

A Torsion-Rank Obstruction to Turaev's Alexander Polynomial Realization Conjecture

Alper Ferudun*

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Abstract

We give counterexamples to the sufficiency assertion in Turaev's Conjecture 12.26 in Ohtsuki's problem list. If a closed connected oriented 3-manifold has first homology $\mathbb{Z} \oplus T$, and its Alexander polynomial is nonzero modulo a prime p , its order of vanishing at $t = 1$ modulo p is at least $\dim_{\mathbb{F}_p}(T/pT)$. The proof uses an integral square presentation of the infinite cyclic cover's first homology, obtained by cutting along a primitive dual surface. Consequently, whenever T has p -rank at least three, the pair $(\mathbb{Z} \oplus T, t + t^{-1} + |T| - 2)$ satisfies the conjecture's symmetry and augmentation conditions but is not realizable. The example $T = (\mathbb{Z}/2)^3$ gives the polynomial $t + t^{-1} + 6$. We also give a Smith-normal-form strengthening and torus-bundle examples distinguishing a prescribed torsion group from its order.

1 The realization question

Let M be a closed connected oriented 3-manifold with $b_1(M) = 1$. Put

$$T = \operatorname{Tor} H_1(M; \mathbb{Z}), \quad \Lambda = \mathbb{Z}[t, t^{-1}].$$

Choose a primitive epimorphism $\phi : \pi_1(M) \rightarrow \mathbb{Z}$, and denote the associated infinite cyclic cover by $\widetilde{M}$. The Alexander polynomial Δ_M is the order over Λ of $H_1(\widetilde{M}; \mathbb{Z})$, defined up to multiplication by $\pm t^j$. Equivalently it generates the smallest principal ideal containing the zeroth Fitting ideal of this module. We use the ordinary polynomial associated to the torsion-free abelian quotient, not the full abelian Reidemeister torsion in the group ring of $H_1(M; \mathbb{Z})$.

Conjecture 12.26, attributed to Turaev in Ohtsuki's list [3, p. 542], asks whether a rank-one finitely generated abelian group H and a Laurent polynomial Δ can occur simultaneously as $(H_1(M; \mathbb{Z}), \Delta_M)$ precisely when

$$\Delta(t) = t^k \Delta(t^{-1}) \quad (k \in 2\mathbb{Z}), \quad \Delta(1) = \pm |\operatorname{Tor} H|. \tag{1}$$

The source notes that these conditions are necessary and are sufficient when $\operatorname{Tor} H$ is cyclic. Prescribing the torsion isomorphism type is essential: characterizing polynomials while allowing the first homology group to vary is a different problem. Alcaraz's characterization for $b_1 = 1$ concerns the latter problem [1].

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>. Unrefereed preprint. CC BY 4.0.

Theorem 1.1. *Let T be a finite abelian group. Suppose that, for some prime p ,*

$$r_p(T) := \dim_{\mathbb{F}_p}(T/pT) \geq 3.$$

Then the pair

$$H = \mathbb{Z} \oplus T, \quad \Delta(t) = t + t^{-1} + |T| - 2 \quad (2)$$

satisfies (1), but no closed connected oriented 3-manifold realizes this pair. In particular the sufficiency assertion of Conjecture 12.26 is false.

For instance, one may take

$$H = \mathbb{Z} \oplus (\mathbb{Z}/2)^3, \quad \Delta(t) = t + t^{-1} + 6.$$

The proof gives the same conclusion for every $T = (\mathbb{Z}/p)^r$ with p prime and $r \geq 3$. The ingredients are classical cut-surface presentations and elementary determinant facts. We give the integral construction explicitly to justify reduction modulo p ; reduction of a gcd from an arbitrary presentation would not suffice.

2 A square presentation with prescribed specialization

All homology groups in this section have integer coefficients. The following presentation is a standard consequence of cutting along a dual surface; compare Alcaraz [1, Section 4.3]. It also agrees with the first-Betti-number-one principality in McMullen's description of Alexander ideals [2, Theorem 5.1].

Lemma 2.1. *There is a square matrix $P(t)$ over Λ presenting $H_1(\widetilde{M}; \mathbb{Z})$ such that*

$$\text{coker } P(1) \cong T, \quad \det P(t) \doteq \Delta_M(t). \quad (3)$$

Here $\doteq$ denotes equality up to $\pm t^j$.

Proof. Represent the primitive integral class Poincaré dual to ϕ by a connected closed oriented embedded surface S . Integral degree-one classes admit such surface representatives, and tubing components makes the representative connected. The surface is nonseparating because $[S] \neq 0$. Let Y be obtained by cutting M along S ; then Y is connected, with two boundary copies of S . Write g for the genus of S and i_+, i_- for the two maps on homology induced by these boundary inclusions.

Eliminating the product collar from the Mayer–Vietoris sequence for M gives the exact segment

$$\begin{aligned} H_2(Y) &\longrightarrow H_2(M) \longrightarrow H_1(S) \xrightarrow{i_+ - i_-} H_1(Y) \\ &\longrightarrow H_1(M) \xrightarrow{\phi} \mathbb{Z} \longrightarrow 0. \end{aligned}$$

The map to $\mathbb{Z}$ is intersection with S and is the primitive quotient. Integral Poincaré duality gives $H_2(M) \cong H^1(M) \cong \mathbb{Z}$. The class $[S]$ is a generator and is the image of a boundary copy of S in Y . Thus $H_2(Y) \rightarrow H_2(M)$ is onto, and the sequence reduces to

$$0 \longrightarrow \mathbb{Z}^{2g} \xrightarrow{i_+ - i_-} B \longrightarrow T \longrightarrow 0, \quad B = H_1(Y). \quad (4)$$

In particular B has rank $2g$. Choose a decomposition

$$B \cong \mathbb{Z}^{2g} \oplus \bigoplus_{j=1}^s \mathbb{Z}/m_j, \quad m_j > 1,$$

with $q = 2g + s$ generators and s torsion relations.

Stacking copies of Y constructs $\widetilde{M}$. The corresponding Mayer–Vietoris sequence over Λ contains

$$\begin{aligned} H_1(S) \otimes \Lambda &\xrightarrow{i_+ - ti_-} B \otimes \Lambda \longrightarrow H_1(\widetilde{M}) \\ &\longrightarrow H_0(S) \otimes \Lambda \xrightarrow{1-t} H_0(Y) \otimes \Lambda. \end{aligned}$$

The final map is injective because S and Y are connected. Consequently $H_1(\widetilde{M})$ is the cokernel of $i_+ - ti_-$. Lift the $2g$ surface relations to the chosen q generators of B and append its s torsion relations. There are q generators and q relations, giving the required square matrix $P(t)$. The surface relations may have torsion coordinates; no block-diagonal splitting is needed.

Putting $t = 1$ in this construction presents the cokernel in (4), so $\text{coker } P(1) = T$. Hence $|\det P(1)| = |T| \neq 0$, and $\det P(t)$ is nonzero. For a square presentation the zeroth Fitting ideal is generated by the determinant, proving (3). The convention for a zero-by-zero matrix is determinant one. $\square$

3 The torsion rank obstruction

Proposition 3.1. *Let p be a prime. The reduction $\overline{\Delta}_M$ of Δ_M modulo p satisfies*

$$(t-1)^{r_p(T)} \mid \overline{\Delta}_M(t) \quad \text{in } \mathbb{F}_p[t, t^{-1}]. \quad (5)$$

If $\overline{\Delta}_M \neq 0$, then

$$\text{ord}_{t=1} \overline{\Delta}_M \geq r_p(T). \quad (6)$$

Proof. Use the $q \times q$ matrix of Lemma 2.1 and reduce it modulo p , obtaining $\overline{P}(t)$. Right exactness of tensor product gives

$$\text{coker } \overline{P}(1) \cong T/pT.$$

Thus $\overline{P}(1)$ has rank $q - r_p(T)$. An invertible constant row operation over $\mathbb{F}_p$ makes its last $r_p(T)$ rows zero. Apply that same operation to $\overline{P}(t)$. Every entry of those rows vanishes at one, so every entry is divisible by $t-1$ in the Laurent polynomial ring. Factoring $t-1$ from each such row proves divisibility of the determinant by $(t-1)^{r_p(T)}$. By Lemma 2.1 this is (5). The unit $\pm t^j$ does not change divisibility or multiplicity at one. If the determinant is nonzero, the multiplicity inequality follows. $\square$

Proof of Theorem 1.1. Put $N = |T|$. The Laurent polynomial in (2) is symmetric with exponent $k = 0$, and its value at one is N . It therefore satisfies (1). But $p \mid N$, so

$$\overline{\Delta}(t) = t + t^{-1} - 2 = t^{-1}(t-1)^2 \quad \text{in } \mathbb{F}_p[t, t^{-1}].$$

This is nonzero and has multiplicity exactly two at one, also in characteristic two. A realizing manifold would violate (6), since $r_p(T) \geq 3$. $\square$

Remark 3.2. The integral presentation is indispensable. For the abstract Λ -module $\Lambda/(p, t-1)$, the order over Λ is one: the gcd of p and $t-1$ is one. After reduction modulo p its order is $t-1$. Thus orders need not commute with reduction for an arbitrary finitely presented module. Lemma 2.1 supplies the topological reason that the determinant argument applies here.

4 An integral Smith form obstruction

Let $d_1 \mid \cdots \mid d_r$ be the nontrivial invariant factors of T , and define an ideal of Λ by

$$J_T = \prod_{i=1}^r (d_i, t-1).$$

For $T = 0$ this product is the unit ideal.

Proposition 4.1. *Every realizable pair $(\mathbb{Z} \oplus T, \Delta_M)$ satisfies*

$$\Delta_M \in J_T = \left(\left(\prod_{i=1}^{r-j} d_i \right) (t-1)^j : 0 \leq j \leq r \right). \quad (7)$$

Proof. Apply unimodular integer row and column operations to $P(1)$ to obtain Smith form. Its nonunit diagonal entries are precisely $d_1, \dots, d_r$, by (3). Perform the same operations on $P(t)$. In the row corresponding to d_i , every entry lies in the ideal $(d_i, t-1)$: evaluation at one is zero except for the diagonal entry d_i . Every term of the determinant therefore lies in the product of these r ideals. The operations change the determinant by a sign; the Laurent unit ambiguity also preserves ideal membership.

To obtain the displayed generators, expand the product of ideals. For fixed j , the product of the smallest $r-j$ invariant factors divides the product of any other $r-j$ of them, and is itself one of those products. This proves the equality of ideals. $\square$

For $T = (\mathbb{Z}/p)^r$, this condition reads $\Delta_M \in (p, t-1)^r$. The example $T = (\mathbb{Z}/2)^3$ can therefore also be excluded integrally. With $s = t-1$,

$$t(t+t^{-1}+6) = t^2 + 6t + 1 = 8 + 8s + s^2.$$

The coefficient of s^2 is odd, whereas an element of $(2, s)^3 = (8, 4s, 2s^2, s^3)$ has even quadratic coefficient in its expansion at $s = 0$. Localization at $t = 1 + s$ does not alter this conclusion, since $1 + s$ is a unit modulo this ideal. The condition (7) is necessary only; we do not assert that it gives a sufficient realization criterion.

5 Torus bundle controls

The polynomial itself is not the obstruction. For every integer $N > 0$, consider the oriented torus bundle whose monodromy is induced by

$$F_N = \begin{pmatrix} 0 & -1 \\ 1 & 2-N \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

Its infinite cyclic cover is $T^2 \times \mathbb{R}$, and its Alexander module is presented by $tI - F_N$. Thus

$$\det(tI - F_N) = t^2 + (N-2)t + 1 = t(t+t^{-1}+N-2).$$

The bundle's first homology is $\mathbb{Z} \oplus \mathrm{coker}(F_N - I)$. The determinant of $F_N - I$ is N , and its entries have gcd one. Its Smith form is therefore $\mathrm{diag}(1, N)$, giving torsion $\mathbb{Z}/N$. In particular the polynomial $t+t^{-1}+6$ occurs with cyclic torsion of order eight, but cannot occur with $(\mathbb{Z}/2)^3$.

The multiplicity bound is sharp for some noncyclic torsion. The torus bundle with monodromy $-I$ has

$$T \cong (\mathbb{Z}/2)^2, \quad \Delta_M \doteq (t+1)^2.$$

Modulo two the root at one has multiplicity two, exactly $r_2(T)$. These examples are consistent with the known cyclic case and illustrate why a polynomial-only characterization does not answer the prescribed-homology question.

6 Verification and scope

The proof above applies to the entire family without a computational hypothesis. The accompanying exact-arithmetic script checks the determinant assertions on all 6,817 two-by-two pencils over $\mathbb{F}_2$ and $\mathbb{F}_3$, on 480 additional larger pencils, and on 240 Smith-form pencils. It also checks 80 instances of the counterexample family and 64 cyclic torus-bundle controls. These are regression checks, not a verification of the topological argument or a substitute for the proof.

The result disproves the sufficiency direction of the stated conjecture; it does not characterize all realizable pairs. The cut-surface and Fitting-ideal methods are classical. The bounded literature search did not locate this explicit counterexample family, but no absolute priority claim is made. This preprint has not undergone independent peer review or formal verification. AI tools assisted with literature retrieval, symbolic reasoning, drafting and arithmetic checks; responsibility for the manuscript's claims remains with the author.

References

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