

Free Knots and Links with Two Minimal Representatives: Explicit Examples for a Problem of Fenn, Ilyutko, Kauffman and Manturov

Alper Ferudun*

October 14, 2026

Abstract

Problem 51 of the list of unsolved problems in virtual knot theory and combinatorial knot theory by Fenn, Ilyutko, Kauffman and Manturov (2014) asks to prove or disprove the conjecture that some free link has several minimal representatives, that is, non-isomorphic diagrams with the least number of vertices. We prove that the free knot with the Gauss word $abae c b d f d g e g c f$ has exactly two minimal representatives: this diagram and the diagram $abace b d f d e g c g f$, obtained from it by one third Reidemeister move, both with seven vertices. The same holds for the two-component free link given by the words $e 1 e 3 5 2 6 4$ and $1 2 3 4 5 6$. Equivalence and non-isomorphism are elementary. The lower bound of seven vertices follows from Manturov's parity bracket, whose invariance we prove: each diagram has exactly one even crossing, and its bracket has two different terms, one of them an irreducible graph on the six odd crossings. That there are exactly two minimal representatives is computer-assisted; it rests on the computation of the bracket of all diagrams with seven vertices by independent programs. Kauffman and Manturov anticipated the mechanism: they observed that a diagram with a triangle is equivalent to its image under the third move, that the two can have the same unoriented state, and that minimal diagrams with triangles can be constructed; their minimality criterion, applied to a suitable explicit diagram, also answers the question. The problem list still states the conjecture. To our knowledge this is the first explicit written example with a self-contained proof; no explicit example was found in the literature accessible to us. Minimal representatives that are not related by third moves are not treated. This is an unrefereed note.

1 Introduction

Free knots and links were introduced by Manturov [Man09a, Man09b] as a strong simplification of virtual knots. A free link is a framed 4-valent graph modulo three moves analogous to the Reidemeister moves; equivalently, a virtual link modulo crossing changes and virtualisations [Man09a, Lemma 1], [Man09b, Section 3], or a Gauss diagram without arrows and signs modulo the corresponding moves [FIKM14, Section 2.2]. In the language of words, free knots had been studied earlier by Turaev, as Gauss words up to homotopy [Tur07]: an oriented free knot is a homotopy class of Gauss words [Gib11, Section 1], see also [Man14]. The main tool is parity: the crossings of every diagram are divided into even and odd ones compatibly with the moves, and the parity bracket smooths the even crossings, keeps the odd ones, and takes values in graphs modulo the second move [Man09a, Man09b, Man11].

In the problem list of Fenn, Ilyutko, Kauffman and Manturov [FIKM14], item 51 of Section 8.1 asks to prove or disprove the conjecture on the non-uniqueness of the minimal representative of a free link, stated as “there exists a free link having several minimal representatives”. The item gives no

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
Unrefereed preprint. CC BY 4.0.

further context. It is record AMR-105-0051 of the UnsolvedMath collection [ula26], where it is listed as open. A minimal representative is a diagram with the least number of vertices [Man09b, Corollary 1]. The uniqueness of the minimal diagram is proved, or asserted, for special classes: an irreducibly odd diagram (every crossing odd for the Gaussian parity, no decreasing second move) occurs as a smoothing of every diagram of its free knot [Man11, Theorem 3], hence it is the only minimal diagram, as Kauffman and Manturov point out [KM15, Section 5]; for a family of minimal diagrams obtained from permutations, Boothby, Henrich and Leaf attribute to Manturov the uniqueness of the minimal diagram up to purely virtual Reidemeister moves (proof of their main theorem in [BHL14]); and Ilyutko and Manturov extend an invariant of Manturov [Man13] to free graph-links so that, under certain conditions on a representative, the representative can be recovered from the value of the invariant (abstract of [IM20]). The minimality criterion of Kauffman and Manturov does not claim uniqueness, and they conjecture that a framed 4-graph of girth at least four is the unique minimal diagram of its free knot [KM15, Conjecture 1].

1.1 Results

The answer is positive, already for free knots (free links with one component). Diagrams are written as families of cyclic Gauss words (Section 2); the diagrams of Theorem A are drawn in Figure 1.

Theorem A (a free knot). *Let K_1 and K_2 be the framed 4-graphs with one unicursal component given by the Gauss words*

$$K_1 = abaecbdfdegc f, \quad K_2 = abacebdfdegc f.$$

- (i) K_2 is obtained from K_1 by one third Reidemeister move on the triangle (e, c, g) ; so K_1 and K_2 represent the same free knot $\mathcal{K}$.
- (ii) K_1 and K_2 are not isomorphic.
- (iii) Every diagram of $\mathcal{K}$ has at least seven vertices; so K_1 and K_2 are minimal representatives of $\mathcal{K}$.
- (iv) Every minimal representative of $\mathcal{K}$ is isomorphic to K_1 or to K_2 ; so $\mathcal{K}$ has exactly two minimal representatives up to isomorphism.

Theorem B (a two-component free link). *Let D_1 and D_2 be the framed 4-graphs with two unicursal components given by the families of Gauss words*

$$D_1 = \{e1e35264, 123456\}, \quad D_2 = \{e526e413, 124356\}.$$

Then (i)–(iv) of Theorem A hold for D_1 , D_2 and the free link $\mathcal{L}$ that they represent, with the triangle $(3, 4, e)$ in (i).

Parts (i)–(iii) of both theorems are proved in the text without computation (Sections 2 to 5). Part (iv) is computer-assisted: by (iii) and the invariance of the parity bracket it reduces to the computation of the bracket of all diagrams with seven vertices, which was made by several independent programs (Section 6). A program that checks (i), (ii) and the brackets used in (iii) in two seconds is part of the reproducibility package (Section 8). Section 7 contains remarks.

1.2 Credits

The notions and the parity bracket are Manturov's; for the link we use his bracket for the parity in which a crossing is odd if it is formed by two different components [Man09b, Theorem 4]. We prove the

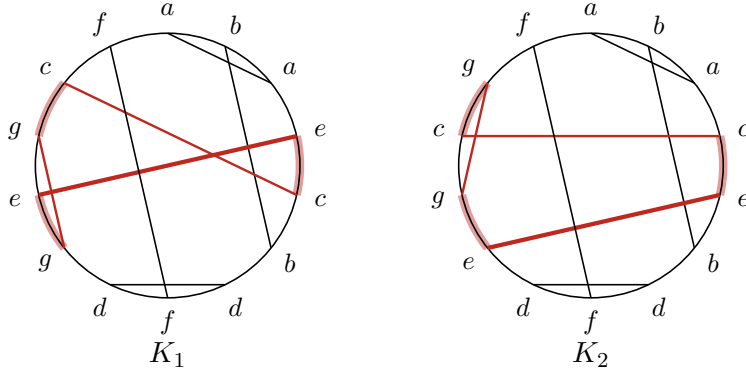


Figure 1: The chord diagrams of the two minimal representatives K_1 and K_2 of the free knot $\mathcal{K}$; the endpoints are the positions $0, 1, \dots, 13$ of the Gauss words, read clockwise from the top. The three shaded arcs of the circle are the edges of the triangle (e, c, g) ; the chords e, c, g are drawn in colour, the even chord e thick. The third move exchanges the two endpoints on each shaded arc.

invariance of the bracket for both parities used, so that the note is self-contained. The mechanism was anticipated by Kauffman and Manturov. Right after their minimality criterion [KM14, Corollary 3.4] they remark that the unoriented states (Remark 7.3) of two diagrams that differ by a third move of the unoriented type coincide, so that the state does not determine the diagram [KM14, Section 3]. In [KM15, Section 5] they observe that a diagram with a triangle is equivalent to its image under the third move, and, after their minimality theorem for the G_2 bracket [KM15, Theorem 3], they remark that free knot diagrams satisfying its conditions and “having some triangles” can be constructed. They give no such diagram and do not discuss whether the image under the third move is isomorphic to it. Their criterion, applied to a suitable explicit diagram with a triangle, also answers the problem (Remark 7.3). The problem list [FIKM14] of September 2014, of which Kauffman and Manturov are coauthors, still states the non-uniqueness as a conjecture. To our knowledge, the note gives the first explicit written example with a self-contained proof of non-isomorphism, minimality and the exact number of minimal representatives; see “Scope and priority” at the end.

2 Free links and Gauss words

2.1 Definitions

We use the definitions of Manturov [Man09b, Section 2] (see also [Man09a, Section 1]). A *4-graph* is a topological space with finitely many components, each of which is a circle or a finite graph all of whose vertices have valency four; loops and multiple edges are allowed. It is *framed* if at every vertex the four half-edges are split into two pairs of *opposite* half-edges. A circle component is a *free loop*. A *unicursal component* is a free loop or a class of edges, two edges being equivalent if one can pass from one to the other through vertices, leaving every vertex through the half-edge opposite to the half-edge of arrival. Isomorphisms are homeomorphisms that respect opposite half-edges. Each move changes one fragment of the graph and leaves the rest, with its framing, unchanged [Man09b, Figs. 1–3]: (R1) addition or removal of a *loop*, an edge joining two non-opposite half-edges of one vertex; (R2) addition or removal of a *bigon*, two vertices joined by two edges that are non-opposite at both vertices; the increasing move “does not impose any conditions on the edges” [Man09b, Section 2], so any two edges may be used, also two stretches of one edge or an edge of a free loop; (R3) the third move on a *triangle*, three vertices joined pairwise by three edges such that at each of them the two edges of the triangle are non-opposite. The moves preserve the number of unicursal components. A *free link* is a class of

framed 4-graphs modulo isomorphisms and the moves; its elements are its *diagrams* or *representatives*, and their vertices are also called crossings. A *free knot* is a free link with one unicursal component. The moves are given by pictures of planar fragments; for an abstract graph one embeds it in a surface, preserving the framing, performs the move there and forgets the surface [Man09a, Section 1].

2.2 Gauss words

Lemma 2.1. *Framed 4-graphs with k unicursal components correspond bijectively, up to isomorphism, to families of k cyclic words in which every letter occurs exactly twice in total, up to renaming the letters, rotating and reversing each word, and permuting the words. Free loops correspond to empty words.*

Proof. Traverse every component that is not a free loop, passing straight through each vertex, and record the vertices passed. Each vertex is passed twice in total, once along each pair of opposite half-edges; these two passages are its *occurrences*. Conversely, the words determine the graph: each occurrence carries two opposite half-edges, incoming and outgoing, and the edges join the outgoing half-edge of each occurrence to the incoming half-edge of the next occurrence in the same word, cyclically. The choices in reading off the words are exactly the ambiguities listed. $\square$

For a knot the word is its chord diagram (the Gauss diagram without arrows and signs); two chords are *linked* if their endpoints alternate. Two occurrences are *adjacent* if they are consecutive in a word, cyclically; adjacent occurrences are joined by an edge. A word xy has two edges, both joining the same two occurrences, and a word of length one has one edge, joining the two half-edges of its only occurrence. Two half-edges at x are opposite if and only if they belong to the same occurrence of x ; hence

$$\text{two edges are non-opposite at } x \iff \text{they are attached to different occurrences of } x. \quad (1)$$

Lemma 2.2 (moves in words). (R1) *A loop is a letter x whose two occurrences are adjacent; the decreasing move deletes x .* (R2) *A bigon is a pair of letters $x \neq y$ such that one occurrence of x is adjacent to one occurrence of y and the other occurrence of x to the other occurrence of y (patterns $\dots xy \dots xy \dots$ or $\dots xy \dots yx \dots$, in one word or in two words); the decreasing move deletes x and y , and the increasing move inserts xy into one edge and xy or yx into another edge or into the same edge, possibly of an empty word.* (R3) *A triangle is a set of three letters x, y, z with adjacent pairs of occurrences of x and y , of y and z and of z and x , which together use each of the six occurrences exactly once; the third move exchanges the two letters within each of the three pairs.*

Proof. By (1) the conditions are the definitions. Removing a loop at x deletes x from its word. Each edge of a bigon lies on a strand passing straight through x and y , and removing the bigon joins each of the two strands up without these vertices. The three edges of a triangle lie on three strands, each passing two of the vertices consecutively; so a neighbourhood of the triangle consists of three strands with six external ends, each crossing the other two. With the framing it is isomorphic to the planar fragment of the third move, which is replaced by the fragment in which each strand meets its two crossings in the opposite order, with the same external ends. In the words this exchanges the two letters within each of the three pairs. $\square$

We write fragments as *tangles*: strands between labelled boundary points through the vertices of the fragment. For the second move these are $(0, u, v, 1), (2, u, v, 3)$ (the *parallel* bigon) or $(0, u, v, 1), (2, v, u, 3)$ (the *antiparallel* one), replaced by $(0, 1), (2, 3)$; for the third move, with suitable names of the boundary points, the tangle

$$T = \{(0, a, b, 1), (2, b, c, 3), (4, c, a, 5)\} \quad \text{is replaced by} \quad T' = \{(0, b, a, 1), (2, c, b, 3), (4, a, c, 5)\}, \quad (2)$$

the vertex a of T' corresponding to a of T , and so on. Both are invariant under $a \rightarrow b \rightarrow c \rightarrow a$, $(0, 1) \rightarrow (2, 3) \rightarrow (4, 5) \rightarrow (0, 1)$.

Smoothings [Man09b, Section 2.1]. If p, q, r, s are the half-edges at v , with p opposite to r and q to s , the two smoothings at v remove v and join the half-edges in the non-opposite pairs (p, q) , (r, s) , respectively (p, s) , (q, r) . We write $\overline{B}$ for the word B read backwards.

Lemma 2.3 (smoothings in words). *If v occurs twice in one word $W = vAvB$, the two smoothings at v replace W by the two words A and B , respectively by the single word $A\overline{B}$. If v occurs in two words vA and vB , they replace these words by the single word AB , respectively $A\overline{B}$. Other words are unchanged.*

Proof. In $W = vAvB$ the first occurrence of v has its incoming half-edge at the end of B and its outgoing one at the start of A , the second one at the end of A and at the start of B . One non-opposite pairing joins the start of A to the end of A and the start of B to the end of B , which closes A and B into two cycles; the other joins the two ends and the two starts, which gives the cycle through A and then through B backwards. The case of two words is the same computation. $\square$

For example, for $W = vv$ the two smoothings give two free loops, respectively one. The formulas were also compared with a computation from half-edges in 12,063 random cases (Section 8).

2.3 The question

Definition 2.4. A *minimal representative* of a free link is a diagram of it with the least number of vertices. A free link has *several* minimal representatives if it has two that are not isomorphic.

This is Manturov’s minimality “with respect to the number of vertices” [Man09b, Corollary 1], for framed 4-graphs up to isomorphism. The conventions that could change the answer are Manturov’s: free loops are components of diagrams; representatives are taken up to isomorphism, that is, words up to renaming, rotation, reversal and permutation; and the increasing second move may be applied to any two edges. The following readings are not the question. (a) *Free knots*: the question for them is stronger, and Theorem A answers it too. (b) *Strong minimality*: a minimal representative that “lives inside” every equivalent representative [Man14], for instance as a smoothing as in [Man11, Theorem 3], is unique if it exists (a diagram with as many vertices that has it as a smoothing is isomorphic to it), so non-uniqueness cannot refer to it; by our theorems, $\mathcal{K}$ and $\mathcal{L}$ have no such representative. (c) *Minimal representatives not related by third moves*: in our theorems they are related by one third move; this stronger question remains open (Remark 7.1). (d) *Virtual and flat virtual links* are a different subject. (e) *Degenerate readings*: if vertices were labelled, or if representatives were planar virtual diagrams up to planar isotopy, the question would be trivial (rename the vertices, or immerse the graph differently). The source excludes these readings: its free knots and links are those of Gauss diagrams without arrows and signs [FIKM14, Section 2.2], with references to [Man09a, Man09b], where diagrams are framed graphs; and in [KM15, Section 5], by two of the authors of [FIKM14], uniqueness of the minimal diagram is meant up to isomorphism: the conjecture there is restricted to girth at least four, which excludes triangles, right after the remark that a diagram with a triangle is equivalent to its image under the third move.

3 Parity and the parity bracket

Definition 3.1 ([Man09b, Section 5]). Let $\mathcal{D}$ be the class of diagrams of free links with a fixed number of components. A *parity* on $\mathcal{D}$ assigns to every vertex of every diagram in $\mathcal{D}$ a value, even or odd, invariant under isomorphisms, such that for every move $D \rightarrow D'$: (P0) every vertex that does

not take part in the move has the same parity in D and in D' , and for a third move the corresponding vertices of T and T' in (2) have the same parity; (P1) the vertex of a loop is even; (P2) the two vertices of a bigon have the same parity; (P3) the number of odd vertices of a triangle is even.

Lemma 3.2. (a) Gaussian parity [Man09a, Section 1], [Man11, Section 3]: on one-component diagrams, a vertex is odd if its chord is linked with an odd number of chords. This is a parity, and every one-component diagram has an even number of odd vertices. (b) Mixed parity [Man09b, Section 6]: on two-component diagrams, a vertex is odd if its two occurrences lie on different components. This is a parity.

Proof. (a) Linking depends only on the cyclic order of four endpoints, so the parity is invariant under isomorphisms, and a move does not change the linking of two chords not taking part in it. The chord of a loop has adjacent endpoints and is linked with no chord: (P1). The endpoints of the chords u, v of a bigon form two adjacent pairs, each lying in one of the two arcs cut out by a third chord w ; so w is linked with both or with neither. Hence u and v are linked with equally many chords, and the count of w changes by 0 or 2: (P2), (P0). Exchanging two adjacent endpoints of different chords toggles the linking of these two chords and of no other pair; the third move does this for the pairs $\{a, b\}$, $\{b, c\}$, $\{c, a\}$, so the counts of a, b, c change by 0 or ± 2 and the others do not change: (P0). A chord d outside the triangle has in each of its arcs an even number of the six endpoints of a, b, c , which form three adjacent pairs; so d is linked with an even number of a, b, c . Adding the counts of a, b, c , the links among them count twice and the links with outside chords give an even number; so the number of odd chords among a, b, c is even: (P3). The sum of all counts is twice the number of linked pairs, hence even. (b) The moves do not change the component of a strand: (P0). A loop lies on one strand: (P1). Both vertices of a bigon are crossings of the same two strands: (P2). The three strands of a triangle lie on two components, so either all lie on one component (no odd vertex) or two lie on one and the third on the other (two odd vertices): (P3). $\square$

For three components the mixed parity would violate (P3). Two framed 4-graphs are R2-equivalent if they are related by second moves and isomorphisms. Let $\tilde{V}$ be the vector space over $\mathbb{Z}/2$ spanned by the R2-equivalence classes of framed 4-graphs with any number of components, modulo the span of the classes that contain a graph with a free loop (the space $\tilde{\mathbb{Z}}_2\mathfrak{G}$ of [Man09b, Section 3.1]); the class of G in $\tilde{V}$ is $[G]$. A graph without bigons is R2-irreducible.

Lemma 3.3 ([Man09b, Theorem 1]). *Deleting bigons as long as possible, in any order, ends at an R2-irreducible graph $N(G)$ that is unique up to isomorphism; two graphs are R2-equivalent if and only if their normal forms are isomorphic. So the classes $[N]$ of the R2-irreducible graphs N without free loops, one for each isomorphism class, form a basis of $\tilde{V}$, and $[G] = 0$ if $N(G)$ has a free loop, $[G] = [N(G)]$ otherwise.*

Proof. (Manturov calls the statement trivial; we include a proof.) Each deletion removes two letters, so the process ends. Let $\{a, b\} \neq \{c, d\}$ be bigons of G . If they are disjoint, deleting a, b does not separate adjacent occurrences of other letters, so $\{c, d\}$ remains a bigon and both orders give $G - \{a, b, c, d\}$. If they are $\{a, b\}$ and $\{a, c\}$ with $b \neq c$, each occurrence of a is adjacent to an occurrence of b and to one of c ; an occurrence in a word of length at most two has a single neighbouring occurrence, so both occurrences of a lie in words of length at least three, and by the definition of a bigon their neighbours are different occurrences of b and different occurrences of c . So G contains two disjoint blocks bac (each possibly reversed) using all occurrences of a, b, c ; deleting $\{a, b\}$ leaves c in both blocks, deleting $\{a, c\}$ leaves b , and the results are isomorphic by renaming. By induction on the number of vertices all maximal deletion sequences from G end at isomorphic graphs: for two first deletions $G \rightarrow G_1$, $G \rightarrow G_2$, the normal forms of G_1 and G_2 are unique by induction, and they agree since G_1 and G_2 both lead to $G - \{a, b, c, d\}$, respectively are isomorphic. An increasing move from G to G' is undone

move and even vertices	states (strands; “o” a closed circle)	mod-2 sum
R1, v even	$\{0-1\}; \{0-1\} + \circ$	$\{0-1\} = T'$
R2 parallel, u, v even	$\circ; \{0-2, 1-3\}$ twice; $\{0-1, 2-3\}$	$\{0-1, 2-3\} = T'$
R2 antiparallel, u, v even	$\{0-1, 2-3\}; \{0-3, 1-2\}$ twice; $\circ$	$\{0-1, 2-3\} = T'$
R2, u, v odd	T itself (no smoothing)	$T \sim_{R2} T'$
R3, a, b, c even, T	$\circ; \mu_1$ three times; $\mu_2, \mu_3, \mu_4, \mu_5$ once	$\mu_1 + \dots + \mu_5$
R3, a, b, c even, T'	$\circ; \mu_2$ three times; $\mu_1, \mu_3, \mu_4, \mu_5$ once	$\mu_1 + \dots + \mu_5$
R3, c even, T	$(0, a, b, 1), (2, b, 4), (3, a, 5); (0, a, b, 1), (2, b, a, 5), (3, 4)$	see the proof
R3, c even, T'	$(0, b, a, 1), (2, a, 4), (3, b, 5); (0, b, a, 1), (2, 5), (3, b, a, 4)$	see the proof

Table 1: The local tables for Theorem 3.5. For the first move $T = (0, v, v, 1)$ and $T' = \{0-1\}$; for the second move T is the parallel or the antiparallel bigon and $T' = \{0-1, 2-3\}$; for the third move T, T' are as in (2). The matchings of the boundary points are $\mu_1 = \{0-5, 1-2, 3-4\}$, $\mu_2 = \{0-3, 1-4, 2-5\}$, $\mu_3 = \{0-1, 2-5, 3-4\}$, $\mu_4 = \{0-3, 1-2, 4-5\}$, $\mu_5 = \{0-5, 1-4, 2-3\}$.

by a deletion, so $N(G') \cong N(G)$, which gives the second claim. Deletions never remove an empty word, so a class contains a graph with a free loop if and only if the normal form of its members has one, and the last claims follow. $\square$

Second moves preserve the number of components, so basis vectors with different numbers of components are different.

Definition 3.4. Fix a parity on $\mathcal{D}$. For $D \in \mathcal{D}$ with set $E(D)$ of even vertices and $s: E(D) \rightarrow \{1, 2\}$, let D_s be the graph obtained by smoothing every even vertex v in the way $s(v)$ (the two smoothings of each vertex numbered arbitrarily), the odd vertices being kept. The *parity bracket* is

$$\{D\} = \sum_{s: E(D) \rightarrow \{1, 2\}} [D_s] \in \tilde{V}.$$

For the mixed parity this is Manturov’s bracket, whose invariance is [Man09b, Theorem 4]. For the Gaussian parity it is the analogue that keeps all smoothings (the bracket of [Man09a] keeps only the smoothings with one component), the specialisation $a = 1$ modulo 2 of the even Kauffman bracket of [Man11, Theorem 6], stated there for virtual knots. We prove the invariance for every parity.

Theorem 3.5. *For a parity on $\mathcal{D}$, if $D, D' \in \mathcal{D}$ differ by one move, then $\{D\} = \{D'\}$. So $\{\cdot\}$ is an invariant of free knots for the Gaussian parity, and of two-component free links for the mixed parity.*

Proof. The move replaces a tangle T by a tangle T' with the same boundary points: $D = O \cup T$ and $D' = O \cup T'$ with a common outside part O . By (P0) the even vertices of O are the same for D and D' ; fix a way s_O of smoothing them. Smoothing is local, so smoothing also the even vertices of T in a way s_T gives $O_{s_O} \cup T_{s_T}$. If T_{s_T} contains a closed circle, this graph has a free loop and its class is 0; if two smoothed tangles agree up to renaming their inner vertices (boundary fixed), or differ by a second move inside, the glued graphs have the same class. So the contribution of s_O is determined by the mod-2 sum $\sigma(T)$ of the smoothed tangles, where those with a closed circle are omitted and tangles are identified up to renaming and second moves inside; it suffices to show $\sigma(T) = \sigma(T')$ in the cases allowed by the axioms: R1 with an even vertex (P1); R2 with two even or two odd vertices (P2); R3 with three even vertices or exactly one, which we may call c by the cyclic symmetry of (2) (P3, P0).

Table 1 lists the states, obtained by following the strands through the smoothed vertices. For R1 and R2 the sums are read off. For R3 with three even vertices, the state that joins the two edges of the

triangle at each vertex closes them into a circle, and the other seven states give five matchings, one of them three times; both sums are $\mu_1 + \dots + \mu_5$ (Manturov's count of "seven types" and "five types" in the proof of [Man09b, Theorem 4]). For R3 with only c even, the smoothing of c that joins the two edges of the triangle at c gives $(0, a, b, 1), (2, b, a, 5), (3, 4)$ in T and $(0, b, a, 1), (2, 5), (3, b, a, 4)$ in T' ; both contain the bigon $\{a, b\}$ and become $\{0-1, 2-5, 3-4\}$ after a second move. The other smoothing gives $(0, a, b, 1), (2, b, 4), (3, a, 5)$ and $(0, b, a, 1), (2, a, 4), (3, b, 5)$, which agree after exchanging the names a and b . The argument does not use O ; so it also covers second moves on two stretches of one edge or on a free loop, triangles whose external ends are joined to each other directly, and diagrams with free loops. $\square$

The axioms are needed: for a triangle with two even or three odd vertices the two sums differ. The tables were also computed from half-edges by four independent programs, and the invariance was tested on all moves between small diagrams and on random moves (Section 8).

4 A minimality criterion

The following is Manturov's argument for irreducibly odd diagrams [Man11, Corollary 1], adapted to a bracket with two terms.

Proposition 4.1. *Fix a parity. Let D be a diagram with exactly one even vertex e and m odd vertices, and let D^1, D^2 be its smoothings at e . Assume that (a) D^1 is R2-irreducible without free loops, and (b) $[D^2] \neq 0$ and $[D^2] \neq [D^1]$. Then every diagram of the free link of D has at least $m + 1$ vertices, so D is minimal; and every minimal representative D' has exactly one even vertex e' , at which one of its smoothings is isomorphic to D^1 .*

Proof. By (a), (b) and Lemma 3.3, $\{D\} = [D^1] + [D^2]$ is a sum of two different basis vectors, $[D^1]$ given by D^1 with m vertices. Let D' be a diagram of the same link; then $\{D'\} = [D^1] + [D^2]$ by Theorem 3.5. Every term $[D'_s]$ is 0 or the basis vector given by $N(D'_s)$, which has at most as many vertices as D' has odd vertices. As $[D^1]$ occurs, D' has at least m odd vertices. If D' had at most m vertices, all would be odd, and $\{D'\} = [D']$ would be 0 or one basis vector. So D' has at least $m + 1$ vertices. If it has $m + 1$, then by the same argument exactly one of them, e' , is even, and $\{D'\} = [D'^1] + [D'^2]$ for its smoothings at e' ; each term is 0 or a basis vector, so $\{[D'^1], [D'^2]\} = \{[D^1], [D^2]\}$. The smoothing with class $[D^1]$ has m vertices, as has its normal form D^1 ; so no bigon is deleted, and it is isomorphic to D^1 . $\square$

Proposition 4.2. *In the situation of Proposition 4.1, every minimal representative D' is obtained from D^1 by adding a vertex E in one of the following ways: (1) a word W of D^1 is replaced by $EAEB$, where $\overline{AB}$ is W read from some starting point in some direction (A or B may be empty); (2) two words A, B are replaced by $EAEB$; (3) a word W is replaced by the two words EA and EB , where $\overline{AB}$ is W read from some starting point in some direction.*

Proof. By Proposition 4.1 a smoothing of D' at some vertex E is isomorphic to D^1 . If E occurs twice in one word $EAEB$, the smoothings are $\{A, B\}$ and $\overline{AB}$ (Lemma 2.3): types (2) and (1). If E occurs in two words EA, EB , they are $\overline{AB}$ and $\overline{EB}$; as $\overline{EB}$ is EB rotated and reversed, this is type (3). $\square$

In type (3) the vertex E is a crossing of two components; this does not happen for an even vertex of the Gaussian or the mixed parity. Type (2) lowers the number of components by one. So in our applications, where D^1 has as many components as D , only type (1) occurs. The proposition is used only as a cross-check of part (iv).

5 Proof of parts (i)–(iii)

5.1 The free knot

Number the positions of $K_1 = abae c b d f d g e g c f$ by $0, \dots, 13$; the chords are $a: 0, 2$; $b: 1, 5$; $e: 3, 10$; $c: 4, 12$; $d: 6, 8$; $f: 7, 13$; $g: 9, 11$. A chord x is linked with y if exactly one endpoint of y lies between those of x . So a is linked with b ; b with a, c, e ; c with b, e, f ; d with f ; e with b, c, f, g ; f with c, d, e ; g with e . For the Gaussian parity, e is the only even chord of K_1 ($m = 6$).

(i) The adjacent pairs at the positions 3–4 (ec), 9–10 (ge) and 11–12 (gc) use the occurrences $e: 3, 10$, $c: 4, 12$, $g: 9, 11$ once each, so they form a triangle (Lemma 2.2); its external ends at the positions 10 and 11 are joined directly. Exchanging the letters within the three pairs gives $abace b d f d e g c g f = K_2$.

(ii) In K_2 ($a: 0, 2$; $b: 1, 5$; $c: 3, 11$; $e: 4, 9$; $d: 6, 8$; $f: 7, 13$; $g: 10, 12$) the chord a is linked with b ; b with a, c, e ; c with b, f, g ; d with f ; e with b, f ; f with c, d, e ; g with c ; so e is again the only even chord. An isomorphism preserves linking, hence parity, so it maps the only even chord of K_1 to that of K_2 and preserves the numbers of endpoints on its two sides, as an unordered pair. These are 6, 6 in K_1 and 4, 8 in K_2 . So $K_1 \not\cong K_2$.

(iii) Read from the first e , $K_1 = eAeB$ with $A = cbdfdg$ and $B = gcfaba$. By Lemma 2.3 the smoothings at e are

$$D^1 = A\overline{B} = T_6 := cbdfdgabafcg, \quad D^2 = \{cbdfdg, gcfaba\}.$$

The adjacent pairs of T_6 are $cb, bd, df, fd, dg, ga, ab, ba, af, fc, cg, gc$. A bigon $\{x, y\}$ needs two adjacencies of x and y through different occurrences of x and of y . The pairs $\{d, f\}$, $\{a, b\}$, $\{c, g\}$ are adjacent twice, but always through the occurrence of f at position 3, of b at 7, of g at 11; the other pairs are adjacent at most once. So T_6 is R2-irreducible, with six vertices, one component and no free loop. In D^2 the letters g, c are cyclically adjacent in the first word and adjacent in the second, through different occurrences of both; deleting this bigon gives $T_4 = \{bdfd, faba\}$. In T_4 the pairs $\{b, d\}$ and $\{d, f\}$ are adjacent twice in the first word but through its single b , respectively f , and likewise $\{a, b\}$, $\{a, f\}$ in the second word; no other pair is adjacent twice. So T_4 is R2-irreducible, with four vertices, two components and no free loop. Hence $\{K_1\} = [T_6] + [T_4]$ is a sum of two different basis vectors, and Proposition 4.1 with $m = 6$ shows that every diagram of $\mathcal{K}$ has at least seven vertices. This proves (i)–(iii) of Theorem A.

5.2 The two-component link

Let $D_1 = \{X, Y\}$ with $X = e1e35264$ and $Y = 123456$. Each of $1, \dots, 6$ occurs once in each word and e twice in X : for the mixed parity, e is the only even vertex ($m = 6$).

(i) The adjacent pairs $e3$ (positions 2–3 of X), $4e$ (positions 7–0 of X) and 34 (positions 2–3 of Y) use each occurrence of $e, 3, 4$ once, so they form a triangle. Exchanging the letters gives $\{413e526e, 124356\}$, which is D_2 , the first word read from its position 3.

(ii) An isomorphism preserves components, hence the mixed parity; so it maps e to e and preserves the numbers of letters on the two arcs of the component of e between its occurrences, as an unordered pair. These are 1, 5 for D_1 and 3, 3 for D_2 . So $D_1 \not\cong D_2$.

(iii) We have $X = eAeB$ with $A = 1$, $B = 35264$, and the smoothings at e are

$$D^1 = G_6 := \{146253, 123456\}, \quad D^2 = \{1, 35264, 123456\}.$$

In G_6 each letter occurs once in each word, so a bigon is a pair adjacent in both words; the adjacent pairs 14, 46, 62, 25, 53, 31 of the first word and 12, 23, 34, 45, 56, 61 of the second have no unordered pair in common. So G_6 is R2-irreducible, with six vertices, two components and no free loop. In

D^2 , the letters 4, 3 are cyclically adjacent in the second word and adjacent in the third, so $\{3, 4\}$ is a bigon; deleting it gives $\{1, 526, 1256\}$, where $\{2, 5\}$ is a bigon; deleting it gives $H_2 = \{1, 6, 16\}$. The only pair of letters of H_2 is $\{1, 6\}$, and the occurrence of 1 in the word 1 is adjacent only to itself; so H_2 is R2-irreducible, without free loops, with three components. Hence $\{D_1\} = [G_6] + [H_2]$ is a sum of two different basis vectors, and Proposition 4.1 proves (iii). This proves (i)–(iii) of Theorem B.

6 Part (iv): the complete list of minimal representatives

Part (iv) is computer-assisted. A minimal representative K' of $\mathcal{K}$ has seven vertices by (iii), and $\{K'\} = [T_6] + [T_4]$ by Theorem 3.5. A minimal representative D' of $\mathcal{L}$ has seven vertices and $\{D'\} = [G_6] + [H_2]$; by the proof of Proposition 4.1 it has exactly one even vertex e' , a crossing of one component with itself, and six odd ones, which occur once in each component. So, after renaming, $D' = \{X, 123456\}$, where X contains e' twice and $1, \dots, 6$ once and can be read off from one of the $8!/2 = 20,160$ linear arrangements of these letters. Part (iv) therefore follows from two finite statements:

- (E1) among the 5,283 isomorphism classes of chord diagrams with seven chords, exactly two have the bracket $[T_6] + [T_4]$, those of K_1 and K_2 ;
- (E2) among the diagrams $\{X, 123456\}$, X one of the 20,160 arrangements, exactly two isomorphism classes have the bracket $[G_6] + [H_2]$, those of D_1 and D_2 .

Both were established by two independent programs, written from the definitions in two independent verification runs, without each other's code and without the code with which the results were first obtained. The first enumerates the chord diagrams with at most seven chords up to rotation and reflection (1, 2, 5, 17, 79, 554, 5,283 classes for 1, $\dots$, 7 chords, numbers confirmed by Burnside's lemma), computes their brackets and finds the bracket of K_1 only for the classes of K_1 and K_2 ; it examines the 20,160 arrangements of (E2) and finds exactly the classes of D_1 and D_2 . The second computes the bracket of all chord diagrams with at most seven chords and of all two-component diagrams with at most seven vertices, free loops allowed (1, 2, 5, 14, 54, 273, 1,962, 19,045 classes for 0, $\dots$, 7 vertices; the 18,885 seven-vertex classes with fewer than six odd vertices need no bracket, since all their terms have fewer than six vertices), and finds exactly K_1 , K_2 and D_1 , D_2 . Neither finds a diagram with at most six vertices with one of these brackets, in agreement with (iii). The programs of this note repeat (E1) and (E2) with a third implementation, with the same result. A further verification run, made on the text of this note, established (E1) and (E2) again with a fourth implementation of its own, and confirmed the numbers 5,283 and 19,045 of classes with seven vertices by Burnside's lemma and the number 18,885 (the 20,160 arrangements form 132 classes, and 28 classes have seven odd vertices). As a cross-check, the type-(1) lists of Proposition 4.2 have 312 members in 14 classes for T_6 and 168 members in 8 classes for G_6 , and in each list exactly two classes have the right bracket. This proves part (iv) of Theorems A and B, as a computer-assisted statement.

7 Remarks

Remark 7.1 (third moves). K_1 and K_2 have one triangle each, and the third move on it leads from one to the other; D_1 has one triangle, D_2 has two, and the third moves on both give diagrams isomorphic to D_1 . So the minimal representatives of $\mathcal{K}$, and those of $\mathcal{L}$, form one class under third moves. Whether some free link has minimal representatives that are not related by third moves remains open; we did not search for such an example systematically.

Remark 7.2 (girth). The underlying multigraph of K_1 has the double edges $a-b$, $d-f$, $e-g$ (from the patterns aba , dfd , geg) and the triangle (e, c, g) , so its girth is two; for K_2 the double edges are $a-b$, $d-f$, $c-g$. Even if only bigons of the second move were counted, the girth would be three. So K_1 , K_2 do not satisfy the hypothesis of the conjecture [KM15, Conjecture 1] (girth at least four), and Theorem A is consistent with it. By (P3) a triangle never has three odd vertices, so irreducibly odd diagrams have no triangle, in agreement with the uniqueness of their minimal diagrams.

Remark 7.3 (the criterion of Kauffman and Manturov). Orient a one-component diagram K and replace every crossing by two trivalent vertices joined by an edge, one receiving the two incoming and the other emitting the two outgoing half-edges; this is the unoriented state K_{us} of [KM14, KM15]. If K_{us} has neither bigons nor quadrilaterals, every diagram equivalent to K has a state containing K_{us} , and K is minimal [KM14, Corollary 3.4], [KM15, Corollary 4]. Right after the criterion they remark that two diagrams that differ by a third move of the unoriented type have the same unoriented state, with a hexagon [KM14, Section 3]; for such a pair the criterion gives the minimality of both diagrams at once. The criterion does not apply to K_1 , K_2 : each pattern xyx of the word gives a quadrilateral, and their states have six and five quadrilaterals (so they are not isomorphic). It applies to $X_1 = 01203241536546$ and to $X_2 = 01203415624635$, whose states have girth six and are isomorphic; X_1 has one triangle, the image of X_1 under the third move is isomorphic to X_2 , and X_2 is not isomorphic to X_1 . So [KM14, Corollary 3.4] applied to X_1 , X_2 also gives a free knot with two non-isomorphic minimal representatives with seven vertices. This was observed in one of the verification runs; the program of this note checks the girth, the move and the non-isomorphism, and the last verification run also the isomorphism of the two states. The parity bracket of X_1 is zero (five even chords): the two methods are complementary. We did not determine all minimal representatives of the free knot of X_1 .

Remark 7.4 (other examples). Among the 5,283 chord diagrams with seven chords, eight classes satisfy the hypotheses of Proposition 4.1 and have a non-isomorphic image under a third move; they form four pairs. Two pairs are alone with their brackets among all 5,283 classes: K_1 , K_2 , and $abacdcebfgdgef$, $abacdefbfcgdge$, whose free knot therefore also has exactly two minimal representatives (Section 6). The other four classes share one bracket.

Remark 7.5 (an unused figure). The source archive of [FIKM14] contains a figure that is not referenced in its text (file `non_uniq.eps`, dated 6 June 2014), with two chord diagrams with seven chords and a graph with seven vertices. Read from the path data, the chord diagrams are $ADGEGACBCFEDFB$ and $AFEDFACBCDGEGB$: they have the same labelled intersection graph and are not isomorphic, both brackets are zero (so neither represents $\mathcal{K}$), and the second has the bigon $\{A, F\}$. The figure seems to concern chord diagrams with the same intersection graph; we draw no conclusion from it.

8 Computations and reproducibility

Parts (i)–(iii) are proved without computation, and part (iv) rests on (E1), (E2); everything else computed is a test. The package contains the programs of this note (standard library only), the programs with which the results were first obtained (`original/`), those of the two verification runs of the first version (`verification_run_A/`, `verification_run_B/`) and those of the verification run of this text (`independent_run_2/`), each with its own library written from the definitions; all with their recorded outputs (Table 2). The library of this note computes smoothings, bigons and triangles from half-edges, not from the word formulas of Section 2, which it tests; `make_figure.py` generated Figure 1 from the Gauss words and reads the chords back from the coordinates in the source of this note. All recorded outputs were reproduced by running the programs again from copies, up to the lines that report running times.

program	what it checks, and the result	time
<code>check_note.py</code>	Lemma 2.3 against half-edges (12,063 random smoothings); all statements of Section 5; Table 1, and that the sums differ for a triangle with two even or three odd vertices; Proposition 4.2; Remarks 7.3 and 7.5: 12,124 assertions hold	2 s
<code>exhaustive.py</code>	all moves between chord diagrams with ≤ 7 chords (20,869: 6,436 first, 6,699 second, 7,734 third moves) and between two-component diagrams with ≤ 6 vertices (7,876): the bracket never changes. Controls: exchanging one pair of a triangle only changes it in 1,483 of 7,734 cases, the rule “odd = linked with ≥ 2 chords” (not a parity) in 1,037 of 2,348 moves. (E1), (E2); Remark 7.4	15 s
<code>original/</code>	random moves on two-component diagrams: 16,055 (4,408 third moves), no change; a false third move changes the bracket in 597 of 1,726 cases; unique normal forms on 37,387 graphs; the lists of Proposition 4.2, also with NetworkX (47 s); no diagram with ≤ 6 vertices (11,465 labelled chord diagrams, 123,670 two-component diagrams) has the bracket of K_1 , resp. D_1	2 min
<code>verification_run_A/</code>	certificates, local tables; (E1) (7 s; class numbers by Burnside’s lemma) and (E2) (3 s); 2,820 smoothings against the word formulas; all deletion orders on 1,454 graphs; 3,204 random moves (763 third moves), no change; a false move detected in 328 of 2,003 cases; the published minimal diagrams it examined [Man09b, BHL14] have no triangle; Remark 7.5	1 min
<code>verification_run_B/</code>	self-tests (11,165 canonical forms against brute force, 5,000 graphs with all deletion orders, 11,984 smoothings); certificates, local tables; the same 28,745 moves as <code>exhaustive.py</code> , no change (17 s); (E1), (E2) from all diagrams with ≤ 7 vertices (143 s); breadth-first search through all diagrams with ≤ 11 vertices reachable from K_1 and from D_1 (203,942 and 153,865 diagrams, 83 and 190 s), reaching no smaller and no third seven-vertex diagram (evidence only; in a control it recovers a hidden smaller diagram in 114 of 114 cases, 15 min); Remark 7.3 (14 min)	40 min
<code>independent_run_2/</code>	all statements of Section 5 and of Remarks 7.1 and 7.2; Table 1 and its controls; Lemmas 2.2, 2.3 and 3.3 on 3,000, 3,000 and 1,500 random graphs; 3,644 random moves (799 third moves), no change; a false third move detected in 323 of 2,163 cases; class numbers by Burnside’s lemma (5,283; 19,045); (E1), (E2), Proposition 4.2, Remark 7.4; Remark 7.3 (the states of X_1 , X_2 are isomorphic) and Remark 7.5, the latter also from the path data	20 s

Table 2: The programs of the package, with running times on one core of a laptop.

Verification. The results were first obtained, with proofs, with the programs in `original/`. Two independent verification runs, both AI-assisted, followed; they did not see each other. Each re-derived the proofs of parts (i)–(iii) (the word model, the normal-form lemma, the parity axioms, the local tables, the invariance theorem, the criterion and every computation of Section 5) and wrote its own programs. The first established (E1) and (E2), checked published minimal diagrams for triangles and decoded the unused figure; it required the citation of the remark of Kauffman and Manturov on minimal diagrams with triangles, the correct girth of K_1 (two, not three), a discussion of the degenerate readings and a precise description of the figure. The second recomputed everything with its own programs, tested the invariance on all moves in the ranges of Table 2, established (E1) and (E2) from all diagrams with at most seven vertices, and ran the programs of the first version again from a copy, reproducing all their outputs byte for byte; it found the example X_1 , X_2 of Remark 7.3, noticed that the list of candidates lacked type (3) of Proposition 4.2, and required a computational test of the knot bracket, the use of the complete enumeration for part (iv), the published title of [KM15], and the statement that the question for minimal representatives not related by third moves

remains open.

After the note was written, a further independent verification run, also AI-assisted, examined its text. It read the cited sources again in their source archives, compared every citation, theorem number and quotation with them, re-derived all proofs line by line, and wrote its own programs (`independent_run_2/`), with which it recomputed every finite statement of Sections 5 and 7, Table 1 and (E1), (E2), with the class numbers by Burnside’s lemma; it also read the coordinates of the unused figure itself. It ran the programs of the package again from the archive and reproduced their recorded outputs up to running times, and it repeated the literature search. It required the correct number of the criterion of [KM14] (Corollary 3.4, not 4), the citation of the remark of Kauffman and Manturov that diagrams related by a third move can have the same unoriented state, a more careful account of the known uniqueness results, the reference to Turaev’s homotopy of Gauss words, and the record of this run. No run found an error in a proof; all corrections were made.

Scope and priority. The note answers Problem 51 of [FIKM14] in the sense of Definition 2.4: the free knot $\mathcal{K}$ and the two-component free link $\mathcal{L}$ each have exactly two minimal representatives up to isomorphism, (i)–(iii) being proved in the text and (iv) computer-assisted. Minimal representatives not related by third moves are not treated. The problem and the conjecture are due to Fenn, Ilyutko, Kauffman and Manturov [FIKM14]; the notions, the parities and the parity bracket to Manturov [Man09a, Man09b, Man11], including the invariance for the mixed parity [Man09b, Theorem 4]. Kauffman and Manturov gave a criterion [KM14, Corollary 3.4] and remarked right after it that diagrams related by a third move can have the same unoriented state [KM14, Section 3]; the criterion, applied to the diagram X_1 of Remark 7.3, also answers the problem. They also observed that a diagram with a triangle is equivalent to its image under the third move and remarked that minimal free knot diagrams with triangles can be constructed [KM15, Section 5]. So the method is not new; to our knowledge, the note gives the first explicit written example with a self-contained proof of non-isomorphism, minimality and the exact number of minimal representatives. The problem list of September 2014, of which Kauffman and Manturov are coauthors, still states the non-uniqueness as a conjecture. We claim no priority.

What was read, in the arXiv versions whose numbering we use (the journal versions were not compared): [FIKM14] (version 1, Sections 2.2 and 8.1, and the files of the source archive); [Man09b] (version 1) in full; [Man09a] (version 2, Sections 1–2); [Man11] (version 1, Sections 3 and 5); [KM14] (version 3, Section 3, also in the PDF of that version, which numbers its statements by section); [KM15] (version 2, Sections 4–5); of [BHL14] the abstract and the proof of the main theorem (version 4); of [Man14] and [IM20] the abstracts; the arXiv versions of [Gib11] (version 2) and [Tur07] (version 2) were searched for statements on the uniqueness of minimal representatives and contain none. Not read, because not accessible to us: the book [MI12], the survey [IMN13], the papers [Man10, Man13], the text of [IM20] and the chapter [Pet21]; a remark on the problem could be contained in them. The searches, on 14 October 2026, used the arXiv interface (free knots and links, minimal diagrams and representatives, parity brackets, framed 4-graphs, Gauss words and phrases, nanowords), Crossref, zbMATH, OpenAlex (the 20 works citing [FIKM14], the 13, 7 and 1 works citing [KM14], [KM15] and [BHL14], and the works citing Manturov’s papers on parity) and a few web searches; they were repeated in part by the last verification run. About twenty further papers on free knots, parities and Gauss words were read in the first version and in the verification runs. No explicit example of a free link with two non-isomorphic minimal representatives, and no statement that the problem is settled, was found in the literature accessible to us; a search that finds nothing is not a proof of novelty.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has

not been peer reviewed.

References

- [BHL14] Tomas Boothby, Allison Henrich, and Alexander Leaf. Minimal diagrams of free knots. *Journal of Knot Theory and Its Ramifications*, 23(6):1450032, 2014. The abstract and the proof of the main theorem of the arXiv version 4 were read. [arXiv:1008.3163](#), [doi:10.1142/S0218216514500321](#).
- [FIKM14] Roger Fenn, Denis P. Ilyutko, Louis H. Kauffman, and Vassily O. Manturov. Unsolved problems in virtual knot theory and combinatorial knot theory. *Banach Center Publications*, 103:9–61, 2014. We read the arXiv version (v1, the only version), whose section and item numbers we use; the journal version was not compared. [arXiv:1409.2823](#), [doi:10.4064/bc103-0-1](#).
- [Gib11] Andrew Gibson. Homotopy invariants of Gauss words. *Mathematische Annalen*, 349(4):871–887, 2011. The arXiv version 2 was read in part (Sections 1–2) and searched; the journal version was not compared. [arXiv:0902.0062](#), [doi:10.1007/s00208-010-0536-0](#).
- [IM20] Denis P. Ilyutko and Vassily O. Manturov. An invariant of graph-links valued in graphs and an almost classification of graph-links. *Journal of Knot Theory and Its Ramifications*, 29(9):2050063, 2020. Only the abstract was read. [doi:10.1142/S0218216520500637](#).
- [IMN13] Denis P. Ilyutko, Vassily O. Manturov, and Igor M. Nikonov. Parity in knot theory and graph-links. *Journal of Mathematical Sciences*, 193(6):809–965, 2013. Not accessible to us; not read. [doi:10.1007/s10958-013-1499-y](#).
- [KM14] Louis H. Kauffman and Vassily O. Manturov. A graphical construction of the $\text{sl}(3)$ invariant for virtual knots. *Quantum Topology*, 5(4):523–539, 2014. We read the arXiv version 3, whose numbering by sections we use (the criterion is Corollary 3.4); the journal version was not compared. [arXiv:1207.0719](#), [doi:10.4171/QT/57](#).
- [KM15] Louis H. Kauffman and Vassily O. Manturov. Graphical constructions for the $\text{sl}(3)$, C_2 and G_2 invariants for virtual knots, virtual braids and free knots. *Journal of Knot Theory and Its Ramifications*, 24(6):1550031, 2015. We read the arXiv version 2 (titled there with $\text{so}(3)$ in place of C_2), whose numbering we use; the journal version was not compared. [arXiv:1406.7331](#), [doi:10.1142/S0218216515500315](#).
- [Man09a] Vassily O. Manturov. On free knots, 2009. Version 2 read. [arXiv:0901.2214](#).
- [Man09b] Vassily O. Manturov. On free knots and links, 2009. Version 1 (the only version) read; theorem numbers as in that version. [arXiv:0902.0127](#).
- [Man10] Vassily O. Manturov. Parity in knot theory. *Sbornik: Mathematics*, 201(5):693–733, 2010. Not accessible to us; not read. [doi:10.1070/SM2010v201n05ABEH004089](#).
- [Man11] Vassily O. Manturov. Free knots and parity. In *Introductory Lectures on Knot Theory*, Series on Knots and Everything, pages 321–345. World Scientific, 2011. We read the arXiv version (v1), whose theorem numbers we use; the published chapter was not compared. [arXiv:0912.5348](#), [doi:10.1142/9789814313001_0012](#).

- [Man13] Vassily O. Manturov. An almost classification of free knots. *Doklady Mathematics*, 88(2):556–558, 2013. Not accessible to us; not read. doi:10.1134/S1064562413050189.
- [Man14] Vassily O. Manturov. Reidemeister moves and groups, 2014. Only the abstract was read. arXiv:1412.8691.
- [MI12] Vassily O. Manturov and Denis P. Ilyutko. *Virtual Knots: The State of the Art*. Series on Knots and Everything. World Scientific, 2012. Not accessible to us; not read. doi:10.1142/8438.
- [Pet21] Nicolas Petit. Virtual strings and free knots. In Colin Adams et al., editors, *Encyclopedia of Knot Theory*, pages 335–342. CRC Press, Boca Raton, FL, 2021. Zbl 1465.57073. Not accessible to us; not read.
- [Tur07] Vladimir Turaev. Topology of words. *Proceedings of the London Mathematical Society*, 95(2):360–412, 2007. The arXiv version 2 was searched; the journal version was not compared. arXiv:math/0503683, doi:10.1112/plms/pdm014.
- [ula26] ulamai. UnsolvedMath. Hugging Face dataset, <https://huggingface.co/datasets/ulamai/UnsolvedMath>, 2026. Version 1.6.0; record AMR-105-0051.