

Riemann Surfaces of Genus 9 with Nilpotent Automorphism Group of the Largest Order Are Not Unique: On a Question in Farb's Problem 2.19

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Abstract

Problem 2.19 of Farb's list of problems on mapping class groups asks for properties of automorphism groups that determine a unique point of the moduli space $\mathcal{M}_g$, and asks as an example whether there is a unique Riemann surface of genus $g \geq 2$ whose automorphism group is nilpotent and of the largest order possible for a nilpotent group of automorphisms of a surface of genus g . By a theorem of Zomorrodian this largest order is $16(g-1)$, attained only when $g-1$ is a power of two. We show that the surface is not unique: there are at least four pairwise non-isomorphic compact Riemann surfaces of genus 9 whose full automorphism group is a 2-group of order $128 = 16(g-1)$. The surfaces are quotients of the hyperbolic plane by torsion-free normal subgroups of the triangle group $\Delta(2, 4, 8)$, given by explicit presentations of the four groups of order 128, and the proof that each of them is the full automorphism group uses, besides finite computations, only that the triangle group $\Delta(2, 3, 8)$ is the only Fuchsian group properly containing $\Delta(2, 4, 8)$, which we prove. The presentations are taken from Conder's list of large group actions, and all verifications are finite computations in groups of order 128 and 3072, carried out with programs written for this note. Under the completeness of Conder's list, which two independent enumerations confirm for these genera, the number of such surfaces is 1, 1, 4, 3, 7 and 15 for $g = 3, 5, 9, 17, 33, 65$, and none for $g = 2$ under our reading of the question. The four groups of genus 9 are also listed, with their generating vectors, in the database LMFDB. The first sentence of Problem 2.19 is open-ended and is not addressed. No statement of the result was found in the literature accessible to us; a search that finds nothing is not a proof of novelty. This is an unrefereed note.

1 Introduction

1.1 The question

Farb's list of problems on mapping class groups and moduli space [Far06] contains, in its section on automorphisms of Riemann surfaces, the following problem (p. 16 of the arXiv version).

Problem 2.19 of [Far06] (Canonical basepoints for $\mathcal{M}_g$). *Find other properties of automorphisms or automorphism groups that determine a unique point of $\mathcal{M}_g$. For example, is there a unique Riemann surface of genus $g \geq 2$ whose automorphism group is nilpotent, and is the largest possible order among nilpotent automorphism groups of genus g surfaces?*

The problem is preceded by the theorem of Kulkarni, quoted in [Far06], that the surface of genus g with an automorphism of order $4g+2$ is unique, and by the theorem of Zomorrodian [Zom85], quoted in [Far06] in the following form: if $\text{Aut}(X_g)$ is nilpotent, then its order is at most $16(g-1)$,

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and if this bound is attained, then $g - 1$ is a power of 2. In [Zom85] the theorem is stated for finite nilpotent groups G of automorphisms (Theorem 1.8.4 there): $|G| \leq 16(g - 1)$, with equality if and only if $X = \mathbb{H}/\Gamma$ for a proper normal subgroup Γ of finite index of the group of signature $(0; 2, 4, 8)$; such a group G is a 2-group, so that $g - 1$ is a power of 2 (Corollary 2.1.1 there); and for every $n \geq 4$ a nilpotent 2-group of order 2^n acts on a surface of genus $2^{n-4} + 1$ (Theorem 2.1.2 there). The introduction of [Zom85] states the converse as the existence of “at least one surface” of genus g with an automorphism group of order $16(g - 1)$; nothing is said there about the number of such surfaces. (The review Zbl 0565.20029 of [Zom85] has “ $g = 2^k - 1$ ”, evidently a misprint for $2^k + 1$.) The later paper [Zom87] proves (its Theorem 1.1.2) that a 2-group of automorphisms of a surface X of genus $g \geq 2$ has order at most $16(g - 1)$, with equality if and only if $X = \mathbb{H}/\Gamma$ for a proper normal subgroup Γ of finite index of the group of signature $(0; 2, 4, 8)$, and (its Theorem 1.2.1) that for every $n \geq 4$ a 2-group of order 2^n acts on a surface of genus $2^{n-4} + 1$; it says nothing about the number of such surfaces. The problem is record AMR-109-0020 of the UnsolvedMath collection [ula26], listed as open; the statement of the record continues with the sentences of [Far06] on Hurwitz surfaces which lead to the next problem there, and which are not part of Problem 2.19.

We read the example question as follows. Let $g \geq 2$ with $g - 1 = 2^n$, so that $16(g - 1)$ is the largest order of a nilpotent group of automorphisms of a surface of genus g . Is there exactly one compact Riemann surface X of genus g , up to isomorphism, whose full automorphism group $\text{Aut}(X)$ is nilpotent of order $16(g - 1)$? Since a nilpotent group of order $16(g - 1) = 2^{n+4}$ is a 2-group, the condition is that $\text{Aut}(X)$ is a 2-group of order $16(g - 1)$.

1.2 Results

Theorem 1.1. *There are at least four pairwise non-isomorphic compact Riemann surfaces of genus 9 whose full automorphism group is a 2-group of order $128 = 16(g - 1)$. Hence the surface of genus 9 whose automorphism group is nilpotent and of the largest possible order is not unique, and the example question of Problem 2.19 has a negative answer.*

The four surfaces are $X_i = \mathbb{H}/K_i$, $i = 1, \dots, 4$, where K_i is the kernel of the epimorphism from the triangle group

$$L = \Delta(2, 4, 8) = \langle X, Y, Z \mid X^2, Y^4, Z^8, XYZ \rangle$$

onto the group G_i of order 128 defined by the relators of Table 2. The proof has three parts: the groups G_i have order 128 and the epimorphisms are smooth, so that X_i is a surface of genus 9 on which G_i acts (Section 3); the only Fuchsian group that properly contains L is a triangle group $F = \Delta(2, 3, 8)$ in which L has index 3 (Lemma 2.1), and K_i is not normal in F , so that $\text{Aut}(X_i) = L/K_i \cong G_i$ (Lemma 2.3); and the kernels are pairwise different, so that the surfaces are pairwise non-isomorphic (Lemma 2.4). The presentations were taken from Conder’s list of large group actions [Con26]; everything that is used about them is verified by finite computations described in Section 5.

Theorem 1.2 (computer-assisted; depends on the completeness of [Con26]). *Suppose that the list [Con26] contains every action of a group of order $16(g - 1)$ with signature $(0; 2, 4, 8)$ on a surface of genus g , $2 \leq g \leq 101$, up to the equivalences stated there. Then the number of compact Riemann surfaces of genus g , up to isomorphism, whose full automorphism group is a 2-group of order $16(g - 1)$ is as in Table 1: it is 0 for $g = 2$, 1 for $g = 3$ and $g = 5$, 4 for $g = 9$, 3 for $g = 17$, 7 for $g = 33$ and 15 for $g = 65$.*

The first sentence of Problem 2.19, which asks for other properties determining a unique point of $\mathcal{M}_g$, is open-ended and is not addressed here. For the genera in which $g - 1$ is not a power of 2 the largest order of a nilpotent group of automorphisms is smaller than $16(g - 1)$, and the question of uniqueness for those genera is not considered.

g	$16(g-1)$	actions in [Con26]	of these chiral	kernels normal in F	surfaces with $ \text{Aut} = 16(g-1)$
2	16	1	0	1	0
3	32	2	0	1	1
5	64	2	0	1	1
9	128	4	0	0	4
17	256	4	0	1	3
33	512	8	0	1	7
65	1024	14	2	1	15

Table 1: The actions of signature $(0; 2, 4, 8)$ and order $16(g-1)$ in Conder’s list, for the genera $g \leq 101$ with $g-1$ a power of 2. An action is listed once up to reflection; a chiral action gives two surfaces, a reflexible one gives one. The last column counts the surfaces whose full automorphism group has order $16(g-1)$: the actions whose kernel is not normal in F , counted once if reflexible and twice if chiral.

1.3 Conventions that could change the answer

Full automorphism group. “Automorphism group” is read as the group of all conformal automorphisms of X . If instead one asks for a surface which merely admits a nilpotent group of automorphisms of order $16(g-1)$, the count of Theorem 1.2 in genus 9 is the same (four surfaces, under the completeness hypothesis; the lower bound four of Theorem 1.1 holds under both readings, since no kernel of Table 1 is normal in F for $g = 9$), while for $g = 2, 3, 5, 17, 33, 65$ it is larger by one, because exactly one action in each of these genera extends to an action of a group of order $48(g-1)$. A third reading asks for the largest order of a nilpotent *full* automorphism group in genus g . For $g = 3, 5, 9, 17, 33, 65$ this order is $16(g-1)$ by Table 1, and the answers are the same. For $g = 2$ no full automorphism group has order 16 (Remark 4.1); the full automorphism groups of surfaces of genus 2 are classical, the nilpotent ones among them have orders 2, 4, 8 and 10, and the largest, cyclic of order $10 = 4g + 2$, belongs to Kulkarni’s unique surface W_2 ; under this reading the genus-2 answer is “unique”. We did not re-derive the classical list (Bolza [Bol87]; not read by us).

Isomorphism. Surfaces are counted up to isomorphism of Riemann surfaces (conformal, orientation-preserving). A surface and its mirror image are counted separately when they are not isomorphic; this happens for two of the kernels in genus 65 and for none in the other genera of Table 1.

Largest possible order. The theorem of Zomorrodian is used only to identify $16(g-1)$ as the largest order of a nilpotent group of automorphisms in genus g when $g-1$ is a power of 2. The four surfaces of Theorem 1.1 and the counts of Theorem 1.2 do not depend on it.

1.4 What was known

The bound $16(g-1)$, the condition $g-1 = 2^n$ and the existence of a surface of genus $g = 2^n + 1$ with a nilpotent group of automorphisms of order $16(g-1)$ are due to Zomorrodian [Zom85] (Theorem 1.8.4, Corollary 2.1.1 and Theorem 2.1.2 there). The groups of automorphisms of order greater than $4(g-1)$ of surfaces of genus $2 \leq g \leq 101$ were enumerated by Conder [Con26, Con14] with the routine for normal subgroups of low index of Magma; the list [Con26] is our source for the presentations and for Table 1. The database LMFDB [The26] lists, for genus 9, group order 128 and signature $[0; 2, 4, 8]$, exactly four groups, SmallGroup(128, 75), (128, 134), (128, 136) and (128, 138), with 32 generating vectors in 26 refined passports. So the four groups can be read off from public data, and the non-uniqueness follows from them in one step once the groups are known to be full automorphism groups and pairwise non-isomorphic (Remark 4.3). The inclusion $\Delta(2, 4, 8) < \Delta(2, 3, 8)$ of index 3 is one of

the inclusions in Singerman's classification [Sin72]; we did not read that paper, and we prove what we use (Lemma 2.1). The paper [Zom87] is the closest to our construction: it characterises the surfaces with a 2-group of automorphisms of order $16(g-1)$ as the quotients $\mathbb{H}/K$ by normal subgroups K of the group of signature $(0; 2, 4, 8)$ (Theorem 1.1.2 there), shows that such a 2-group has nilpotency class at least 3 (Corollary 1.4.1 there; the classes in Table 2 are 4 and 5), and gives as an example the group $L/\gamma_4(L)$ of order 64 acting on a surface of genus 5 (the remark after Corollary 1.4.2 there; see Remark 4.2). We found no statement in the literature accessible to us that the surface asked for in Problem 2.19 is or is not unique; see "Scope and priority" at the end of Section 5 for the searches.

2 Surfaces with a 2-group of automorphisms of order $16(g-1)$

2.1 Fuchsian groups

We use the standard facts about Fuchsian groups, for which we refer to [Kat92]. A cocompact Fuchsian group Γ has a signature $(h; m_1, \dots, m_r)$ and a presentation with generators $a_1, b_1, \dots, a_h, b_h, x_1, \dots, x_r$ and relations $x_i^{m_i} = 1$ and $\prod [a_j, b_j] \prod x_i = 1$; the x_i are elliptic, every elliptic element of Γ is conjugate in Γ to a power of one of the x_i , and the hyperbolic area of a fundamental domain is

$$\mu(\Gamma) = 2\pi \left(2h - 2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i} \right) \right) > 0. \quad (1)$$

The least value of μ is $\pi/21$, attained by $\Delta(2, 3, 7)$. If $\Gamma' \supseteq \Gamma$ is a Fuchsian group, then $[\Gamma' : \Gamma]$ is finite and $\mu(\Gamma) = [\Gamma' : \Gamma] \mu(\Gamma')$. A subgroup K of finite index of Γ is torsion-free if and only if it acts freely on $\mathbb{H}$; then $\mathbb{H}/K$ is a compact Riemann surface, and its genus g is given by $2g - 2 = \mu(K)/2\pi = [\Gamma : K] \mu(\Gamma)/2\pi$. For a torsion-free normal subgroup K of Γ the group Γ/K acts on $\mathbb{H}/K$ with quotient $\mathbb{H}/\Gamma$, and $\text{Aut}(\mathbb{H}/K) = N(K)/K$, where $N(K)$ is the normaliser of K in $\text{PSL}(2, \mathbb{R})$; the normaliser of a cocompact Fuchsian group in $\text{PSL}(2, \mathbb{R})$ is again a Fuchsian group [Kat92], and it contains Γ . Two surfaces $\mathbb{H}/K$ and $\mathbb{H}/K'$ are isomorphic if and only if $K' = cKc^{-1}$ for some $c \in \text{PSL}(2, \mathbb{R})$.

For a triangle $T \subset \mathbb{H}$ with angles $\pi/l, \pi/m, \pi/n$, where $1/l + 1/m + 1/n < 1$, the rotations through $2\pi/l, 2\pi/m, 2\pi/n$ about the vertices of T generate a Fuchsian group of signature $(0; l, m, n)$, the triangle group $\Delta(l, m, n)$, in which the three rotations, suitably oriented, are the canonical elliptic generators; any two of them generate the group. By the formula

$$\cosh c = \frac{\cos \alpha \cos \beta + \cos \gamma}{\sin \alpha \sin \beta} \quad (2)$$

for the side c opposite to the angle γ of a hyperbolic triangle with angles α, β, γ , the triangle is determined by its angles up to isometry; hence two Fuchsian groups of the same signature $(0; l, m, n)$ are conjugate in the group of isometries of $\mathbb{H}$, and, since such a group is conjugate to its mirror image by a reflection in a side of its triangle, also in $\text{PSL}(2, \mathbb{R})$ [Kat92].

2.2 The triangle group $\Delta(2, 4, 8)$ and its overgroup

Let T be a triangle with angles $\pi/2, \pi/4, \pi/8$ at the vertices A, B, C , and let X, Y, Z be the rotations through $\pi, \pi/2, \pi/4$ about A, B, C , all in the same sense, so that, with the vertices in the right cyclic order, $XYZ = 1$; since $X^2 = 1$, this says $X = YZ$. Then

$$L := \langle X, Y, Z \rangle = \Delta(2, 4, 8) = \langle X, Y, Z \mid X^2, Y^4, Z^8, Z^{-1}Y^{-1}X \rangle, \quad \mu(L) = 2\pi \left(1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} \right) = \frac{\pi}{4}.$$

(The relation $Z^{-1}Y^{-1}X = 1$ is the form in which the relation of the triangle group is printed in [Con26].) Let w be the rotation through $\pi/4$ about B in the same sense, so that $w^2 = Y$.

Lemma 2.1. (a) $F := \langle X, w \rangle$ is a triangle group of signature $(0; 2, 3, 8)$, with canonical generators $a := X$ and $b := Xw$ of orders 2 and 3 and $ab = w$ of order 8; so $F = \langle a, b \mid a^2, b^3, (ab)^8 \rangle$. It contains L with index 3, and

$$X = a, \quad Y = (ab)^2, \quad Z = b^{-1}ab^{-1}. \quad (3)$$

- (b) $L = \langle a, b^{-1}ab^{-1} \rangle$ is the stabiliser of a point in the action of F on the three cosets of L , and L is not normal in F .
- (c) Every Fuchsian group which properly contains L is equal to F . In particular the normaliser of L in $\mathrm{PSL}(2, \mathbb{R})$ is L .

Proof. (a) By (2) the side AB of T , opposite to the angle $\pi/8$, and the side c' between the vertices with angles $\pi/2$ and $\pi/8$ of a triangle T' with angles $\pi/2, \pi/8, \pi/3$ satisfy

$$\cosh AB = \frac{\cos(\pi/8)}{\sin(\pi/4)} = \sqrt{2} \cos \frac{\pi}{8}, \quad \cosh c' = \frac{\cos(\pi/3)}{\sin(\pi/8)} = \frac{1}{2 \sin(\pi/8)},$$

and the two values agree, because $2\sqrt{2} \sin(\pi/8) \cos(\pi/8) = \sqrt{2} \sin(\pi/4) = 1$. So there is a triangle T' with angles $\pi/2$ at A , $\pi/8$ at B and $\pi/3$ at a third vertex C' , and we place it on the same side of the line AB as T . The rotations through $\pi, \pi/4$ and $2\pi/3$ about A, B, C' in the same sense as before are X, w and a rotation R of order 3, and since the vertices A, B, C' are in the same cyclic order as A, B, C , the relation of the triangle group of T' reads $XwR = 1$. This triangle group, of signature $(0; 2, 3, 8)$, is generated by X and w , so it is F , and with $a := X$ and $b := Xw = R^{-1}$ it has the generators a of order 2, b of order 3 and $ab = w$ of order 8, and the presentation $\langle a, b \mid a^2, b^3, (ab)^8 \rangle$ (eliminate R from the presentation of the triangle group). Then $(ab)^2 = w^2 = Y$ and

$$b^{-1}ab^{-1} = w^{-1}X X w^{-1}X = w^{-2}X = Y^{-1}X = Z,$$

by $X^2 = 1$ and $X = YZ$. So $L = \langle X, Y, Z \rangle \leq F$, and $\mu(L)/\mu(F) = (\pi/4)/(\pi/12) = 3$ gives the index.

(b) The map $a \mapsto (12), b \mapsto (123)$ defines a transitive action of F on $\{1, 2, 3\}$, since the images satisfy the relations of F . The stabiliser F_3 of the point 3 has index 3 and contains a and $b^{-1}ab^{-1}$, the latter because $3 \cdot b^{-1} = 2, 2 \cdot a = 1, 1 \cdot b^{-1} = 3$ (right actions). With the transversal $1, b, b^2$ of the cosets of F_3 (the cosets of the points 3, 1, 2), the Schreier generators of F_3 are a (at the point 3), $bab^{-2} = bab = (b^{-1}ab^{-1})^{-1}$ (at 1), $b^2ab^{-1} = b^{-1}ab^{-1}$ (at 2), and the trivial elements $b^i b b^{-i-1}$. So $F_3 = \langle a, b^{-1}ab^{-1} \rangle$, which contains X, Z and $Y = XZ^{-1}$, hence L ; as both have index 3, $F_3 = L$. The stabiliser of a point of a transitive action with image S_3 is not normal (the image of L is $\{1, (12)\}$).

(c) Let F' be a Fuchsian group with $L < F'$, and $m = [F' : L] \geq 2$. Then $\mu(F') = \pi/(4m) \geq \pi/21$, so $m \leq 5$. A signature with $h \geq 1$ has $\mu \geq \pi$, and one with $h = 0$ and $r \geq 4$ periods has $\mu \geq 2\pi(-2 + 3 \cdot \frac{1}{2} + \frac{2}{3}) = \pi/3$; both exceed $\pi/8$. So F' is a triangle group $(0; l, m', n)$ with $2\pi(1 - 1/l - 1/m' - 1/n) = \pi/(4m)$, that is, $1/l + 1/m' + 1/n = 1 - 1/(8m)$ for some $m \in \{2, 3, 4, 5\}$. Running through the triples with $l \leq m' \leq n$ and $(l, m') \in \{(2, 3), (2, 4), (2, 5), (2, 6), (3, 3)\}$: for $l \geq 3$ the sum is at most $1/3 + 1/3 + 1/4 < 15/16$ unless $(l, m', n) = (3, 3, 3)$, which is not hyperbolic; for $l = 2$ and $m' \geq 7$ it is at most $1/2 + 2/7 < 15/16$; the triples $(2, 2, n)$ are not hyperbolic; for $l = 2$ and $m' \in \{3, 4, 5, 6\}$ the value $1/n = 1 - 1/(8m) - 1/2 - 1/m'$ is the reciprocal of an integer only for $m' = 3, m = 3, n = 8$. So the single solution is $(l, m', n) = (2, 3, 8), m = 3$. So F' has signature $(0; 2, 3, 8)$ and contains L with index 3. Let (a', b') be a canonical pair of generators of F' and $u := a'b'$, of order 8. In the action of F' on the three cosets of L , the images of a' and b' have orders dividing 2 and 3, generate a transitive group, and their product has order dividing 8; as in (b) the image of b' is a 3-cycle, that of a' a transposition, and that of u a transposition. The images of $u, b'ub'^{-1}$ and $b'^{-1}ub'$ are the three transpositions, so two of them move the coset L ; we replace u by that conjugate, which

is again of order 8. Then $u \notin L$, and u^2 lies in the kernel of the action, hence in L . The element u^2 is elliptic of order 4, so it is conjugate in L to $Y^{\pm 1}$ or to $Z^{\pm 2}$. If $u^2 = \gamma Z^{\pm 2} \gamma^{-1}$ with $\gamma \in L$, then u and $\gamma Z^{\pm 1} \gamma^{-1}$ are rotations of order 8 about the same point with the same square, so they differ by the rotation through π about that point, which is $(\gamma Z \gamma^{-1})^4 \in L$; then $u \in L$, a contradiction. Hence $u^2 = \gamma Y^{\pm 1} \gamma^{-1}$, so u is a rotation of order 8 about the fixed point of $\gamma Y \gamma^{-1}$, and it generates $\gamma \langle w \rangle \gamma^{-1}$. Since L has prime index in F' and $u \notin L$, $F' = \langle L, u \rangle = \gamma \langle L, w \rangle \gamma^{-1} = \gamma F \gamma^{-1} = F$. The last sentence follows because the normaliser of L is a Fuchsian group containing L , and L is not normal in F by (b). $\square$

2.3 From groups to surfaces

Let G be a finite group and $\varphi: L \rightarrow G$ an epimorphism which is *smooth*: $\varphi(X)$, $\varphi(Y)$, $\varphi(Z)$ have the orders 2, 4, 8. Then $K := \ker \varphi$ is torsion-free (an elliptic element of L is conjugate to a non-trivial power of X , Y or Z , whose image is non-trivial), and $X_K := \mathbb{H}/K$ is a compact Riemann surface of genus g with $2g - 2 = |G| \mu(L)/2\pi = |G|/8$, on which $G = L/K$ acts. If $|G| = 16(g - 1)$, this is consistent. Conversely:

Lemma 2.2. *Let X be a compact Riemann surface of genus $g \geq 2$ and $G \leq \text{Aut}(X)$ a 2-group of order $16(g - 1)$. Then $X \cong \mathbb{H}/K$ for a torsion-free normal subgroup K of L with $L/K \cong G$, and the epimorphism $L \rightarrow L/K$ is smooth.*

Proof. Write $X = \mathbb{H}/K$ with K torsion-free, and let Γ be the preimage of G in $N(K)$, a Fuchsian group of some signature $(h; m_1, \dots, m_r)$ with $\Gamma/K = G$ and $\mu(\Gamma) = |G|^{-1} \mu(K) = 2\pi(2g - 2)/|G| = \pi/4$. The periods m_i are orders of elements of G , hence powers of 2. By (1), $h = 0$ and $\sum_{i=1}^r (1 - 1/m_i) = 2 + 1/8$. For $r \geq 4$ the sum is at least $4 \cdot \frac{1}{2} = 2$ with equality only for $(2, 2, 2, 2)$, and the next values $(2, 2, 2, 4)$, $(2, 2, 2, 8)$, $(2, 2, 4, 4)$ give $2 + \frac{1}{4}$, $2 + \frac{3}{8}$, $2 + \frac{1}{2}$, all larger than $2 + \frac{1}{8}$, and larger periods give larger sums; so $r = 3$, and $1/m_1 + 1/m_2 + 1/m_3 = 7/8$ with powers of 2 forces $\{m_i\} = \{2, 4, 8\}$: without a period 2 the sum is at most $3/4$, with two periods 2 it exceeds 1, and $1/m_2 + 1/m_3 = 3/8$ with powers of 2 gives $\{4, 8\}$. So Γ has signature $(0; 2, 4, 8)$, and Γ is conjugate in $\text{PSL}(2, \mathbb{R})$ to L , because triangle groups with the same signature are conjugate (the triangle is determined by its angles). Replacing K by a conjugate, $\Gamma = L$, and the quotient map is smooth because K is torsion-free. $\square$

Lemma 2.3. *Let K be a torsion-free normal subgroup of L of finite index. Then $N(K) \in \{L, F\}$, and $N(K) = F$ if and only if $bKb^{-1} = K$. If $N(K) = L$, then $\text{Aut}(X_K) = L/K$, of order $16(g - 1)$ with g the genus of X_K (a 2-group when $g - 1$ is a power of 2). If $N(K) = F$, then $\text{Aut}(X_K) = F/K$ has order $48(g - 1)$ and is not nilpotent.*

Proof. $N(K)$ is a Fuchsian group containing L , so $N(K) \in \{L, F\}$ by Lemma 2.1(c), and $|L/K| = \mu(K)/\mu(L) = 16(g - 1)$. Since K is normal in L and $F = \langle L, b \rangle$, K is normal in F if and only if $bKb^{-1} = K$. In the second case the images of a , b , ab in F/K have the orders 2, 3, 8, because K is torsion-free; in a nilpotent group elements of coprime orders commute, so the product of elements of orders 2 and 3 has order 6. $\square$

Lemma 2.4. *Let K, K' be torsion-free normal subgroups of L of finite index with $N(K) = N(K') = L$. Then $X_K \cong X_{K'}$ if and only if $K = K'$.*

Proof. If $K' = cKc^{-1}$ with $c \in \text{PSL}(2, \mathbb{R})$, then $L = N(K') = cN(K)c^{-1} = cLc^{-1}$, so c normalises L , and $c \in L$ by Lemma 2.1(c). Hence $K' = K$. $\square$

i	additional relators of G_i	class	$ Z(G_i) $	elements of order 2, 4, 8	r_i
1	$z^2y^{-1}zy^2z^{-2}x$	5	2	19, 60, 48	$z^2y^{-1}zy^2z^{-2}x$
2	$zyxy^{-1}z^{-1}x$	5	2	27, 52, 48	$zyxy^{-1}z^{-1}x$
3	$yz^{-1}xy^2z^{-1}yz^{-1}, (z^{-1}yz^{-2})^2$	4	4	31, 64, 32	$(z^{-1}yz^{-2})^2$
4	$(z^{-1}yz^{-2})^2, z^2xyy^{-2}z^{-1}yz^{-1}x$	5	2	27, 52, 48	$(z^{-1}yz^{-2})^2$

Table 2: The groups $G_i = \langle x, y, z \mid x^2, y^4, z^8, z^{-1}y^{-1}x, \text{ additional relators} \rangle$ of order 128, from [Con26] (genus 9, type (2, 4, 8); the generators $G.1, G.2, G.3$ there are written x, y, z). The invariants were computed from the permutation representations (Section 5); all four groups have exponent 8 and abelianisation $\mathbb{Z}/2 \times \mathbb{Z}/4$. The last column is the relator r_i used in Proposition 3.1(c).

3 The four groups of order 128

The relation $z^{-1}y^{-1}x = 1$ of the presentations in [Con26] says $x = yz$, which is the relation $X = YZ$ of L . So for each i the map $X \mapsto x, Y \mapsto y, Z \mapsto z$ defines a homomorphism $\varphi_i: L \rightarrow G_i$, which is onto because x, y, z generate G_i ; we write $K_i := \ker \varphi_i$.

Proposition 3.1 (computer-assisted). *For $i = 1, \dots, 4$:*

- (a) G_i has order 128, and x, y, z have the orders 2, 4, 8 in G_i . So φ_i is a smooth epimorphism, K_i is torsion-free of index 128, and $X_i := \mathbb{H}/K_i$ is a compact Riemann surface of genus 9 on which G_i acts.
- (b) For $i \neq j$ the relator r_j of Table 2 (a word in X, Y, Z which lies in K_j) is not in K_i ; the same holds for the image of r_j under the automorphism ρ of L with $\rho(X) = X^{-1}, \rho(Y) = Y^{-1}, \rho(Z) = YZ^{-1}Y^{-1}$. So the kernels $K_1, \dots, K_4$ are pairwise different, and no K_i is the image of a $K_j, j \neq i$, under ρ .
- (c) $br_i b^{-1} \in L \setminus K_i$, where r_i is the relator of Table 2. So $bK_i b^{-1} \neq K_i$, and K_i is not normal in F .

Proof. (a) A coset enumeration of G_i over the trivial subgroup produces a table with 128 rows and the permutations $\bar{x}, \bar{y}, \bar{z}$ of $\{1, \dots, 128\}$ induced by right multiplication. These permutations satisfy all relators of G_i , have the orders 2, 4, 8, and generate a group which is transitive and of order 128 (verified with a Schreier–Sims computation); so this group is a regular permutation group which is a quotient of G_i of order 128; so $|G_i| \geq 128$. The inequality $|G_i| \leq 128$ rests on the completeness of the coset enumeration (the program `tc.py`, cross-checked with the independent coset enumeration of SymPy, which also gives 128). Hence $|G_i| = 128$ and the orders are as stated. Parts (b) and (c) use only the permutations; and Theorem 1.1 is unchanged if G_i is replaced by the permutation group of order 128, which is a smooth quotient of L in any case. The rest is the discussion before Lemma 2.2 with $2g - 2 = 128/8$.

(b) The word r_j (and $\rho(r_j)$) is evaluated in the permutation representation of G_i and is found to be a non-identity permutation; so $\varphi_i(r_j) \neq 1$ and $r_j \notin K_i$.

(c) By Lemma 2.1(b), F acts on $\{1, 2, 3\}$ with L the stabiliser of 3, and with the transversal $1, b, b^2$ every element of L , written as a word in a, b , is rewritten as a word in the Schreier generators $a = X$ and $b^{-1}ab^{-1} = Z$ (Schreier’s lemma; the rewriting reads the word letter by letter, following the point 3 through the action: a letter a read at the point 3, 1, 2 contributes X, Z^{-1}, Z , and a letter $b^{\pm 1}$ contributes nothing). The word $br_i(X, Y, Z)b^{-1}$, with $X = a, Y = (ab)^2, Z = b^{-1}ab^{-1}$ substituted, is first checked to return the point 3 to 3, so that it lies in L , and its rewritten form is evaluated in

the permutation representation of G_i and is a non-identity permutation. So $br_i b^{-1} \in L \setminus K_i$, while $r_i \in K_i$. $\square$

Proof of Theorem 1.1. By Proposition 3.1(a) each X_i is a compact Riemann surface of genus 9 on which the 2-group $G_i = L/K_i$ of order 128 acts. By Proposition 3.1(c) and Lemma 2.3, $N(K_i) = L$ and $\text{Aut}(X_i) = L/K_i \cong G_i$. By Proposition 3.1(b) and Lemma 2.4 the four surfaces are pairwise non-isomorphic. By the theorem of Zomorrodian [Zom85], as quoted in Section 1.1, $128 = 16(g-1)$ is the largest order of a nilpotent group of automorphisms of a surface of genus 9. $\square$

4 The other genera

The list [Con26] gives, for each genus $2 \leq g \leq 101$, the actions of groups G with $|G| > 4(g-1)$ on surfaces of genus g as quotients Γ/K of Fuchsian groups Γ with triangular or quadrangular signature, each action by defining relators of G in the canonical generators of Γ , up to reflection (and, for signatures $(2, q, q)$, up to duality), together with the information whether the action is reflexible or chiral. We extracted the actions with signature $(2, 4, 8)$ and $|G| = 16(g-1)$ for $g-1 = 2^n$, $g \leq 101$; their numbers are in Table 1, and for each of them we ran the computations of Proposition 3.1. In every case (a) holds with $16(g-1)$ in place of 128; the kernels of one genus are pairwise different, also up to ρ ; the reflexivity recorded in the list was confirmed ($\rho(K) = K$ if and only if the relators, with ρ applied, hold in G); and the normality of K in F was decided in both directions: for the kernels marked “normal” in Table 1 a coset enumeration of $\langle a, b \mid a^2, b^3, (ab)^8, R(a, (ab)^2, b^{-1}ab^{-1}) \rangle$, R the relators of G , gives a group of order $48(g-1) = 3|G|$, so that the normal closure of R in F has the same index as K and equals K ; for the others a relator r with $brb^{-1} \in L \setminus K$ was found as in Proposition 3.1(c). Since a chiral kernel K and its mirror image $\rho(K)$ give non-isomorphic surfaces when $N(K) = L$ (Lemma 2.4; $\rho(K)$ is the kernel of the mirror action, with $N(\rho(K)) = L$ as well, and $\rho(K) \neq K$), the number of surfaces is the number of non-normal kernels, counted twice when chiral. By Lemma 2.2, every surface of genus g whose full automorphism group is a 2-group of order $16(g-1)$ is $\mathbb{H}/K$ for a torsion-free normal subgroup K of L of index $16(g-1)$, which under the hypothesis is one of the listed kernels or its mirror image. This proves Theorem 1.2, under the completeness hypothesis on the list; the hypothesis is needed for the counts, not for the lower bounds. Without it, the certificates prove that there are at least 1, 1, 4, 3, 7 and 15 such surfaces for $g = 3, 5, 9, 17, 33, 65$. Two independent enumerations of all normal subgroups K of L with L/K a 2-group of order at most 1024, made in two of the verification runs with programs written for the purpose (Section 5; the programs and their outputs are in the reproducibility package), find exactly the listed kernels and the mirror images of the two chiral ones in genus 65; so for these genera the counts do not depend on the hypothesis.

Remark 4.1 (genus 2). For $g = 2$ the list contains one action of signature $(2, 4, 8)$, of the group of order 16 with the relators $(yz^{-1})^2$ and $z^{-1}xy^{-1}z^{-2}$, and its kernel is normal in F : the surface is the well-known surface of genus 2 with 48 automorphisms. So no surface of genus 2 has a nilpotent full automorphism group of order 16, although a nilpotent group of order 16 acts on that surface. This is the example with which the existence proof of [Zom85] begins (p. 253 there): the only group of order 16 generated by elements of orders 2 and 8 whose product has order 4 is $\langle a, b \mid a^2, (ab)^4, b^8, aba = b^3 \rangle$, and it acts on a surface of genus 2; the paper speaks of a group of automorphisms, not of the full automorphism group.

Remark 4.2 (genus 5 and the example of [Zom87]). The two actions of genus 5 in [Con26] have groups of order 64 of nilpotency class 4 and 3 (computed from the permutation representations). A smooth quotient of L of class at most 3 is a quotient of $L/\gamma_4(L)$, which has order 64 by [Zom87, Theorem 1.4.1]; so the action of class 3 is the action of $L/\gamma_4(L)$, the example of [Zom87], and its kernel is $\gamma_4(L)$. This

kernel is normal in F : the quotient of F by the normal closure of the relators of this action has order $192 = 3 \cdot 64$ (Section 4), so the surface $\mathbb{H}/\gamma_4(L)$ has 192 automorphisms and its full automorphism group is not nilpotent; the surface of genus 5 whose full automorphism group has order 64 is the one of the action of class 4.

Remark 4.3 (the groups). The four groups of Table 2 are pairwise non-isomorphic. For G_1 and G_3 this is seen from the invariants in the table; for the others, an isomorphism $G_j \rightarrow G_i$ would map (y, z) to a pair of elements of G_i of orders 4 and 8, with a product of order 2, which generates G_i and satisfies the relators of G_j ; the program `noniso.py` tests all such pairs and finds none for every $i \neq j$. So the four surfaces are non-isomorphic already because their automorphism groups are, and Lemma 2.4 is needed only for the counts of Theorem 1.2. The four groups are, as abstract groups, the four groups of order 128 listed in the LMFDB for this genus and signature (Section 1.4); in the order of Table 2 they are `SmallGroup(128, 136)`, `(128, 134)`, `(128, 75)` and `(128, 138)`, as identified through the generating vectors of the LMFDB in two of the verification runs.

Remark 4.4 (what remains open). The first sentence of Problem 2.19 asks for properties determining a unique point of $\mathcal{M}_g$ in general; nothing is said about it here. For the example question we have answered the uniqueness for the genera of Table 1 only, assuming the completeness of [Con26] for $g \neq 9$. A description of the surfaces of genus $2^n + 1$ with a 2-group of automorphisms of order 2^{n+4} for all n , and the number of those with full automorphism group of that order, are not known to us.

5 Computations and reproducibility

All statements of Section 3 and Section 4 rest on finite computations with permutations of at most 3072 points, in integers, with programs written for this note; they are in the reproducibility package with their command lines and recorded outputs.

<code>parse_conder.py</code>	reads the actions of signature $(2, 4, 8)$ and order $16(g - 1)$, and of signature $(2, 3, 8)$ and order $48(g - 1)$, from the file [Con26] into <code>actions_248_238.json</code>
<code>tc.py</code>	a coset enumeration (Todd–Coxeter) for finite presentations, with the permutations induced on the cosets
<code>words.py</code>	words in the generators; the parser for the relators as printed in [Con26]
<code>certificates.py</code>	Proposition 3.1(a)–(c) for every action with $g \leq 65$: orders, the relations, transitivity and order of the permutation group (Schreier–Sims of SymPy), distinct kernels also up to ρ , the relator r with $brb^{-1} \in L \setminus K$ (or the statement that none exists among the relators)
<code>nilp_check.py</code>	the same facts by coset enumeration, the reflexivity, and the orders of the quotients of F by the normal closures of the relators (normality of K in F)
<code>crosscheck_perm.py</code>	the orders of the coset tables against the orders of the permutation groups
<code>invariants.py</code>	the invariants of Table 2
<code>noniso.py</code>	the four groups of genus 9 are pairwise non-isomorphic (Remark 4.3)

The file [Con26] was read as text, and its header (version of April 2026, with a note on corrections concerning reflexivity) was read in full. The actions it lists are used only as candidates: everything asserted about them is re-derived from the printed relators by the programs above. The completeness of the list is used only in Theorem 1.2.

Verification. The result was found and the note written with the programs of the table above. Two independent verification runs, both AI-assisted, followed; they did not see each other. The first run compared the reading of the question with Farb’s text, re-derived Lemmas 2.1 to 2.4 and the logic of Proposition 3.1, and recomputed the certificates with programs of its own; it found one gap, in Lemma 2.1(c) (the element u had to be chosen so that it moves the coset L), and asked for references for two standard facts, for the statement of what rests on the coset enumeration, and for wording on the conditional counts. The second run wrote its own programs from the statements of the note (coset enumeration of SymPy, permutation certificates, normality in $\Delta(2, 3, 8)$ by two methods, reflexivity, invariants, an exhaustive isomorphism test) and reproduced every number of Tables 1 and 2 for all 35 actions with $g \leq 65$; it matched the 32 generating vectors of the LMFDB for genus 9 with the four kernels, showed that the four groups are pairwise non-isomorphic, and enumerated all normal subgroups of L with a 2-group quotient of order at most 1024, which gave exactly the kernels of Conder’s list and the mirror images of the two chiral ones. It checked the quotations of the zbMATH reviews and the bibliography, and searched the literature without finding a statement on the uniqueness; it required the citation of the LMFDB as prior public data and an exact quotation of the review of [Zom85]. All corrections were made.

A third independent run, also AI-assisted, examined the final text. It re-derived the proofs of Lemmas 2.1 to 2.4, Proposition 3.1 and the deduction of Theorem 1.2, and compared the statements about the sources with Farb’s text, the zbMATH reviews, Conder’s file, the LMFDB records and Crossref. With programs written independently of those in the package it rebuilt the four groups of Table 2, found the kernels pairwise different, also up to reflection, and none of them normal in $\Delta(2, 3, 8)$, recomputed the invariants, showed by canonical Cayley tables over all generating pairs of orders 4 and 8 that the four groups are pairwise non-isomorphic, and identified them with the groups of the LMFDB; for all genera of Table 1 it recomputed the orders, the reflexivity and the normality in $\Delta(2, 3, 8)$, and its own enumeration of the normal subgroups of L with a 2-group quotient of order at most 1024 again found exactly the listed kernels and the two mirror images. It re-ran `certificates.py` and `noniso.py` from a copy and reproduced the recorded outputs. It asked for the correction of a clause of Lemma 2.3 (the order $16(g - 1)$ of L/K is a power of 2 only when $g - 1$ is) and of a sentence on the LMFDB records; both were made. Its programs and outputs are in the package (`independent_run_2/`), those of the second run in `verification_run_B/`.

Scope and priority. The result of this note is Theorem 1.1, the four surfaces of genus 9, and, under the completeness of Conder’s list, the counts of Theorem 1.2. The question is Farb’s [Far06]; the bound and the existence are Zomorrodian’s [Zom85]; the actions are Conder’s [Con26]; the inclusion $\Delta(2, 4, 8) < \Delta(2, 3, 8)$ is Singerman’s [Sin72]. What is ours is the observation that the question is decided by the kernels of the actions in Conder’s list, the proof of Lemma 2.1, and the certificates. New, to our knowledge and relative to the literature we found, is the negative answer to the example question. We claim no priority.

What was read: [Far06], pp. 15–16 of version 1 of the arXiv version, and the bibliography entries for Kulkarni and Zomorrodian there; the paper [Zom85] in full, in the copy of the publisher (free archive of the journal), on 14 October 2026, and its review Zbl 0565.20029; the paper [Zom87] in full, in the copy of the publisher (open access), on 14 October 2026; the header and the entries of genus 2, 3, 5, 9, 17, 33, 65 of [Con26] (the file was processed as data); the records of the LMFDB [The26] for genus 9 and group order 128 (through the database’s interface, as data). Not read: [Sin72], which was not accessible to us; [Con14]; [Kat92], cited for standard facts; the classical classification of the automorphism groups of surfaces of genus 2 (Section 1.3). The searches were made on 13 October 2026: queries to Crossref and zbMATH Open for the sources, the list of Zomorrodian’s papers in zbMATH, and the LMFDB; the verification runs searched further (arXiv, OpenAlex, zbMATH, web) and found

no statement on the uniqueness; the author of the problem was not contacted. No statement that the surface is or is not unique was found in the literature accessible to us; a search that finds nothing is not a proof of novelty.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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