

Determinants of Prime Alternating Achiral Knots: A Partial Answer to a Problem of Stoimenow

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Abstract

Stoimenow asked (Problem 12.25 of the problem list edited by Ohtsuki, 2002; Conjecture 4.2 of his paper in Comm. Anal. Geom. 13 (2005)) whether every odd sum of two squares $n \notin \{1, 9, 49\}$ is the determinant of a prime alternating achiral knot. He proved this for squares and, with rational knots, for all $n > 1$ without prime factors $q \equiv 3 \pmod{4}$, and reported a check up to 2000. We give a partial answer; the question remains open. Gluing a plane network in a disk to its dual network, turned over by a half-turn of the sphere, is a form of a classical construction of self-dual maps; when the resulting graph is nonseparable and has an odd number of spanning trees, it is the checkerboard graph of a prime alternating achiral knot. For two and three boundary points this gives Stoimenow's determinant formulas back; for four it gives a new one. With it we show: (i) if n is the determinant of such a knot, so are $s^4 n$ for odd $s \geq 3$ and $3^6 n$; hence so is every $n = j^4 m$ with j odd and $m > 1$ composed of primes $\equiv 1 \pmod{4}$, and the question reduces to $n = d^2 m$ with $d > 1$ a squarefree product of primes $\equiv 3 \pmod{4}$; (ii) $5q^2$ is the determinant of such a knot for every prime $q \equiv 3 \pmod{4}$, which extends Stoimenow's remark on $q \leq 11$; (iii) so is every number in an explicit range of the form $d^2(A^2 + B^2)$, by three-terminal graphs; (iv) so is every $n \leq 10^6$ not covered by Stoimenow's results (16,703 numbers), by explicit certificates checked by computer. For larger n the general case is open, and we explain why these families do not settle it. This is an unrefereed note.

1 Introduction

1.1 The problem

Knots are smooth embeddings $S^1 \rightarrow S^3$ up to isotopy. A knot is *achiral* (amphicheiral) if it is isotopic to its mirror image; no condition on orientations is imposed [Sto05, §1]. A knot is *prime* if it is non-trivial and not a connected sum of two non-trivial knots, and *alternating* if it has an alternating diagram. The determinant is $\det K = |\Delta_K(-1)|$, the order of the first homology of the double branched cover [Sto05, §2].

Problem (Stoimenow [Oht02, Problem 12.25], [Sto05, Conjecture 4.2]). *Let n be an odd natural number, $n \notin \{1, 9, 49\}$, which is a sum of two squares. Is n the determinant of a prime alternating achiral knot?*

What is known (from [Sto05], read in the arXiv version v3, and [Oht02]). By Hartley and Kawachi [HK79], using Goeritz matrices, the determinant of an achiral knot is a sum of two squares. Conversely, every odd sum of two squares is the determinant of an achiral knot, which can be chosen alternating,

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or prime [Sto05, Theorem 4.1]; the Remark to the problem in [Oht02, p. 541] adds “but not always both”. An odd n is the determinant of an achiral rational knot if and only if $n = a^2 + b^2$ with $\gcd(a, b) = 1$ [Sto05, Theorem 4.2]; rational knots are prime and alternating, so every $n > 1$ without prime factors $q \equiv 3 \pmod{4}$ is the determinant of a prime alternating achiral knot. The number 1 is not, since the only alternating knot of determinant 1 is the unknot [Sto05, §4.2]; nor are 9 and 49 [Sto05, Proposition 4.1]; and an odd square is if and only if it is not 1, 9 or 49 [Sto05, Theorem 4.4]. The conjecture is reported to hold for $n \leq 2000$ [Sto05, after Conjecture 4.2]; no list of witnesses is given. In [Sto05, §4.2] a family of diagrams with determinant $f_1^2 + f_2^2$, formula (12) there, is shown never to give $5p^2$ with $p \equiv 3 \pmod{4}$ prime, and it is remarked there that the cases $p \leq 11$ are realized by knots with Conway polyhedron 8^* . A graph-theoretic form of the problem, with planar self-dual graphs without cut vertex, which Stoimenow calls slightly stronger, is [Sto05, Conjecture 6.1].

An odd n is a sum of two squares if and only if every prime $q \equiv 3 \pmod{4}$ divides n to an even power, and an $n > 1$ has a representation with coprime parts if and only if moreover no such prime divides n . So the open cases form the *residual set*

$$\mathcal{R} = \{ n \text{ odd} : n = a^2 + b^2, n \text{ not a square, } q^2 \mid n \text{ for some prime } q \equiv 3 \pmod{4} \}.$$

Its smallest members are 45, 117, 153, 245, 261, 333, 369, 405, 477, 549; $\mathcal{R}$ has 39 members below 2000 and 16,703 below 10^6 .

1.2 Results

A *network* with $m \geq 2$ terminals is a connected graph drawn in a closed disk D that meets ∂D exactly in m of its vertices, the terminals. For $m = 2$ (terminals s, t) its *pair* is $p(N) = (\tau(N), F(N))$, where τ counts spanning trees and $F(N)$ counts spanning forests with two components, one containing s and the other t ; $\tilde{N}$ is N with an edge st added outside D . Every network has a dual network N^* (Definition 2.4). Put the network X in the northern hemisphere of S^2 with its terminals on the equator, and let ψ be the half-turn of S^2 of Section 3; it exchanges the hemispheres and maps the midpoints of the boundary arcs to the terminals. Let $G(X) = X \cup \psi(X^*)$.

Theorem 1.1. *For every network X with $m \geq 2$ terminals, ψ maps $G(X)$ onto its dual, up to isotopy. If $G(X)$ is nonseparable and $\tau(G(X))$ is odd, the alternating diagram with checkerboard graph $G(X)$ represents a prime alternating achiral knot of determinant $\tau(G(X))$. In particular:*

- (a) *for $m = 2$, $G(X) = X \parallel X^*$ and $\tau(G(X)) = \tau(X)^2 + F(X)^2$;*
- (b) *if X is a star whose arms are networks $N_0, \dots, N_{m-1}$ with pairs (x_j, y_j) , then $G(X)$ is the wheel W_m with the spokes replaced by the N_j and the rims by dual networks, and with $z_j = x_j + iy_j$,*

$$m = 3 : \quad \tau(G(X)) = |\Lambda_3|^2, \quad \Lambda_3 = x_1 \overline{z_0} z_2 + y_1 \operatorname{Im}(z_0 z_2),$$

$$m = 4 : \quad \tau(G(X)) = |\Lambda_4|^2, \quad \Lambda_4 = x_1 x_3 z_0 z_2 + x_1 y_3 y_2 z_0 + y_1 x_3 y_0 z_2 + y_1 y_3 (z_0 z_2 + y_0 y_2).$$

We say that n is *realized* if it is the determinant of one of the knots of Theorem 1.1.

Theorem 1.2 (fourth-power scaling). *If n is realized, then $s^4 n$ is realized for every odd $s \geq 3$, and so is $3^6 n$. Hence every $n = j^4 m$ with j odd and $m > 1$ composed of primes $\equiv 1 \pmod{4}$ is realized (in particular every $n \in \mathcal{R}$ in which each prime $q \equiv 3 \pmod{4}$ occurs to an exponent divisible by 4), and so is every $n = 3^{2e} m$ with $e \geq 2$ and m as before. If every $n = d^2 m$ with $d > 1$ a product of distinct primes $\equiv 3 \pmod{4}$ and $m > 1$ composed of primes $\equiv 1 \pmod{4}$ is realized, the answer to the problem is yes.*

Theorem 1.3. *Let Q be odd, and either $Q = a^2 + ab + b^2$ with $a, b \geq 1$ coprime, or $Q = 3Y^2 - X^2$ with $Y > |X|$ and $\gcd(X, Y) = 1$. Then $5Q^2$ is realized by a four-terminal wheel of Theorem 1.1(b). Every prime $q \equiv 3 \pmod{4}$ is of one of these forms; so $5q^2$ is the determinant of a prime alternating achiral knot for every prime $q \equiv 3 \pmod{4}$.*

Theorem 1.4. *Let $d \geq 3$ be odd, A even and B odd with $\gcd(A, B) = 1$ and $\gcd(d, AB) = 1$, and let k be an odd divisor of A with $B < dk$ and $d < A/k$. Then $d^2(A^2 + B^2)$ is realized by a three-terminal wheel. For $k = 1$: every such $d^2(A^2 + B^2)$ with $B < d < A$.*

Theorem 1.5 (computation). *Every $n \in \mathcal{R}$ with $n \leq 10^6$ is realized, with $m \in \{2, 3, 4\}$ and series-parallel networks; for each of these 16,703 numbers an explicit certificate is given (Section 6).*

Together with [Sto05, Theorems 4.2 and 4.4], Theorem 1.5 answers the problem positively for all $n \leq 10^6$.

1.3 What is new, what is not, and what stays open

The problem, the results of [Sto05] quoted above and the formulas of the cases $m = 2$ and $m = 3$ of Theorem 1.1 are Stoimenow's: case (a) is a checkerboard-graph form of his construction (9), and $\Lambda_3 = f_1 + if_2$ is his formula (12) (Proposition 3.2). Gluing a graph to its dual to obtain self-dual maps is classical [ST50, AR92, SC92, SS96], and Tutte [Tut79] studied spanning trees of self-dual maps; the first statement of Theorem 1.1 is a general form of this known construction principle. The four-terminal wheels of Theorem 1.3 are self-dual knots on the polyhedron 8^* , so Theorem 1.3 extends Stoimenow's remark from $q \leq 11$ to all q . New, as far as we know, are the application to the problem, the formula for Λ_4 and Theorems 1.2 to 1.5. No printed answer to the problem was found in the sources we searched ("Scope and priority" at the end); a search that finds nothing is not a proof of novelty, and we claim no priority. The problem stays open: the cases $n = d^2m > 10^6$ with $d > 1$ squarefree that are not covered by Theorems 1.3 and 1.4 remain, and Section 7 explains why our families do not cover them.

2 Checkerboard graphs and networks

Graphs are finite and may have multiple edges; $\tau(G)$ is the number of spanning trees, computed by Kirchhoff's theorem [Kir47]. A graph is *nonseparable* if it is connected, has at least two edges, and its edge set has no partition into two non-empty parts whose spanned subgraphs have at most one common vertex; such a graph has no loops and no bridges. If G and H are nonseparable and f is an edge of H , replacing an edge e of G by $H \setminus f$, with the ends of f put at those of e (a 2-sum), gives a nonseparable graph; and a plane graph is nonseparable if and only if its dual is (both standard). For a connected plane graph G , $T \mapsto (E \setminus T)^*$ is a bijection between the spanning trees of G and of G^* .

Checkerboard graphs. Let G be a connected graph in $S^2 \subset \mathbb{R}^3$. The diagram $D(G)$ has one crossing c_e on each edge e , and its curve is the medial graph of G ; its regions are black (one around each vertex of G) and white (one in each face), and at c_e the two black corners belong to the ends of e . We fix one of the two crossing types relative to the colouring, as in [Sto05, §6], and use it at every crossing. A diagram on S^2 represents the link lying near S^2 with over-strands farther from the origin; switching all crossings amounts to the inversion $x \mapsto x/|x|^2$ and gives the mirror image. G is the checkerboard (Tait) graph of $D(G)$.

Lemma 2.1. *Let G be a connected plane graph.*

(a) *$D(G)$ is alternating and $\det D(G) = \tau(G)$ [Sto05, Lemma 6.1].*

(b) $D(G^*)$ is $D(G)$ with all crossings switched, hence a diagram of the mirror image [Sto05, §6].

(c) $D(G)$ is a knot diagram if and only if $\tau(G)$ is odd.

(d) For an orientation-preserving homeomorphism h of S^2 , $h(D(G)) = D(h(G))$.

Proof. (b) The medial curves of G and G^* coincide and the colours are exchanged, so the type of every crossing changes. (c) By (a) and [Sto05, p. 16]: the determinant of a link with μ components is divisible by $2^{\mu-1}$, and that of a knot is odd. (d) h maps black regions to the regions around the vertices of $h(G)$, and it preserves the crossing type because it preserves the orientation. $\square$

Lemma 2.2. *If the plane graph G is nonseparable, $D(G)$ is reduced and prime as a diagram. If moreover $\tau(G)$ is odd, $D(G)$ represents a prime knot.*

Proof. Let γ be a simple closed curve meeting $D(G)$ transversally in two points that are not crossings. The colour changes at each of them, so γ is an arc in a black region, around a vertex v , and an arc in a white region. The black region of a vertex $w \neq v$ does not meet γ ; it lies on one side of γ together with the crossings of all edges at w . So the edges whose crossings lie inside γ and those outside span subgraphs with at most the vertex v in common, and as G is nonseparable one side contains no crossing: $D(G)$ is prime. If a crossing c_e were nugatory, a closed curve would meet $D(G)$ only in c_e , leaving it through two opposite corners, which then belong to one region. If they are black, the ends of e coincide and e is a loop; if white, the faces on the two sides of e coincide and e is a bridge. Neither occurs. Menasco's theorem [Men84] (an alternating link is prime if a reduced alternating diagram of it is prime; see [Sto05, §4.2]) shows that the link is prime, and it is a knot by Lemma 2.1(c). It is non-trivial, since a reduced alternating diagram with $c \geq 1$ crossings has crossing number c [Kau87, Mur87, Thi87]. $\square$

Networks. Let N be a network in a disk D with terminals $t_0, \dots, t_{m-1}$ in counterclockwise order, and let a_k be the arc of ∂D from t_k to t_{k+1} (indices mod m).

Lemma 2.3. *The arcs $a_0, \dots, a_{m-1}$ lie in m distinct regions of $D \setminus N$.*

Proof. If a_k and a_l , $k \neq l$, lay in one region, an arc in that region would join a point of a_k to a point of a_l and cut D into two parts. Each part contains a terminal (t_{k+1} on one side and t_k on the other), and the connected graph N would meet both parts without meeting the arc. $\square$

Definition 2.4. Let R_k be the region of $D \setminus N$ containing a_k and μ_k the midpoint of a_k . The *dual network* N^* has one vertex in each region of $D \setminus N$, the vertex of R_k being μ_k , and for each edge e of N an edge e^* in D which crosses e once, meets no other edge of N , and joins the vertices of the regions on the two sides of e . It is a network with terminals $\mu_0, \dots, \mu_{m-1}$ (it is connected, as any two regions are joined by a path in D avoiding the vertices of N).

Lemma 2.5. *The dual network of N^* , with terminals at the midpoints t_k of the arcs from μ_{k-1} to μ_k , is N up to an isotopy of D fixing ∂D .*

Proof. On the sphere $D \cup D'$ consider the connected plane graph $H = N \cup \partial D$. Its dual is $H^* = N^* \cup \{\sigma\} \cup \{c_0^*, \dots, c_{m-1}^*\}$, with $\sigma \in D'$ and c_k^* joining σ to μ_k across a_k . Each face of H^* contains exactly one vertex of H . The faces not incident with σ are the regions of $D \setminus N^*$ not meeting ∂D , and contain the non-terminal vertices of N ; the face between c_{k-1}^* and c_k^* contains t_k , and its part in D is the region of $D \setminus N^*$ along the arc from μ_{k-1} to μ_k . Every edge of N crosses exactly its dual edge. So N is a dual network of N^* . $\square$

For $m = 2$ we write $s = t_0$, $t = t_1$, and $N \cdot N'$ and $N \parallel N'$ for the series and parallel compositions. The networks obtained from single edges by these compositions are the *series-parallel* networks; for coprime $a, b \geq 1$ the *rational* network $R(a, b)$ is a single edge if $a = b = 1$, $R(a - b, b) \parallel e$ if $a > b$, and $R(a, b - a) \cdot e$ if $b > a$ (e an edge).

Lemma 2.6 (substitution). *Let S be a graph, and let G be obtained by replacing every edge $e = uv$ of S by a two-terminal network N_e with terminals u, v . Then*

$$\tau(G) = \sum_T \prod_{e \in T} \tau(N_e) \prod_{e \notin T} F(N_e),$$

the sum over the spanning trees T of S . If S is nonseparable and every $\widehat{N}_e$ is nonseparable, G is nonseparable.

Proof. Let T' be a spanning tree of G . All edges at a non-terminal vertex of N_e lie in N_e , so every component of $T' \cap N_e$ contains a terminal: $T' \cap N_e$ is a spanning tree of N_e or a spanning forest with two components separating u from v . The set T of the edges e of the first kind is a spanning tree of S (a path of T' between vertices of S passes through N_e from end to end only if $e \in T$, and a cycle in T would give one in T'). Conversely every spanning tree T of S , with a spanning tree of N_e for $e \in T$ and a separating forest for $e \notin T$, gives a spanning tree of G . The last statement holds because G arises from S by 2-sums. $\square$

Corollary 2.7. $p(N \cdot N') = (\tau\tau', \tau F' + F\tau')$ and $p(N \parallel N') = (\tau F' + F\tau', FF')$, where $p(N) = (\tau, F)$, $p(N') = (\tau', F')$. Hence $p(R(a, b)) = (a, b)$. For every series-parallel network N , $\widehat{N}$ is nonseparable.

Proof. Apply Lemma 2.6 to the path and the digon with two edges, and to the graphs obtained by identifying s with t (for $F = \tau(N/st)$): a digon, and two loops at one vertex. The pair of $R(a, b)$ follows by induction. $(N \cdot N')^\wedge$ and $(N \parallel N')^\wedge$ arise from a triangle and from three parallel edges by replacing two edges by N and N' ; induction and 2-sums give the last statement. $\square$

Lemma 2.8. *For a two-terminal network N , $p(N^*) = (F(N), \tau(N))$, and $\widehat{N}^*$ is nonseparable if $\widehat{N}$ is.*

Proof. Draw the edge $e = st$ of $\widehat{N}$ outside D . Then $(\widehat{N})^* = N^* \cup \{e^*\}$ with e^* joining the terminals of N^* , that is $(\widehat{N})^* = \widehat{N}^*$; this gives the second statement. Under $T \mapsto (E \setminus T)^*$ the spanning trees of $\widehat{N}$ without e (there are $\tau(N)$ of them) correspond to those of $\widehat{N}^*$ containing e^* (there are $\tau(N^*/e^*) = F(N^*)$), and the trees containing e (there are $\tau(N/st) = F(N)$) to those avoiding e^* (there are $\tau(N^*)$). $\square$

3 The construction

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere, C its equator, and Σ_+ , Σ_- the closed northern and southern hemispheres; C is oriented counterclockwise as seen from the north pole. For $m \geq 2$ let $t_j \in C$ be the point at angle $2\pi j/m$ and μ_k the point at angle $(2k+1)\pi/m$, the midpoint of the arc a_k from t_k to t_{k+1} (indices mod m). Let ψ be the rotation of $\mathbb{R}^3$ by π about the line through the origin and the point of C at angle $\pi/(2m)$. It preserves S^2 and its orientation, exchanges Σ_+ and Σ_- , satisfies $\psi^2 = \text{id}$, and acts on C by $\theta \mapsto \pi/m - \theta$; hence

$$\psi(t_j) = \mu_{-j}, \quad \psi(\mu_k) = t_{-k}.$$

A network X with m terminals is placed in Σ_+ with terminals $t_0, \dots, t_{m-1}$, and its dual network X^* (Definition 2.4) in Σ_+ with terminals $\mu_0, \dots, \mu_{m-1}$. Then $\psi(X^*) \subset \Sigma_-$ meets C exactly in the points

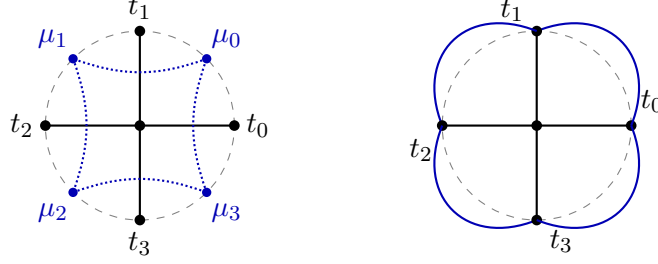


Figure 1: The star X with four single edges, in the stereographic projection from the south pole (Σ_+ is the inner disk). Left: X (solid) and X^* (dotted). Right: $G(X)$, where the four arcs outside the circle form $\psi(X^*) \subset \Sigma_-$. $G(X)$ is the wheel W_4 , the checkerboard graph of the knot 8_{18} , of determinant $45 = 5 \cdot 3^2$.

$\psi(\mu_k) = t_{-k}$, that is, in all terminals of X , and $G(X) = X \cup \psi(X^*)$ is a connected plane graph with $2|E(X)|$ edges. Figure 1 shows the case of a star with four single edges.

Proof of the first statement of Theorem 1.1. Write $G = G(X)$ and let R_k be the region of $\Sigma_+ \setminus X$ along a_k . The arcs between consecutive terminals of $\psi(X^*)$ are again the a_k ; let R'_k be the region of $\Sigma_- \setminus \psi(X^*)$ along a_k . By Lemma 2.3 the R_k are distinct, and so are the R'_k . Hence the faces of G are the regions of $\Sigma_+ \setminus X$ and of $\Sigma_- \setminus \psi(X^*)$ that do not meet C , and the m faces $F_k = R_k \cup a_k \cup R'_k$. We draw G^* as follows. The vertex of F_k is μ_k ; the vertex of a face in Σ_+ is the vertex of X^* in it; and the vertex of a face $\psi(U)$ in Σ_- , where U is a region of $\Sigma_+ \setminus X^*$, is ψ of the vertex of $(X^*)^*$ in U , where $(X^*)^* = X$ by Lemma 2.5. This is consistent: $\psi^{-1}(R'_k)$ is the region of $\Sigma_+ \setminus X^*$ along $\psi(a_k)$, the arc from μ_{-k-1} to μ_{-k} , whose vertex in $(X^*)^*$ is its midpoint t_{-k} , and $\psi(t_{-k}) = \mu_k$. The dual edges of the edges of X are then the edges of X^* , and those of the edges of $\psi(X^*)$ are the edges of $\psi(X)$. So $G^* = X^* \cup \psi(X) = \psi(\psi(X^*) \cup X) = \psi(G)$, up to isotopy. $\square$

Proof of the second statement of Theorem 1.1. The rotations ψ_t by the angle $t \in [0, \pi]$ about the axis of ψ preserve the distance from the origin and join the identity to ψ . So the knot K of $D(G)$ is isotopic to $\psi(K)$, whose diagram is $\psi(D(G)) = D(\psi(G)) = D(G^*)$ by Lemma 2.1(d), and $D(G^*)$ represents the mirror image of K by Lemma 2.1(b). Thus K is achiral; it is alternating with determinant $\tau(G)$ by Lemma 2.1(a), and prime by Lemma 2.2. $\square$

Remark 3.1. An orientation-reversing homeomorphism h of S^2 with $h(G) = G^*$ would serve as well: h , extended radially, maps K to a mirror image of K , and it reverses all crossing types relative to the colouring, so the diagram $h(D(G))$ is $D(G^*)$ with all crossings switched, which is $D(G)$ by Lemma 2.1(b). Then K is again isotopic to its mirror image.

Proof of Theorem 1.1(a). For $m = 2$, $\psi(X^*)$ joins $\psi(\mu_0) = t_0$ to $\psi(\mu_1) = t_1$, so $G(X) = X \parallel X^*$ and, by Corollary 2.7 and Lemma 2.8, $\tau(G(X)) = \tau(X)F(X^*) + F(X)\tau(X^*) = \tau(X)^2 + F(X)^2$. If $\widehat{X}$ is nonseparable, $G(X)$ is the 2-sum of $\widehat{X}$ and $\widehat{X}^*$, hence nonseparable. $\square$

For a rational network $X = R(a, b)$ this gives the determinants $a^2 + b^2$ of [Sto05, Theorem 4.2]; $G(X)$ is then a checkerboard-graph form of Stoimenow's construction (9).

Proposition 3.2. *Let X be a star with centre c and arms $N_0, \dots, N_{m-1}$, where N_j is a two-terminal network from c to t_j with pair (x_j, y_j) . Then $G(X)$ is the wheel W_m (hub c , rim vertices $t_0, \dots, t_{m-1}$) in which the spoke ct_j is replaced by N_j and the rim edge $t_i t_{i+1}$ by $\psi(N_{-i}^*)$, of pair (y_{-i}, x_{-i}) , and $\tau(G(X))$ is the sum of Lemma 2.6 over the spanning trees of W_m . For $m = 2$ it equals $(x_0 x_1)^2 + (x_0 y_1 + y_0 x_1)^2$,*

and for $m = 3, 4$ it equals $|\Lambda_m|^2$ with Λ_3, Λ_4 as in Theorem 1.1(b). If $m \geq 3$ and every $\widehat{N}_j$ is nonseparable, $G(X)$ is nonseparable.

Proof. The regions of $\Sigma_+ \setminus X$ are the inner regions of the arms and the sectors R_k between N_k and N_{k+1} . So X^* is the union of the dual networks N_j^* , where N_j^* joins μ_{j-1} (in R_{j-1}) to μ_j , and $\psi(N_j^*)$ joins t_{1-j} to t_{-j} ; with $j = -i$ this is the rim edge from t_i to t_{i+1} . Its pair is (y_{-i}, x_{-i}) by Lemma 2.8. The wheels W_2 (a triangle with one doubled edge), $W_3 = K_4$ and W_4 have 5, 16 and 45 spanning trees, and the sums of Lemma 2.6 are polynomials of degree $2m$ in the $2m$ variables x_j, y_j . Multiplying out shows that they equal the stated expressions; we checked the three identities by an exact expansion (Part B of the program `check_note.py`, Section 8), and each of the three verification runs checked them independently. The case $m = 2$ is (a) with $X = N_0 \cdot N_1$. For $m \geq 3$ the wheel is nonseparable and every $\widehat{N}_j^*$ is nonseparable by Lemma 2.8, so Lemma 2.6 applies. $\square$

With $(A, B) = (x_0, y_0)$, $(C, D) = (x_2, y_2)$ and $(X, Y) = (y_1, x_1)$ one has $\operatorname{Re} \Lambda_3 = X(AD + BC) + Y(AC + BD) = f_1$ and $\operatorname{Im} \Lambda_3 = Y(AD - BC) = f_2$, the two forms of [Sto05, (12)]. So the three-terminal wheels give exactly the numbers $f_1^2 + f_2^2$ of Stoimenow's family; we compared the determinant formulas, not the diagrams. With single edges, W_3 gives the Borromean rings ($\tau = 16$, a link) and W_4 the knot 8_{18} ($\tau = 45$). The medial graph of W_4 is Conway's polyhedron 8^* (the square antiprism); alternating knots on 8^* with rational tangles have checkerboard graphs obtained from W_4 by substituting rational networks for its edges, and our four-terminal wheels with rational arms are those whose rim networks are the prescribed duals of the arms.

Remark 3.3. The publisher's summary of [Tut79] says that a *central reflex*, a map on the sphere carried to its dual map by reflection in the centre, has a square number of spanning trees, a fact it attributes to earlier work, and that the paper determines the square root directly; we have not read the paper itself. So such self-dualities cannot give the non-square numbers of $\mathcal{R}$; the self-dualities used here are half-turns of S^2 .

4 Fourth-power scaling

Lemma 4.1. *For odd $s \geq 3$ let $Z_s = P_s \parallel (E_{s-1} \cdot e)$, where P_s is a path of s edges, E_{s-1} consists of $s-1$ parallel edges and e is an edge, and let $Z' = E_3 \cdot E_3 \cdot E_3$. Then $p(Z_s) = (s^2, s^2)$ and $p(Z') = (27, 27)$, and both are series-parallel.*

Proof. By Corollary 2.7, $p(P_s) = (1, s)$, $p(E_{s-1}) = (s-1, 1)$, $p(E_{s-1} \cdot e) = (s-1, s)$, so $p(Z_s) = (s + s(s-1), s^2)$; and $p(E_3) = (3, 1)$ gives $p(Z') = (27, 9 + 9 + 9)$. $\square$

Proof of Theorem 1.2. Let $n = \tau(G(X))$ with $G(X)$ nonseparable and n odd, let Z be a series-parallel network with $p(Z) = (k, k)$, k odd, and let X' be X with one edge f replaced by Z , drawn in a thin disk around f . Then X' is a network with the same terminals, and its dual network is X^* with f^* replaced by Z^* : the regions of $\Sigma_+ \setminus X'$ are those of $\Sigma_+ \setminus X$ and the inner regions of Z , and the duals of the edges of Z form Z^* , joining the two regions beside f . So $G(X')$ is $G(X)$ with f replaced by Z and $\psi(f^*)$ by $\psi(Z^*)$, where $p(Z^*) = (k, k)$ by Lemma 2.8. In Lemma 2.6 with skeleton $G(X)$ every term acquires the factor k from f and the factor k from $\psi(f^*)$, so $\tau(G(X')) = k^2 n$; and $G(X')$ is nonseparable, being obtained by two 2-sums. By Theorem 1.1, $k^2 n$ is realized. Lemma 4.1 gives $k = s^2$ and $k = 27$.

Let $m > 1$ be composed of primes $\equiv 1 \pmod{4}$. Then $m = a^2 + b^2$ with coprime $a, b \geq 1$, so m is realized by Theorem 1.1(a) with $X = R(a, b)$ (Corollary 2.7); scaling with $s = j$ gives $j^4 m$ for odd $j \geq 3$, and, as every $e \geq 2$ is $2\alpha + 3\beta$ with $\alpha, \beta \geq 0$, α steps with Z_3 and β with Z' give $3^{4\alpha+6\beta} m = 3^{2e} m$. Finally let $n \in \mathcal{R}$ and write $n = j^4 d^2 m$, where j and d are products of primes $\equiv 3$

(mod 4), each of which divides d at most once, and m is composed of primes $\equiv 1 \pmod{4}$; $m > 1$ because n is not a square. If $d = 1$, n is realized. If $d > 1$ and $d^2 m$ is realized, so is n . The numbers outside $\mathcal{R}$ are covered by [Sto05, Theorems 4.2 and 4.4]. $\square$

5 The numbers $5q^2$ and a three-terminal range

Proof of the first statement of Theorem 1.3. In Proposition 3.2 with $m = 4$ take rational arms with the following pairs; all of them are coprime and positive. For $Q = a^2 + ab + b^2$: $p(N_0) = (a, b)$, $p(N_2) = (b, a)$, $p(N_1) = p(N_3) = (1, 1)$. Then $z_0 z_2 = i(a^2 + b^2)$ and $\Lambda_4 = 2z_0 z_2 + a z_0 + b z_2 + ab = Q + 2iQ$. For $Q = 3Y^2 - X^2$: $p(N_0) = p(N_2) = (1, 1)$, $p(N_1) = (Y, Y + X)$, $p(N_3) = (Y, Y - X)$, which are coprime as $\gcd(Y, Y \pm X) = \gcd(X, Y) = 1$. With $z_0 = z_2 = 1 + i$,

$$\Lambda_4 = (x_1 y_3 + y_1 x_3 + y_1 y_3) + i(2x_1 x_3 + x_1 y_3 + y_1 x_3 + 2y_1 y_3) = (3Y^2 - X^2) + 2i(3Y^2 - X^2).$$

In both cases $\tau = |1 + 2i|^2 Q^2 = 5Q^2$, which is odd, and the wheel is nonseparable by Proposition 3.2. $\square$

Lemma 5.1. *Let $q \equiv 3 \pmod{4}$ be prime. If $q = 3$ or $q \equiv 1 \pmod{3}$, then $q = a^2 + ab + b^2$ with coprime $a, b \geq 1$. If $q \equiv 2 \pmod{3}$, then $q = 3Y^2 - X^2$ with $Y > |X|$ and $\gcd(X, Y) = 1$.*

Proof. We use Thue's lemma: if $q \nmid r$, there are integers x, y with $0 < |x|, |y| < \sqrt{q}$ and $x \equiv ry \pmod{q}$. (Two of the $(\lfloor \sqrt{q} \rfloor + 1)^2 > q$ numbers $u - rv$, $0 \leq u, v \leq \sqrt{q}$, agree mod q ; take differences. Neither x nor y is 0, since otherwise both are.)

For $q = 3$ take $a = b = 1$. If $q \equiv 1 \pmod{3}$, -3 is a square mod q , and Thue's lemma with $r^2 \equiv -3$ gives $q \mid x^2 + 3y^2 < 4q$. The value $2q$ is impossible ($x \equiv y \pmod{2}$ would give $4 \mid x^2 + 3y^2$), and if $x^2 + 3y^2 = 3q$ then $3 \mid x$ and $q = y^2 + 3(x/3)^2$. So $q = x^2 + 3y^2$ with $x, y \geq 1$. Then $a = x - y$, $b = 2y$ give $a^2 + ab + b^2 = x^2 + 3y^2$; $a \neq 0$ since $q \neq 4y^2$, and if $a < 0$ the pair $(-a, a + b)$ has the same value and is positive. As $\gcd(a, b)^2 \mid q$, the pair is coprime.

If $q \equiv 2 \pmod{3}$, then $q \equiv 11 \pmod{12}$ and 3 is a square mod q by quadratic reciprocity. Thue's lemma with $r^2 \equiv 3$ gives $q \mid x^2 - 3y^2$ with $-3q < x^2 - 3y^2 < q$ and $x^2 \neq 3y^2$; the value $-2q$ would give $x^2 \equiv 2 \pmod{3}$. So $3y^2 - x^2 = q$ has solutions; take one, (X, Y) , with $Y > 0$ minimal. Then $X^2 < 3Y^2$, so $|X| < 2Y$. If $|X| \geq Y$, put $\varepsilon = \text{sign } X$; the pair $(X', Y') = (2X - 3\varepsilon Y, 2Y - |X|)$ satisfies $3Y'^2 - X'^2 = 3Y^2 - X^2 = q$ and $0 < Y' < Y$ (equality $|X| = Y$ would give $q = 2Y^2$), a contradiction. So $|X| < Y$, and $\gcd(X, Y)^2 \mid q$ gives $\gcd(X, Y) = 1$. $\square$

This completes the proof of Theorem 1.3. For example $245 = 5 \cdot 7^2$ comes from $(a, b) = (1, 2)$, $605 = 5 \cdot 11^2$ from $(Y, X) = (2, 1)$, and 45 from $a = b = 1$, that is from W_4 and the knot 8_{18} .

Proof of Theorem 1.4. Take $m = 3$ in Proposition 3.2 with rational arms of pairs $p(N_0) = (1, 1)$, $p(N_1) = (d, A/k - d)$ and $p(N_2) = ((dk - B)/2, (dk + B)/2)$. The entries are positive integers, as $dk > B$ are both odd and $A/k > d$. The pairs are coprime: $\gcd(d, A/k - d)$ divides $\gcd(d, A) = 1$, and a common divisor of $(dk \mp B)/2$ divides $\gcd(dk, B) = 1$ (note $\gcd(k, B) \mid \gcd(A, B)$). With $z_0 = 1 + i$ one has $\bar{z}_0 z_2 = dk + iB$ and $\text{Im}(z_0 z_2) = dk$, so $\Lambda_3 = d(dk + iB) + (A/k - d)dk = d(A + iB)$, and $\tau = d^2(A^2 + B^2)$, which is odd. $\square$

For instance $k = 1$, $d = 3$, $B = 1$ and $A = 2c$ give all $9(4c^2 + 1)$ with $c \geq 2$ and $3 \nmid c$, such as $153 = 9 \cdot 17$.

6 All residual numbers up to 10^6

The file `certificates.csv` of the reproducibility package has one line `n;family;specs` for each $n \in \mathcal{R}$, $n \leq 10^6$. A spec is either `a/b`, with coprime $a, b \geq 1$, meaning the rational network $R(a, b)$, or `S(x,y)` or `P(x,y)`, the series or parallel composition of the networks x and y . In family F1 (15,330 numbers) the single spec is a network B and the graph is $G(B) = B \parallel B^*$ of Theorem 1.1(a); in families F2 (771 numbers) and F3 (602) the specs are the arms N_0, N_1, N_2 , resp. $N_0, \dots, N_3$, of a star as in Proposition 3.2. Table 1 lists the first lines.

n	factorization	family	specs
45	$3^2 \cdot 5$	F3	1/1 1/1 1/1 1/1 (W_4 , the knot 8_{18})
117	$3^2 \cdot 13$	F1	P(1/3, 1/3) (pair (6, 9))
153	$3^2 \cdot 17$	F2	1/1 3/1 1/2
245	$5 \cdot 7^2$	F3	1/2 1/1 2/1 1/1
261	$3^2 \cdot 29$	F1	S(1/1, P(1/3, 1/3)) (pair (6, 15))
333	$3^2 \cdot 37$	F2	1/2 1/1 1/5
405	$3^4 \cdot 5$	F1	S(1/1, S(3/1, 3/2)) (pair (9, 18))

Table 1: The first certificates. For F1 the pair $(\tau(B), F(B))$ is given; then $n = \tau(B)^2 + F(B)^2$.

Proof of Theorem 1.5. All networks in the certificates are series-parallel, so the graph of each certificate is nonseparable, by Corollary 2.7 and Lemma 2.8 with Theorem 1.1(a) for F1 and with Proposition 3.2 for F2 and F3. By Theorem 1.1 it suffices that its number of spanning trees is the stated n . The program `verify_certs.py` checks exactly the following: it computes $\mathcal{R} \cap [1, 10^6]$ by its own sieve; for every line it parses the specs, checks that every leaf `a/b` is a coprime pair of positive integers, computes the pair of every network by Corollary 2.7, and checks that $\tau(B)^2 + F(B)^2$ (F1), $|\Lambda_3|^2$ (F2) or $|\Lambda_4|^2$ (F3) equals n ; and it checks that the certified numbers are exactly the members of $\mathcal{R} \cap [1, 10^6]$. All 16,703 lines pass, in a few seconds. By Proposition 3.2 this proves the theorem. $\square$

Independently of Proposition 3.2 and of the proof of Theorem 1.1, `check_note.py` recomputes every determinant by the tree sum of Lemma 2.6, and builds every certificate graph $G(X)$ as a plane map (a rotation system, with X^* obtained by tracing faces), checking Euler characteristic 2, Kirchhoff's count $\tau(G(X)) = n$, nonseparability, an orientation-preserving isomorphism to the dual map, and one component of $D(G(X))$, in about 16 seconds; the second and third verification runs did the same with their own programs. The certificates were found by a search: for F1 over the 973,177 pairs of series-parallel networks with entries at most 1000, and for F2 and F3 over arms with entries in smaller ranges; only 9 numbers needed larger ranges for the arms, among them $79037 = 7^2 \cdot 1613$ (F2, 1/1|13/1|6/13) and $84113 = 19^2 \cdot 233$ (F3, 1/2|2/1|14/3|3/1). No network in the certificates has a pair entry above 1000.

Remark 6.1. The graphs used are planar, self-dual and without cut vertex; with Theorem 1.1(a) for $X = R(a, b)$ and [Sto05, Theorem 6.2] for squares, the “if” part of [Sto05, Conjecture 6.1] also holds for $n \leq 10^6$.

7 What remains open

By Theorem 1.2 it would suffice to realize the numbers d^2m with $d > 1$ a product of distinct primes $\equiv 3 \pmod{4}$ and $m > 1$ composed of primes $\equiv 1 \pmod{4}$; those above 10^6 not covered by Theorems 1.3 and 1.4 (and their scalings) are open. All our determinants have the form $|\Lambda|^2$ with $\Lambda \in \mathbb{Z}[i]$, and as

the primes $q \equiv 3 \pmod{4}$ are Gaussian primes, $|\Lambda|^2 = d^2 m$ forces $\Lambda = d\gamma$ with $|\gamma|^2 = m$. We explain why the three families do not give this in general.

Proposition 7.1. *Let B be a two-terminal network with $\widehat{B}$ nonseparable and $\tau(B) = q$ prime. Then $q \nmid F(B)$. Hence for $j \geq 1$ neither (q, qj) nor (qj, q) is the pair of a network B with $\widehat{B}$ nonseparable.*

Proof. As $\widehat{B}$ is nonseparable, the blocks of B form a chain from s to t , and $\tau(B)$ is the product of their numbers of spanning trees. So one block H , entered at u and left at $v \neq u$, has $\tau(H) = q$, and the others are ℓ bridges. Then $F(B) = q\ell + \tau(H/uv)$ (in a separating forest exactly one bridge is missing, or H carries a forest separating u from v), and it remains to show $q \nmid \tau(H/uv)$.

Let L be the Laplacian of H , V its vertex set, $\mathbb{Z}_0^V = \{x \in \mathbb{Z}^V : \sum x = 0\}$ and $K = \mathbb{Z}_0^V / L\mathbb{Z}^V$, a group of order $\tau(H) = q$ by Kirchhoff's theorem. For $a \in \mathbb{Z}_0^V$ let $\varphi_a \in \mathbb{Q}^V$ solve $L\varphi_a = a$ (unique up to constants), and put $\langle a, b \rangle = a \cdot \varphi_b \pmod{\mathbb{Z}}$. This is well defined on K , symmetric, and non-degenerate: if $a = Lg$ with $g \in \mathbb{Z}^V$ then $a \cdot \varphi_b = g \cdot b \in \mathbb{Z}$; and if $a \cdot \varphi_b = \varphi_a \cdot b \in \mathbb{Z}$ for all b , then φ_a has integral differences, so $a \in L\mathbb{Z}^V$. For $\delta = e_u - e_v$ one has $\langle \delta, \delta \rangle = \varphi_\delta(u) - \varphi_\delta(v) = \tau(H/uv)/\tau(H)$, the effective resistance between u and v [Kir47]. The class of δ is not 0: if $\delta = Lf$ with $f \in \mathbb{Z}^V$, the set A where f is maximal satisfies $A \neq V$ and $1 \geq \sum_{w \in A} (Lf)(w) \geq (\text{number of edges leaving } A)$, so H would have a bridge. (This is the injectivity of the Abel–Jacobi map of a 2-edge-connected graph [BN07, Theorem 1.8].) So $[\delta]$ generates $K \cong \mathbb{Z}/q$, and $\langle \delta, \delta \rangle \equiv 0$ would give $\langle \delta, \cdot \rangle \equiv 0$, contradicting non-degeneracy. Thus $\tau(H/uv)/q \notin \mathbb{Z}$. The last statement follows by Lemma 2.8. $\square$

Corollary 7.2. *Let $q \equiv 3 \pmod{4}$ be prime and $c \geq 1$ such that $1 + c^2$ has no other representation as a sum of two squares (for instance $1 + c^2$ prime). Then $q^2(1 + c^2)$ is not $\tau(B)^2 + F(B)^2$ for any network B with $\widehat{B}$ nonseparable.*

Proof. From $\tau^2 + F^2 = q^2(1 + c^2)$ and $q \equiv 3 \pmod{4}$ we get $q \mid \tau$ and $q \mid F$, so $\{\tau/q, F/q\} = \{1, c\}$, which Proposition 7.1 excludes. $\square$

If $X \parallel X^*$ is nonseparable, so is $\widehat{X}$: a partition of the edges of $\widehat{X}$ as in the definition of separability has a part not containing the edge st , and this part is separated in the same way in $X \parallel X^*$. So $45 = 3^2 \cdot 5$, $245 = 7^2 \cdot 5$ and $2313 = 3^2 \cdot 257$ are not realized by Theorem 1.1(a); they are realized by wheels. For the three-terminal wheels, Stoimenow showed that $f_1^2 + f_2^2 \neq 5p^2$ for primes $p \equiv 3 \pmod{4}$ unless a pair is $(0, p)$, $(p, 0)$ or (p, p) , and remarked that no alternating arborescent tangle has such a pair [Sto05, §4.2]; for networks this also follows, as they have no pair with a zero entry, and (p, p) is excluded by Proposition 7.1. Theorem 1.4 obtains $\Lambda_3 = d\gamma$ only under size conditions relating $\gamma = A + iB$ and d . The four-terminal wheels give $d\gamma$ for the d represented by binary quadratic forms attached to γ ; Theorem 1.3 is the case $\gamma = 1 + 2i$, and for other γ we have not found a family covering every d .

A uniform answer would need an operation that keeps primeness and multiplies Λ . Connected sum multiplies determinants but is not prime. A symmetric gluing along an edge fixed by the self-duality is not available either, since such an edge cannot exist when the determinant is odd: if $h(G) = G^*$ with $h(e) = e^*$, then h maps $G \setminus e$ onto $G^* \setminus e^* = (G/e)^*$, so $\tau(G) = \tau(G \setminus e) + \tau(G/e) = 2\tau(G/e)$ is even. In the computation of Theorem 1.5, 91.8% of the numbers are already realized by Theorem 1.1(a); we found no counterexample, but the computation gives no proof beyond 10^6 .

8 Computations and reproducibility

The proof of Theorem 1.5 is a computation, and the identities of Proposition 3.2 are finite polynomial expansions checked by computer; the other proofs do not depend on computation. The package has three parts. *The program of this note*, `check_note.py` (standard library only, about 20 seconds),

checks the finite and exact parts: the residual set; the identities of Proposition 3.2, the comparison with [Sto05, (12)] and the substitutions of Theorems 1.3 and 1.4, by exact polynomial arithmetic; Corollary 2.7 and Lemmas 2.8 and 4.1 on explicit graphs; Lemma 5.1 and $\Lambda_4 = (1+2i)q$ for all 39,322 primes $q \equiv 3 \pmod{4}$ below 10^6 ; Theorem 1.4 on 3,000 random parameter sets; all certificates, as in Section 6; the first statement of Theorem 1.1 on 400 random plane networks with $2 \leq m \leq 6$, not only series-parallel ones (the wrong gluing $\mu_k \mapsto t_{k+1}$, used as a control, gives a self-dual map only for $m = 2$, where it is another half-turn); Theorem 1.2 on 180 substitutions in certificate maps; and Proposition 7.1 on 300 random graphs. No check failed. *The programs with which the results were first obtained* (`original/`) include `verify_certs.py` (the check in the proof of Theorem 1.5), `map` and `sympy` checks, a check on 400 certificates of the condition of [FMP17, Theorem 1.1] for achiral knots (for each prime $q \equiv 3 \pmod{4}$ dividing n , the q -primary part of the homology of the double branched cover is not cyclic), and the generator `gen_certs.py`, which needs about 4 GB of memory. Their module `planemap.py` lists the darts of a dual vertex in clockwise order, so the orientation labels of its isomorphism test are interchanged; its programs test both orientations, either of which gives achirality (Remark 3.1), rather than identifying the one that occurs. *The programs and outputs of the three verification runs* are in `verification_run_A/` and `verification_run_B/` (on the first write-up of the results) and in `independent_run_2/` (on the text of this note).

Verification. The results were first obtained with the programs in `original/`. Two independent verification runs, both AI-assisted, followed; each wrote its own programs. The first run checked the statement against [Oht02, Sto05], re-derived the proofs, and recomputed the residual set, all certificates, the identities, and plane maps of 1,500 certificates and of instances of Theorems 1.2 to 1.4, with a failing control. The second run built $G(X)$ by face tracing in a map library of its own and checked all 16,703 certificate maps in about 7 seconds (Euler characteristic, Kirchhoff count, no cut vertex, isomorphism to the dual map, one component, writhe 0, and the condition of [FMP17, Theorem 1.1]); it tested Theorem 1.1 on 530 networks that are not series-parallel and Lemma 5.1 for all primes below 10^6 , used mutation tests to show that its checks reject wrong input, and re-ran the programs of `original/`, which reproduced their recorded outputs byte for byte. Neither run found an error in a statement or a proof; their corrections were made: the remark of [Sto05, §4.2] on 8^* , the literature on self-dual maps and [Tut79] instead of a novelty claim for the construction, the attribution of “but not always both” to [Oht02], the tree counts 16 and 45, full references, and the orientation convention of `planemap.py`. Then `check_note.py` was written for the exact text of this note; all its checks pass.

A third independent verification run, also AI-assisted, then checked the text of this note line by line. It read [Oht02, p. 541] and Sections 1, 2, 4 and 6 of [Sto05] again, re-derived every proof, and wrote its own programs (`independent_run_2/`). With its own plane-map library, whose orientation convention for duals it validated geometrically, it built $G(X)$ for 500 random plane networks with $2 \leq m \leq 6$ (367 of them not series-parallel) and for all 16,703 certificates. In every case it found Euler characteristic 2, an orientation-preserving isomorphism to the dual map, Kirchhoff’s count as stated and, for odd counts, one component; the control gluing $\mu_k \mapsto t_{k+1}$ failed for every $m \geq 3$. It also expanded the tree sums of W_2 , W_3 , W_4 exactly, made 300 gadget substitutions in certificate maps, checked Lemma 5.1 and Theorem 1.3 for all primes $q \equiv 3 \pmod{4}$ below 10^6 , Theorem 1.4 on 3,000 parameter sets and Proposition 7.1 on 300 random graphs, and recounted the 973,177 series-parallel pairs. It re-ran all programs of the package from an extracted copy of the source archive; they reproduced their recorded outputs byte for byte apart from timings, and the generator reproduced `certificates.csv`. It found no error in a statement or a proof; its corrections were made: Stoimenow’s remark on arborescent tangles in Section 7, the source for $n = 1$, the argument that $\widehat{X}$ is nonseparable when $X \parallel X^*$ is, the description of the search in Section 6, and the wording of the reduction in the abstract and of

Remark 3.3.

Scope and priority. The note does not answer the problem: it proves Theorems 1.1 to 1.5, and the cases $d^2m > 10^6$ of Section 7 remain open. The problem, Theorems 4.1, 4.2, 4.4 and Proposition 4.1 of [Sto05], the formulas (9) and (12) (the cases $m = 2, 3$ of Theorem 1.1), the remark on $5p^2$ for $p \leq 11$ and the remark on arborescent tangles quoted in Section 7 are Stoimenow's. Gluing a graph to its dual to obtain self-dual maps is classical [ST50, AR92, SC92, SS96], and Tutte [Tut79] studied spanning trees of self-dual maps; the determinant as a number of spanning trees and Menasco's criterion are standard. Only the application to the problem, the formula for Λ_4 and Theorems 1.2 to 1.5 are, to our knowledge, not in print. We read [Oht02, p. 541], Sections 1–6 of [Sto05] (arXiv v3; Section 7 is on polynomial invariants), [FMP17, Theorem 1.1] and [BN07, Theorem 1.8] (arXiv version). We know [ST50, AR92, SC92, SS96, Tut79] only from titles and bibliographic summaries, use [Men84, Kau87, Mur87, Thi87, HK79] as quoted in [Sto05], and judged the follow-up [Sto07] by its title (and, in the second verification run, by its zbMATH review, which concerns leading coefficients). The searches, on 19 October 2026, used arXiv (achiral, amphicheiral or amphichiral knots with determinants or spanning trees; self-dual graphs and spanning trees; the author of the problem), the works citing [Sto05] in OpenAlex (15) and Semantic Scholar (17), those citing [Sto07] in OpenAlex (11), zbMATH, Crossref and six web searches. No printed answer to the problem was found in these sources; determinants of amphichiral knots appear there as obstructions (for instance in [BI24]), but no work found realizes the numbers of $\mathcal{R}$. A search that finds nothing is not a proof of novelty, and we claim no priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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