \sigma_0 \circ R$, and by induction $R(W_m) = W_m$.

(a) Induction on m . If all words of W_m have the same trace function, then so have all words $\sigma_0(w)$, $w \in W_m$, because $\text{tr } \sigma_0(w)(X, Y) = \text{tr } w(XYX^2Y^2, Y)$. For $w \in W_m$ we have $\sigma_1(w) = R(\sigma_0(R(w)))$ with $R(w) \in W_m$, so by Lemma 5.1(b) the word $\sigma_1(w)$ has the same trace function as $\sigma_0(R(w))$.

(b) A positive word which contains x is determined, as a cyclic word, by the cyclic sequence $(k_1, \dots, k_r)$ of the numbers of letters y between consecutive letters x . If w has the sequence $(k_1, \dots, k_r)$, then $\sigma_0(w)$ has $(1, 0, k_1 + 2, \dots, 1, 0, k_r + 2)$ and $\sigma_1(w)$ has $(0, 1, k_1 + 2, \dots, 0, 1, k_r + 2)$. In a sequence of the first kind every 0 is followed by an entry ≥ 2 , in one of the second kind every 0 is followed by 1; and in both cases the entries $k_i + 2$ can be read off in cyclic order. Hence the cyclic word of $\sigma_\varepsilon(w)$ determines ε and the cyclic word of w , and by induction the cyclic word of w_ε determines ε . Positive words are cyclically reduced, two cyclically reduced words are conjugate in the free group only if they are cyclic rotations of each other, and the inverse of a positive word is a negative word. $\square$

6 Conjugacy in a subgroup and in $\text{PSL}(2, \mathbb{Z})$

Lemma 6.1. Let a group Φ act on a tree T such that every vertex stabiliser has at most s elements. Let $G \subset \Phi$ be generated by $g_1, \dots, g_r$, let x_* be a vertex and Y the finite subtree spanned by $x_*, g_1x_*, \dots, g_rx_*$. Then every conjugacy class of Φ which consists of elements of infinite order meets G in at most $s \cdot |V(Y)|$ conjugacy classes of G .

Proof. The union T' of the translates hY , $h \in G$, is a G -invariant subtree, because Y meets $g_i^{\pm 1}Y$, and every vertex of T' is a G -translate of a vertex of Y . For an automorphism g of T let $\text{Min}(g)$ be the set of the vertices x for which $d(x, gx)$ is minimal; then $\text{Min}(cgc^{-1}) = c \text{Min}(g)$.

Claim: $\text{Min}(g') \subset T'$ if $g' \in G$ has infinite order. Such a g' fixes no vertex. Let $x \in \text{Min}(g')$, suppose that $x \notin T'$, and let y be the vertex of T' nearest to x . Then $g'y \neq y$ is the vertex of T' nearest to $g'x$, and the geodesic from x to $g'x$ passes through y and $g'y$; so $d(x, g'x) = 2d(x, y) + d(y, g'y) > d(y, g'y)$, a contradiction.

Now let $g \in G$ have infinite order and fix $x_0 \in \text{Min}(g)$. If $c \in \Phi$ and $g' = cgc^{-1} \in G$, then $cx_0 \in \text{Min}(g') \subset T'$, so $hcx_0 \in V(Y)$ for some $h \in G$, and $hg'h^{-1} = (hc)g(hc)^{-1}$ is conjugate to g' in G . So every class of G in the Φ -class of g contains an element $c'gc'^{-1}$ with $c'x_0 \in V(Y)$, and there are at most $s \cdot |V(Y)|$ elements $c' \in \Phi$ with this property. $\square$

Corollary 6.2. *Let G be a finitely generated subgroup of $\text{PSL}(\Lambda) \cong \text{PSL}(2, \mathbb{Z})$. There is a constant $C = C(G)$ such that every conjugacy class of $\text{PSL}(\Lambda)$ which consists of elements of infinite order meets G in at most C conjugacy classes of G , and every such class of $\text{PGL}(\Lambda)$ in at most $2C$.*

Proof. By a classical theorem, $\text{PSL}(2, \mathbb{Z})$ is the free product of a subgroup $\langle s \rangle$ of order 2 and a subgroup $\langle u \rangle$ of order 3. It acts on the following tree: the vertices are the cosets $h\langle s \rangle$ and $h\langle u \rangle$, the edges are the group elements h , and h joins $h\langle s \rangle$ to $h\langle u \rangle$. This graph is connected because s and u generate, and it has no circuit by the normal form theorem for free products. The vertex stabilisers are the conjugates of $\langle s \rangle$ and $\langle u \rangle$, so Lemma 6.1 gives $C = 3|V(Y)|$. A conjugacy class of $\text{PGL}(\Lambda)$ contained in $\text{PSL}(\Lambda)$ is the union of at most two classes of $\text{PSL}(\Lambda)$, because the index is 2. $\square$

7 Proof of Theorem 1.1

Let $g \geq 2$, $k \geq 1$, $N \geq 1$ and $d = \max(k, 2)$. Take an admissible model $S = S(n, \tau, \tau')$ of genus g (Proposition 3.3) with $a, b, q, A, B, \rho, \Theta, \Lambda$ as in Section 4.

The subgroup. In F_2 let $c_1 = \beta$, $c_{j+1} = [\alpha, c_j]$, $u_0 = c_d$ and $v_0 = \beta u_0 \beta^{-1}$; both lie in $\gamma_d(F_2)$. We use three standard facts on free groups: commuting elements are powers of a common element, every non-trivial element lies in a unique maximal cyclic subgroup, and a non-abelian subgroup generated by two elements is free with these two elements as a basis. First, $c_j \neq 1$ for all j : if $[\alpha, c_j] = 1$, then c_j is a power of α , which is impossible for $c_1 = \beta$, and for $j \geq 2$ the exponent sums of c_j vanish, so $c_j = 1$, contrary to induction. Secondly, u_0 and v_0 do not commute: otherwise the maximal cyclic subgroups $\langle r \rangle \ni u_0$ and $\langle \beta r \beta^{-1} \rangle \ni v_0$ would coincide, so $\beta r \beta^{-1} = r^{\pm 1}$; then β^2 commutes with r , so r and u_0 are powers of β , and $u_0 = 1$ because its exponent sums vanish. Put $u = \rho(u_0)$, $v = \rho(v_0)$ and $G = \langle u, v \rangle \subset \text{PSL}(\Lambda)$. Since ρ is injective, G is free with basis (u, v) , and $G \subset \rho([F_2, F_2])$.

The classes. With the words w_ε of Section 5 put $g_\varepsilon = w_\varepsilon(u, v) \in G$ and $f_\varepsilon = \Theta(w_\varepsilon(u_0, v_0))$.

1. $f_\varepsilon \in \gamma_d(\langle T_a, T_b \rangle) \subset \mathcal{I}(2d) \subset \mathcal{I}(k)$, by (3).
2. $g_\varepsilon \neq 1$ lies in the commutator subgroup of $\langle A, B \rangle$, so it is hyperbolic (Lemma 4.2), and f_ε is pseudo-Anosov with $\lambda(f_\varepsilon) + \lambda(f_\varepsilon)^{-1} = |\text{tr } g_\varepsilon|$ (Lemma 4.1).
3. With lifts $U, V \in \text{SL}(2, \mathbb{R})$ of u, v , the number $\text{tr } w_\varepsilon(U, V)$ does not depend on ε (Lemma 5.2(a)). So all f_ε , $\varepsilon \in \{0, 1\}^m$, have the same dilatation.
4. The 2^m elements g_ε are pairwise non-conjugate in G , and no g_ε is conjugate in G to the inverse of any $g_{\varepsilon'}$ (Lemma 5.2(b); G is free on u, v).
5. Let $C = C(G)$ be the constant of Corollary 6.2; it does not depend on m . Call $\varepsilon, \varepsilon'$ equivalent if $f_{\varepsilon'}$ is conjugate in $\text{Mod}^\pm(S)$ to f_ε or to f_ε^{-1} . Then $g_{\varepsilon'}$ is conjugate in $\text{PGL}(\Lambda)$ to g_ε or to g_ε^{-1} (Lemma 4.4, applied to the affine automorphisms and to an inverse). These two classes of $\text{PGL}(\Lambda)$ meet G in at most $4C$ classes of G , and by step 4 each class of G contains at most one $g_{\varepsilon'}$. So every equivalence class has at most $4C$ elements.

Choose m with $2^m \geq 4CN$ and one ε from each of N equivalence classes. $\square$

Remark 7.1. For conjugacy in Mod_g alone the argument gives at least $2^m/C$ classes and uses only Lemma 4.4(i). Since w_ε has length $\frac{1}{2}(5 \cdot 3^m - 3)$, we have $\log \lambda(f_\varepsilon) \leq c_1 3^m$; so the proof gives values $r \rightarrow \infty$ in $\text{spec}(\mathcal{I}_g(k))$ with multiplicity at least $c_2 r^{0.63}$, where $0.63 < \log 2 / \log 3$. The values tend to infinity with m : by a classical fact, $\text{PSL}(2, \mathbb{Z})$ has only finitely many conjugacy classes of hyperbolic elements with a given trace, and by Corollary 6.2 and step 4 each of them contains at most C of the elements g_ε . A root of f_ε in Mod_g has the same invariant disc, so it is affine, and its derivative is a root of g_ε ; hence f_ε is primitive in Mod_g if g_ε is primitive in $\text{PSL}(\Lambda)$. We have no general proof of this; in all 24 instances of Section 8 it was found by computation (the cyclic words of the elements g_ε are not proper powers).

Remark 7.2. The quotient $\mathbb{H}^2 / \langle A, B \rangle$ is a sphere with two cusps and one funnel (compare [Lei04, Lemma 6.3]); its only simple closed geodesic is the core of the funnel, which belongs to AB^{-1} and not to an element of the commutator subgroup. So the axis of g_ε crosses one of its translates under $\langle A, B \rangle$, and the closed Teichmüller geodesic of f_ε passes twice through one point of moduli space: if X lies on the axis of f_ε and on its image under $h \in \langle T_a, T_b \rangle$, where the image is another geodesic, then X and $h^{-1}X$ lie on the axis, and $h^{-1}X \neq f_\varepsilon^j X$ for all j , because the stabiliser of X is finite and $\langle T_a, T_b \rangle$ is torsion-free.

8 Explicit instances

The statements of this section are computer-assisted: they rest on exact computations with integers. Elements of $\text{PSL}(\Lambda)$ are written as integer matrices M in a basis of Λ . For hyperbolic elements two classical complete invariants of the conjugacy class in $\text{PSL}(2, \mathbb{Z})$ were computed, by independent programs: the cyclic word in $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of a conjugate of $\pm M$ with positive entries (for suitable generators of the two factors, $R = su$ and $L = su^{-1}$, so the products in which both letters occur are the cyclically reduced words for the hyperbolic elements), and the cycle of reduced forms of the quadratic form $v \mapsto \det(v, Mv)$. Both transform in an explicit way under conjugation by a matrix of determinant -1 and under inversion.

Take M_2 or M_3 , $d \geq 2$, and $u_0, v_0, g_\varepsilon, f_\varepsilon$ as in Section 7; by (3), $f_\varepsilon \in \mathcal{I}_g(2d)$. For M_2 and $d = 2$ we have, in the standard basis of $\mathbb{R}^2$, $u = \begin{pmatrix} 4161 & -512 \\ 512 & -63 \end{pmatrix}$ and $v = \begin{pmatrix} 8257 & -512 \\ 67072 & -4159 \end{pmatrix}$, and the common trace $t = |\text{tr } g_\varepsilon|$ is 4508549040455682 for $m = 1$; for M_3 , $d = 2$, $m = 1$ it is 2958790142831846402. Table 1 gives the numbers of classes among the 2^m elements g_ε . Altogether 24 instances were computed: for M_2 all $m \leq 6$ ($d = 2$), $m \leq 5$ ($d = 3$), $m \leq 4$ ($d = 4$), and for M_3 all $m \leq 5$ ($d = 2$), $m \leq 4$ ($d = 3$). In each of them the 2^m elements have one trace and are pairwise non-conjugate in $\text{PSL}(\Lambda)$, also up to inversion; and they are pairwise non-conjugate in $\text{PGL}(\Lambda)$ up to inversion, except in the two instances with $(d, m) = (2, 1)$.

surface	d	classes in	m	2^m	digits of t	$\log \lambda$	$\text{PSL}(\Lambda)$	$\text{PSL}(\Lambda)^\pm$	$\text{PGL}(\Lambda)$	$\text{PGL}(\Lambda)^\pm$
M_2	2	$\mathcal{I}_2(4)$	1	2	16	36.045	2	2	1	1
			2	4	58	133.089	4	4	4	4
			6	64	5116	11778.401	64	64	64	64
M_2	3	$\mathcal{I}_2(6)$	5	32	5065	11660.843	32	32	32	32
M_2	4	$\mathcal{I}_2(8)$	4	16	3185	7331.576	16	16	16	16
M_3	2	$\mathcal{I}_3(4)$	1	2	19	42.531	2	2	1	1
			5	32	2015	4637.803	32	32	32	32
M_3	3	$\mathcal{I}_3(6)$	4	16	2002	4608.459	16	16	16	16

Table 1: Conjugacy classes among the 2^m elements g_ε (computer-assisted); $\pm$ means up to inversion.

By Lemma 4.4 the number of classes in $\mathrm{PSL}(\Lambda)$ is a lower bound for the number of conjugacy classes of Mod_g among the f_ε , and the number in $\mathrm{PGL}(\Lambda)$ up to inversion for the number of classes of $\mathrm{Mod}_g^\pm$ up to inversion. For example, there are 64 pseudo-Anosov classes in $\mathcal{I}_2(4)$ with one dilatation which are pairwise non-conjugate in $\mathrm{Mod}_2^\pm$, also up to inversion; and 32 such classes in $\mathcal{I}_2(6)$, 16 in $\mathcal{I}_2(8)$, 32 in $\mathcal{I}_3(4)$, 16 in $\mathcal{I}_3(6)$.

The coincidence for $m = 1$. On M_2 with $d = 2$ the classes $f_0 = \Theta(u_0 v_0 u_0^2 v_0^2)$ and $f_1 = \Theta(v_0^2 u_0^2 v_0 u_0)$ are not conjugate in Mod_2 , also not after inversion, but they are conjugate in $\mathrm{Mod}_2^\pm$. The reflection $x \mapsto -x$ of the cylinder is an orientation-reversing isometry of M_2 (the map $i \mapsto -1 - i$ commutes with τ and τ') which conjugates τ_a, τ_b to their inverses, and the automorphism $\alpha \mapsto \alpha^{-1}, \beta \mapsto \beta^{-1}$ of F_2 , followed by the conjugation with $\beta\alpha\beta$, exchanges u_0 and v_0 . So the class of $w(u_0, v_0)$ is conjugate in $\mathrm{Mod}_2^\pm$ to that of $w(v_0, u_0)$ for every word w , and exchanging x and y maps xyx^2y^2 to a rotation of y^2x^2yx . The same happens for M_3 . For $m \geq 2$ the words of W_m contain different numbers of letters x and y , so this symmetry does not identify two of them. Hence the constant $4C$ in Section 7.

Verification. Theorem 1.1 and all lemmas are proved without computation, given the cited facts (L1)–(L3); the statements of Section 8 and the last sentence of Remark 7.1 are computer-assisted. The results were first obtained, with proofs and programs, in a first written version. Two independent verification runs, both AI-assisted, followed, each with its own programs and without knowledge of the other’s report. The first run re-derived every lemma and the deduction of Theorem 1.1, compared the statement with the source, tested Lemmas 3.1 and 4.2 on large samples, and recomputed all 24 instances with the invariants from quadratic forms, with the same traces and class counts. The second run rebuilt the seeds with its own code and computed the two twists from the curves alone; with Magnus expansions up to degree 9 it found that on the once-punctured M_2 the twists and c_2, c_3, c_4 have Johnson level exactly 2, 4, 6, 8 (the original program for this test uses the pair (τ_a, τ_b^{-1}) and gives the same levels up to degree 6). It determined the handedness by a test against the holonomy, reproduced Table 1 and the case $m = 1$ with its own conjugacy test, and re-ran the original programs with identical results. It also obtained the non-conjugacy statements for $M_2, d = 2, m \leq 6$ in a second way, which uses neither Lemma 4.4 nor (L1) and (L3): the action of the classes f_ε on the twisted homology of the 15 double covers of M_2 . This computation was repeated by the program `check_note.py` of this note, and the argument is written out in the reproducibility package. Neither run found a mathematical error; their requirements (the presentation as an observation, the notion of multiplicity, the reason for a finitely generated subgroup, short arguments for two steps of Lemma 4.4, references, small corrections) were met.

Written after these two runs were the remark that all intersection points of an admissible model are good, the present form of Proposition 3.3, and `check_note.py`. A third run, also AI-assisted, examined the final text, and these things first. It read the source again, went through all proofs line by line, and compared the cited statements with the texts of [Lei04, FLM08, AT17, Jef24, KLSS07, Mas00]. With programs of its own, written from the statements of this note, it rebuilt the seeds; tested the remark on good points on all 688 admissible triples with $n \leq 12$, and Lemma 3.2 and Proposition 3.3 on 400 sums of seeds and on chains of sums up to genus 30; obtained A and $\pm B$ as the linear parts of the two shears; and recomputed Table 1 and all 24 instances with a program of its own for the cyclic words. For $(d, m) = (2, 1)$ it confirmed in an elementary way that g_0 and g_1 are not conjugate in $\mathrm{PSL}(\Lambda)$, by reduction modulo 11 for M_2 and modulo 7 for M_3 . It recomputed the second certificate with a third cell structure, for M_2 and $d = 2, 3, 4$ (all 15 instances), found the elements g_ε of all 24 instances primitive, and ran the package again from the archive, with the recorded outputs. It found no mathematical error; it required that this paragraph describe the state after its checks, locations in the source which do not depend on the version, the attributions after Lemmas 4.1, 4.2 and 5.1, the form in which the theorem of Arnoux–Yoccoz and Ivanov is quoted, and small changes of wording in

Sections 3, 4, 7 and 8. All programs are in the package, with recorded outputs which were reproduced on 11 October 2026.

Scope and priority. The note answers Question 7.6 of [Far06] in the negative for all $g \geq 2$ and all k ; it does not answer Question 7.7. It is an observation: the source contains both ingredients, the mechanism for $k \leq 2$ [Far06, §7.2] and the remark that elements deep in the lower central series of a group generated by two twists about separating curves lie in $\mathcal{I}_g(k)$ (in the proof of [Far06, Theorem 5.10]; it is the construction in the proof of [FLM08, Proposition 4.8]). The note combines them and adds the passage from conjugacy in the free group to conjugacy in Mod_g , whose ingredients are also known: Thurston’s construction, the fact that the elements of $\langle T_a, T_b \rangle$ outside the conjugates of powers of the twists are pseudo-Anosov, the affine representation of the stabiliser of a Teichmüller disc and the uniqueness of the invariant disc are taken from [Lei04], trace identities of this kind go back to Horowitz [Hor72] and Randol [Ran80], and filling pairs of separating curves are in [AT17, Jef24]. We found Lemma 6.1 and the family of Lemma 5.2 nowhere in this form; both are elementary. No priority is claimed. The answer is apparently not recorded in the literature accessible to us; it may well be known to specialists, and we cannot exclude that the question was meant in a sense which we did not consider.

Read, in the arXiv versions, whose numbering we use: [Far06], Sections 1.2 and 7.1–7.3 and Theorem 5.10 with its proof (pages 6–7, 33–34 and 44–47; the same wording and the same numbers are in the author’s public file of the volume, with which the first and the third verification run compared them); [Lei04], Sections 2.1, 2.4, 3.1, 3.2, 5.1, 6.1 and 6.2; [FLM08] (v2), the introduction, Section 2.3, Theorem 1.2 and Proposition 4.8 with its proof, and this paper contains no statement on multiplicities; [AT17], Lemma 2.2 and Theorem 5.1 with their proofs; [Jef24, Theorem 1.2]; [KLSS07] (v2), introduction and Section 5. The theorems of Randol and Horowitz were taken from [Mas00, Theorem 3.1, Lemma 4.1]. Not read, or not accessible to us: the printed volume of the source and the journal versions of these papers; the original papers of Bers [Ber78], Thurston, Kra and Veech, whose statements were taken from [Lei04]; those of Randol [Ran80] and Horowitz [Hor72]; those of Johnson and of Bass and Lubotzky, which are quoted through [Far06, FLM08]; and [And03, AY81, Iva90]. The searches of 11 October 2026 (arXiv, OpenAlex, zbMATH Open, Crossref, Semantic Scholar; the works which cite the source, by titles and available abstracts; MathOverflow through its public interface; seven web searches) found no other text which states or answers the question; the closest items concern pairs of non-conjugate classes with equal dilatation. A search that finds nothing is not a proof of novelty.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

References

- [And03] James W. Anderson. Variations on a theme of Horowitz. In *Kleinian Groups and Hyperbolic 3-Manifolds*, volume 299 of *London Mathematical Society Lecture Note Series*, pages 307–342. Cambridge University Press, 2003. doi:10.1017/CB09780511542817.015.
- [AT17] Tarik Aougab and Samuel J. Taylor. Pseudo-Anosovs optimizing the ratio of Teichmüller to curve graph translation length. In *In the Tradition of Ahlfors–Bers, VII*, volume 696 of *Contemporary Mathematics*, pages 17–28. American Mathematical Society, 2017. arXiv:1510.00995, doi:10.1090/conm/696/14014.

- [AY81] Pierre Arnoux and Jean-Christophe Yoccoz. Construction de difféomorphismes pseudo-Anosov. *Comptes Rendus de l'Académie des Sciences, Série I*, 292(1):75–78, 1981.
- [Ber78] Lipman Bers. An extremal problem for quasiconformal mappings and a theorem by Thurston. *Acta Mathematica*, 141:73–98, 1978. doi:10.1007/BF02545743.
- [Far06] Benson Farb. Some problems on mapping class groups and moduli space. In Benson Farb, editor, *Problems on Mapping Class Groups and Related Topics*, volume 74 of *Proceedings of Symposia in Pure Mathematics*, pages 11–55. American Mathematical Society, 2006. arXiv:math/0606432, doi:10.1090/pspum/074/2264130.
- [FLM08] Benson Farb, Christopher J. Leininger, and Dan Margalit. The lower central series and pseudo-Anosov dilatations. *American Journal of Mathematics*, 130(3):799–827, 2008. arXiv:math/0603675, doi:10.1353/ajm.0.0005.
- [Hor72] Robert D. Horowitz. Characters of free groups represented in the two-dimensional special linear group. *Communications on Pure and Applied Mathematics*, 25(6):635–649, 1972. doi:10.1002/cpa.3160250602.
- [Iva90] Nikolai V. Ivanov. Stretching factors of pseudo-Anosov homeomorphisms. *Journal of Soviet Mathematics*, 52(1):2819–2822, 1990. doi:10.1007/BF01099245.
- [Jef24] Luke Jeffreys. Meanders, hyperelliptic pillowcase covers, and the Johnson filtration. *Geometriae Dedicata*, 218(4):Paper No. 93, 2024. arXiv:2210.11332, doi:10.1007/s10711-024-00936-w.
- [KLSS07] Ilya Kapovich, Gilbert Levitt, Paul Schupp, and Vladimir Shpilrain. Translation equivalence in free groups. *Transactions of the American Mathematical Society*, 359(4):1527–1546, 2007. arXiv:math/0409284, doi:10.1090/S0002-9947-06-03929-8.
- [Lei04] Christopher J. Leininger. On groups generated by two positive multi-twists: Teichmüller curves and Lehmer’s number. *Geometry & Topology*, 8(3):1301–1359, 2004. arXiv:math/0304163, doi:10.2140/gt.2004.8.1301.
- [Mas00] Joseph D. Masters. Length multiplicities of hyperbolic 3-manifolds. *Israel Journal of Mathematics*, 119(1):9–28, 2000. arXiv:math/9812069, doi:10.1007/BF02810661.
- [Ran80] Burton Randol. The length spectrum of a Riemann surface is always of unbounded multiplicity. *Proceedings of the American Mathematical Society*, 78(3):455–456, 1980. doi:10.1090/S0002-9939-1980-0553396-1.
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Version 1.6.0; record AMR-109-0082.